Optimal Allocation of Prizes in Contests

Benny Moldovanu & Aner Sela (2001)
In 1902, Francis Galton asked one of the earliest versions of the question this paper addresses. His question was how to divide a prize pool among competitors (Biometrika, 1902).

This question remains important - consider organizations such as the X Prize Foundation, which ’designs and manages public competitions intended to encourage technological development that could benefit mankind’. Clearly, they would be interested in knowing how optimally to allocate their available funds to maximize technological development (via a mechanism design perspective).
Further examples of contests where this kind of question is important: promotion 'contests' in firms, grades as prizes for a competition in schooling, athletic competitions, architectural competitions, etc.

Sometimes, the contest designer seeks to maximize total effort, sometimes they seek to maximize the quality of the winning entrant. It seems reasonable to believe that this would change the optimal prize allocation.

Most literature prior to this paper considered contests which awarded a single prize - in other words, the entire prize was allotted to the winner. Again, it seems reasonable to believe that this might not be optimal.
Model 1

1. There is a contest which awards $p$ prizes, where the value of the $j$'th prize is $V_j$, and $V_1 \geq V_2 \geq ... \geq V_p \geq 0$, and all $V_p$ are common knowledge to players.

2. There is a set of risk-neutral contestants $K = \{1, 2, ..., k\}$, and without loss of generality $k \geq p$.

3. Each contestant $i$ undertakes an observable effort $x_i$, which incurs a cost $c_i \cdot \gamma(x_i)$, where $\gamma(\cdot)$ is strictly increasing and $\gamma(0) = 0$.

4. $c_i$ is thus an ability parameter, that reflects how costly effort is for each contestant $i$ - agents in this model are heterogeneous, and $c_i$ is agent $i$'s private information.
1. The contestant who expends the most effort wins $V_1$, the contestant who expends the second highest effort wins $V_2$, and so forth.

2. As is standard, each contestant $i$ chooses $x_i$ to maximize her expected utility given other contestants efforts and the value of the different prizes.

3. The contest designer seeks to maximize the expected value of $\sum_{i=1}^{k} x_i$ - the expected total effort expended by contestants.
Consider the case of linear cost functions. The authors show that there is a unique, symmetric equilibrium in the case of 2 prizes and 3 or more contestants, where the effort function is a function of $V_1$, $V_2$, $k$, and distributions on the ability parameter.

More interestingly, the authors show that the optimal prize allocation is *always* to allocate the entire prize pool to the winning contestant. This result is robust to allowing the designer to offer more than two prizes.
When players have concave cost functions, it is again the case that the optimal allocation of prizes is to allocate the entire prize pool to the winning contestant.

But when players have convex cost functions, it is optimal to allocate a strictly positive amount to the second place contestant, and sometimes (depending on the degree of convexity) to award equal prizes to both the winner and the runner-up.

This result is again robust to changing the number of prizes - if cost functions are sufficiently convex, it is optimal to allocate up to $k - 1$ different prizes (i.e. prizes to all but the worst contestant).
For the specific example of uniformly distributed ability, a $c \cdot \gamma(x_i) = cx_i^2$, 3 players and 2 prizes, the optimal allocation of a prize pool of 1 is approximately 0.63 to the winner and 0.37 to the runner-up.

The intuition for this should be fairly obvious - convex costs imply that, without a runner-up prize, the second-place contestant would strongly prefer to have expended less effort. The provision of multiple prizes allows for equilibria in which non-winning contestants are nonetheless happy with the return to their investment of effort.
The intuition for the first two results should be even more obvious. With linear and concave cost functions, the impact of increasing the first prize is larger than the impact of increasing the second prize - primarily because the winning contestant should in general be the contestant with the lowest cost of effort, and thus the one willing to expend the most effort for a given prize.

With convex costs, however, the curvature means that increasing the first prize has decreasing marginal effects on the incentive for the lowest cost player to increase effort - and thus opens the door for improvements by incentivizing extra effort from higher cost players.
A contest designer could charge an entry fee, which would be expected to exclude contestants with low ability (high cost parameters) from the contest.

Since the impact of higher prizes for non-winning contestants is due to the effort expended by players with higher costs, one might expect the option of charging entry fees to dominate even in the case of convex cost functions - but this turns out not to be necessarily true.

In the example with uniform ability, quadratic costs, 3 players and 2 prizes, the optimal entry fee is 0, and thus the optimal contest design still involves multiple prizes.
An obvious extension would be to relax risk-neutrality. It seems reasonable to believe that the effect of risk aversion would be similar to the effect of convex cost functions - the problem is that determination of equilibria for the risk-averse or risk-loving cases is not guaranteed to be possible.

What if players have the choice of multiple contests? Again, it seems reasonable to believe that offering multiple prizes would become more attractive to the contest designer, as it seems obvious that it would maximize the number of contestants they receive.

Repeated contests - what if we consider the pool of potential contestants to be fixed (i.e. sports leagues). Players who consistently lose contests will stop entering, which is not optimal at least in the example of sports.
1. What if contestants produce a tangible asset with their effort, win a prize for producing the best asset, but then are able to sell the asset as well - i.e., the contest is designed to incentivize effort purely for the sake of effort.

2. What if the cost function is not separable in ability and effort - i.e., what if \( c(x_i) \) cannot be decomposed into \( c \cdot \gamma(x_i) \)?