1. Let $R(p^c, V)$ denote the GNP function and $e(p^c, U)$ the expenditure function for a single agent economy.

(i) Assuming no domestic taxes: $p^e = p^c$. The autarky equilibrium is described by:

\[ R(p^a, V) = e(p^a, U^a); \quad R_p(p^a, V) = e_p(p^a, U^a) \]

where the superscript “$a$” reflects the autarky solution. Let $p^t$ denote the domestic price in the presence of trade (which may differ from the world price due to tariffs). Then:

\[ R(p^t, V) \geq p^t \cdot x^a = p^t \cdot e^a \geq e(p^t, U^a) \]

where the first inequality follows from the definition of (revenue) maximization, the equality follows from the definition of the autarky equilibrium, and the last inequality follows from the definition of the expenditure function. The inequalities will, in general, be strict provided that $\tilde{p}^t \neq \tilde{p}^a$ (i.e., if relative prices differ), though this may not be the case if there is no substitutability in production or consumption. From the budget constraint under trade:

\[ e(p^t, U^t) = R(p^t, V) + TR \geq e(p^t, U^a) + TR \]

where $U^t$ denotes the utility level in the trade situation and $TR$ denotes net tariff revenue. Thus, assuming $TR \geq 0 \rightarrow U^t \geq U^a$ since the expenditure function is increasing in “$U$”. Note that, in general, the inequality will be strict.

This proof says nothing about “optimal” policy and hence does not imply free trade is optimal. It merely says that, provided tariff revenue is non-negative, trade will be welfare-enhancing (not welfare reducing).

Why can subsidies lead to lower welfare? Intuitively, autarky is one extreme - or corner solution and we have seen that some trade must be beneficial. This does not imply that free trade is optimal (it is beneficial); but subsidies lead to larger volumes of trade than free trade and thus may lead to declines in welfare. In other words, the fact that welfare is increasing in trade flows in some domain does not imply it is globally increasing in these flows (as caused by changes in trade policies).

(ii) How were the absence of distortions and domestic taxes used? First, by using the same prices for producers and consumers we assume no domestic taxes. Second, by assuming laissez faire decisions lead to GNP maximization we assume no externalities among producers and by assuming the expenditure function does not depend upon domestic output (for example) we assume the absence of externalities between domestic producers and consumers. Should any such distortion occur, the conclusion might not hold.

(iii) Let goods 1 be numeraire, assume a tax on production good 2, and assume a small country. Let $t$ denote the consumption tax so that, in a closed economy: $p^{f1} = p^c = 1; \quad p^{f2} = p^c + t$. The autarky equilibrium is given by:

\[ (1) e(p^1, p^2) = R(p^1, p^2, V) + t(R_{p^2}); \quad (2) e_{p^2}(p^1, p^2, U^a) = R_{p^2}(p^1, p^2, V); \quad p^1 = 1; \quad p^2 = p^f + t \]

where (1) is the budget constraint and (2) the equilibrium condition for market two. Due to the tax, good (2) is underconsumed/underproduced, its domestic producer price is too low, its consumer price too high.
If we allow trade welfare may fall because we are in a second best situation. *Note that, in a closed economy model, a production and consumption tax are the same; however, in an open economy model the two taxes have different effects.* Intuitively, trade is most likely to reduce welfare in the presence of a production tax on good 2 if the trade leads to further reduction in production of good two, which is already underproduced; *i.e.,* we expect welfare might fall if the world price of good two is below the autarky (consumer) price. The analysis is simplified by the small country assumption, so the world price is exogenous. Thus, with free trade and the consumption tax we have:

\[
p_2' = (p_2^w - t); \quad p_2' = p_2^w; \quad p_2' = (p_2^w - t) \quad (1) \quad e(p_1, p_2^w, U^T) = R(p_1, p_2^w, V) + tR(p_1, p_2^w, V)
\]

where (1), the budget constraint, determines domestic utility in terms of the world price (assumed exogenous) and the tax rate: \( U^T = \phi(p_2^w, t) \). No market-clearing equation is needed because supply has to be equal to demand on world markets, not domestic markets, and because we assume the country is small (so that its policies have no impact on world price).

Clearly, if the world price equals the autarky (consumer) price the opportunity to trade will not affect welfare. Differentiating (1):

\[
e_{p_2'} \cdot dp_2^w + u \cdot dU^T = R_{p_2} \cdot dp_2^w + t \left\{ R_{p_2, p_1} \cdot dp_2^w \right\}; \quad dp_2^w = dp_2' = dp_2^w
\]

Collecting terms yields:

\[
e_{p_2'} dU^T = \left( R_{p_2} - e_{p_2} + tR_{p_2, p_1} \right) \cdot dp_2^w; \quad \text{Thus:}
\]

\[
\frac{dU^T}{dp_2^w} = \left( x_2 \left( p_2^w, t, \ldots \right) + tR_{p_2, p_1} \right) \quad \text{where} \quad x_2 \equiv R_{p_2} - e_{p_2} \quad \text{denotes net exports of good two.}
\]

Further, \( R_{p_2, p_1} > 0 \) since it denotes the slope of the supply demand curve. Thus, for \( t > 0 \),

\[
\left( \frac{dU^T}{dp_2^w} \right) > 0 \quad \text{if} \quad x_2 \geq 0 \quad \text{whereas for} \quad x_2 < 0 \quad \text{the sign is potentially ambiguous. Hence, if the world price is above the domestic consumer price, free trade must be beneficial. However, if} \quad p_2^w < p_2^{*a} \quad \text{then trade (a reduction in} \quad p_2 \quad \text{) could lower welfare. In particular, evaluating the derivative at the autarky price implies that utility is increasing in the world price at that point; *i.e.,* if the world price is slightly below the domestic consumer price, then opening the economy up to trade must reduce welfare. However, as the world price continues to decrease, exports become sufficiently negative so that further decreases in \( p_2^w \) will benefit the country.}
\]

In summary, due to the distortion, welfare may fall with the movement from autarky to (free) trade; in this example, welfare can only fall if the world price of good two is below the autarky price (this is a necessary, not a sufficient, condition).
2. This is an economy with two factors whose supplies are endogenous. I change notation slightly and label the goods \((1,2)\). To start, assume the following more general set of preferences:

\[
U^h = \eta^h \left(C_{1h}^h, C_{2h}^h, \ell_1^h, \ell_2^h\right); \quad \ell_i^h + L_i^h \leq \left(L_i^c\right)^e; \quad h = 1, 2; \quad i = 1, 2
\]

where \(\left(C_{1h}^h, C_{2h}^h\right)\) is the household’s consumption of produced goods, \(\left(\ell_1^h, \ell_2^h\right)\) is household consumption of the two factors, \(\left(L_1^h, L_2^h\right)\) is household sales (to the market) of the two factors, and \(\left(\left(L_1^c\right)^e, \left(L_2^c\right)^e\right)\) is the exogenously given endowment vector. For the special case given, household \(h\) is endowed, and derives utility, only from factor \(h\). Each household’s expenditure function is derived as follows:

\[
\text{Min} \left[p_1 C_{1h}^h + p_2 C_{2h}^h - w_1 L_1^h - w_2 L_2^h\right] \quad \text{s.t.} \quad \eta^h \left(C_{1h}^h, C_{2h}^h, \left(L_1^h\right)^e - L_1^c, \left(L_2^h\right)^e - L_2^c\right) \geq U^h
\]

Note that the input supply has been substituted into the utility function (constraint), so that utility is written as a decreasing function of input sales (rather than as an increasing function of own consumption of inputs). Solving the optimization problem yields the usual MRS equal relative price conditions and the minimized value of the objective function yields the expenditure function:

\[
e^h \left(p_1, p_2, w_1, w_2; U^h, \left(L_1^h\right)^e, \left(L_2^h\right)^e\right); \quad \left(\frac{\partial e^h}{\partial p_i}\right) = C_{1h}^e \left(\tilde{p}, \tilde{w}, U^h\right); \quad \left(\frac{\partial e^h}{\partial w_i}\right) = -L_i^e \left(\tilde{p}, \tilde{w}, U^h\right)
\]

where initially I write the expenditure function to show it depends (negatively) on the endowment vector (ceteris paribus, increasing endowments increases the amount of inputs that can be sold \{decreases amount of goods that must be bought\} in order to reach utility target). Hereafter, the endowment vector is suppressed, since if it is not changing it can be subsumed in the form of the expenditure function. The expenditure function has its usual properties, as illustrated; note that for the special case given in problem 2:

\[
\left(\frac{\partial e^h}{\partial w_i}\right) = 0 \quad \text{for} \quad i \neq h \quad \text{since each household is endowed with, and consumes, only one type of input.}
\]

(a) The autarky equilibrium is described by the following equations:

(1) \(e^h \left(\tilde{p}, \tilde{w}, U^h\right) = 0\);

(2) \(\sum_k e^h_{\lambda_k} \left(\tilde{p}, \tilde{w}, U^h\right) - R_{\lambda_k} \left(\tilde{p}, \tilde{V}\right) = 0\);

(3) \(\sum_k e^h_{\lambda_k} \left(\tilde{p}, \tilde{w}, U^h\right) + V_i = 0\);

(4) \(w_i = R_{\lambda_i} \left(\tilde{p}, \tilde{V}\right)\)

where the first set of equations (there are 2 of them for this model) is the household budget constraint; the second set of equations (again there are 2) denotes goods market equilibrium, the third set (2 of them) is factor market equilibrium, and the fourth set (2 of them) determines factor prices. This set of 8 equations determines the 8 variables (4 prices, 2 Utility levels, and 2 input supplies; note that output levels appear only as functions); naturally, there are only 3 relative prices and one equation is redundant due to Walras’ law. Also note that there are no profits, due to CRS; if there were profits, they would have to be distributed back to households (through the expenditure function constraint).

(b) With lump sum transfers the first set of equations above is modified by allowing expenditures to equal net transfers received; the second set of equations is dropped, as prices are determined on world markets (this does not mean the country is small, only that equation set (2) is not sufficient to determine world prices); equation sets (3) and (4) remain, though they will be evaluated at a different commodity price vector. Let \(\left(U^h\right)^e\) denote the autarkic utility level for household \(h\), \(\left(\tilde{p}', \tilde{w}'\right)\) the commodity, factor price vector in the
presence of trade, and \( T^h \) the transfer to each household required to let them reach the autarkic utility level at the trade price vector; thus:

\[
(1') e^h \left( \tilde{p}^i, \tilde{w}^i, (U^h_i) \right) = T^h; \quad (3') \sum_h e^h_i \left( \tilde{p}^i, \tilde{w}^i, (U^h_i) \right) + V^i = 0; \quad (4) \ w_j = R_{ij} \left( \tilde{p}^j, \tilde{V}^j \right)
\]

The government’s budget constraint is then given by: \( B = -\sum_h T^h \geq 0 \). The problem is to show this latter equality is satisfied (i.e., that the government can afford to compensate each household as a result of the move to the trading situation).

\[
e^h \left( \tilde{p}^i, \tilde{w}^i, (U^h_i) \right) \leq \left[ \sum_i p_i^h \left( c_i^h \right)^a - \sum_i w_i^h \left( L_i^h \right)^a \right] \text{ by definition of the expenditure function. Thus:}
\]

\[
\sum_h T^h = \sum_h e^h \left( \tilde{p}^i, \tilde{w}^i, (U^h_i) \right) \leq \sum_{i,h} p_i^h \left( c_i^h \right)^a - \sum_h w_i^h \left( L_i^h \right)^a = \sum_i p_i^h x_i^a - \sum_i w_i^h V_i^a
\]

where \( x_i^a \) denotes autarky output of good \( i \), and \( V_i^a \) is the autarky use (supply) of input \( i \). Finally, by profit maximization:

\[
\sum_i p_i^h x_i^a - \sum_i w_i^h V_i^a \leq \sum_{i,h} p_i^h x_i^a - \sum_i w_i^h V_i^a = 0 \text{ since } \left( x_i^a, V_i^a \right), \left( x_i^a, V_i^a \right) \text{ are both feasible activities, and the latter is chosen at the trade price vector (the last equality follows from CRS). Thus:
}\]

\[
\sum_h T^h \leq 0, \text{ proving the feasibility of the government plan. If the inequality is strict, then the government can increase transfers to all agents, so that all gain from trade.}
\]

However, if lump sum transfers are not feasible, then not all agents need gain as factor prices change; this is part of the point of the H-O model.

For part (c): Using the special utility function (modified by my relabeling of goods)

\[
U^I = C_1^I C_2^I L_1^I; \quad U^II = C_1^{II} C_2^{II} L_2^{II}; \text{ with endowments: } \left( L_1^{I}, L_1^{I} \right) = (1,0); \quad \left( L_1^{II}, L_2^{II} \right) = (0,1). \text{ From the}
\]

symmetry, expenditure functions are easily derived by: \( \min \left( \sum_i p_i C_i - wL \right) \text{ s.t. } (C_1 C_2 (1-L)) \geq U^h \)

Thus:

\[
e^I (\ldots) = 3 \left( w_1 p_1 p_2 \right)^{1/3} \left( U^I \right)^{1/3} - w_1; \quad e^II (\ldots) = 3 \left( w_2 p_1 p_2 \right)^{1/3} \left( U^II \right)^{1/3} - w_2
\]

(c) Given technology: \( Q_1 = L_1 \), \( Q_2 = L_2 \). Assuming no domestic (production or consumption) taxes implies: \( p_1 = w_1 \); \( p_2 = w_2 \)

\[
(i) \text{ For autarky, demands for good } i: \ D_i = \sum_i e^h_i = \left( \frac{w_1 p_j U^I}{(p_j)^2} \right)^{1/3} + \left( \frac{w_2 p_j U^{II}}{(p_j)^2} \right)^{1/3}; \quad j \neq i
\]
From technology, \( D_i = L_i = -e^i_{w_i} = 1 - \left( \frac{p_1 p_2 U^i}{(w_i)^2} \right)^{\frac{1}{3}} \); \( i = 1, 2 \); finally, net expenditures equal zero.

\[ e^i(\ldots) = 0 \rightarrow U^i = \frac{(w_i)^2}{3} \frac{p_i}{p_j} = \frac{p_i}{3} \; \text{; \( i \neq j \) since \( w_i = p_i \). Using demand equals supply, and the obvious symmetry yields the autarky equilibrium: \( L_i = 2/3 \); \( p_i = p_2 = w_1 = w_2 = 1 \); \( U^i = 3^{-3} \)

(ii) Assume free trade and \( p^w_1 = 1, \ p^w_2 = 2 \). Due to technology, and absent taxes/tariffs, \( w_1 = p^w_1 = 1 \), \( w_2 = p^w_2 = 2 \). Thus, person 2 gains from trade and person 1 loses. In particular, substituting into the expenditure function we have:

\[ e^i(\ldots) = 3(2)^{\frac{1}{3}} \left( U^i \right)^{\frac{1}{3}} - 1 = T^i; \ e^{ii}(\ldots) = 3(4)^{\frac{1}{3}} \left( U^{ii} \right)^{\frac{1}{3}} - 2 = T^{ii} . \]

Assuming no transfers \( (T^i = T^{ii} = 0) \)
implies: \( U^{ii} = 2 \cdot 3^{-3} > 3^{-3} > \frac{1}{2} \cdot 3^{-3} = U^i \). However, it is clear that if transfers are possible s.t. \( (T^i + T^{ii}) = 0 \), then both can gain from trade.

(iii) Now we must distinguish among producer prices, consumer prices and world prices; let \( (p_1, w_1, p_2, w_2) \) represent consumer prices, \( (p^w_1, p^w_2, w_1, w_2) \) denote producer prices and \( (p^w_1, p^w_2) \) world prices; (remember that tariffs are combinations of producer subsidies and consumer taxes so we need not separately specify tariffs). Technology implies: \( w_1 = p^w_1; \ w_2 = p^w_2 \). The government budget constraint is:

\[ B = \sum_i \left( p^w_i - p_i \right) Q_i + \sum_i \left( p_i - p^w_i \right) C_i + \sum_i \left( w_i^i - w_i \right) L_i \geq 0 \]

where \( Q_i \) is aggregate output of good \( i \), \( C_i \) is aggregate consumption of good \( i \), \( L_i \) is aggregate supply of factor \( i \); the first summation represents tax revenue from the production tax, the second from the consumption tax, and the third from the factor tax. Note that with the aggregate household budget constraint:

\[ \sum_i p_i C_i - \sum_i w_i L_i = 0 \], and the zero profit condition for firms: \( \sum_i p_i Q_i - \sum_i w_i L_i = 0 \) this constraint reduces to: \( B = \sum_i p^w_i (Q_i - C_i) \geq 0 \) which is the Balance of Trade constraint. Using the specific structure:

\[ Q_i = L_i = -e^i_{w_i} = 1 - \left( \frac{p_1 p_2 U^i}{(w_i)^2} \right)^{\frac{1}{3}} ; \ C_i = \sum_h e^h_{p_i} = \left( \frac{w_j p_j U^i}{(p^i_j)^2} \right)^{\frac{1}{3}} + \left( \frac{w_j p_j U^{ii}}{(p^i_j)^2} \right)^{\frac{1}{3}} ; \]

Note that, if consumer prices are set at autarky levels, the budget constraint holds as equality - that is, **there will be no gains (or losses) from trade**. This is consistent with the proof in class but the fact there are no gains from trade if consumers face autarky prices is a bit “disappointing”; the reason for the result is that there is no substitutability in production. The underlying logic of the formal proof is that if we insulate consumers from price changes, producers will still adapt optimally; in this case, because the production possibility frontier is a box (given input supplies) there is no supply response and hence no gains from trade if consumers are insulated from price changes.
Next, for this model consider whether - absent lump sum transfers – it is possible to devise domestic policy so that both parties gain from trade. Since there are no lump sum transfers, we must have:

\[ e^h(\ldots) = 0 \rightarrow (U^h) = \frac{w^2}{3}p_1p_2 \left( \frac{\bar{w}_h}{\rho} \right)^2 \]

where I define: \( \rho = \frac{p_2}{p_1} \); \( \bar{w}_h = \frac{w_h}{p_1} \), representing relative prices and real wages in terms of good one as the numeraire. Substituting back into the supply and demand relations yields:

\[ Q_1 = \frac{2}{3}; \quad C_1 = \frac{\bar{w}_1 + \bar{w}_2}{3}; \quad C_2 = \frac{\bar{w}_1 + \bar{w}_2}{3\rho} \]

Thus, substituting yields the government budget constraint:

\[ B = p_i^w \left( \frac{2}{3} - \frac{\bar{w}_1 - \bar{w}_2}{3} \right) + p_2^w \left( \frac{2}{3} - \frac{\bar{w}_1 + \bar{w}_2}{3\rho} \right) \geq 0 \]

Clearly, this is satisfied for \textit{laissez-faire}; \textit{i.e.}, if:

\[ \bar{w}_1 = \frac{w_1}{p_i}; \quad \bar{w}_2 = \frac{w_2}{p_i} = \rho; \quad \rho = \frac{p_2^w}{p_i^w} \]

Given the budget constraint, the welfare problem is:

\[ \text{Max} \left( U^H \right) \quad \text{s.t.} \quad U^I \geq \bar{U}^I; \quad B \geq 0; \quad \text{forming the Lagrangean:} \]

\[ N(\ldots) = \delta_2 \left( \frac{\bar{w}_2}{3\rho} \right)^2 + \delta_2 \left( \frac{\bar{w}_2}{3\rho} - \bar{U}^I \right) + \lambda \left[ \frac{2}{3} - \frac{\bar{w}_1 - \bar{w}_2}{3\rho} \right] + \rho \left[ \frac{2}{3} - \frac{\bar{w}_1 + \bar{w}_2}{3\rho} \right]; \quad \rho = \frac{p_2^w}{p_i^w}; \quad \delta_2 = 1 \]

where \( \delta, \lambda \) are the Lagrangean multipliers for the respective constraints. Optimizing over \( \bar{w}_1, \bar{w}_2, \rho, \lambda, \delta \) and using the constraints yields:

\[ N = \frac{2\delta_1 \bar{w}_1}{3\rho} + \frac{\lambda}{3} \left( 1 + \frac{\rho}{\rho} \right) = 0; \quad \delta = \frac{1}{3\rho} \left( 1 - \lambda \right); \quad \lambda \left( \frac{\bar{w}_1}{\rho} \right)^2 + \frac{\lambda}{3\rho} \left( \frac{\bar{w}_2}{\rho} \right)^2 = 0 \]

These equations, in conjunction with the constraints, yield:

\[ \rho = \frac{\rho}{\rho}; \quad \bar{w}_1 = \sqrt{3\rho^w \bar{U}^I}; \quad \bar{w}_2 = \left( 1 + \rho^w - \bar{w}_1 \right); \quad U^H = \left( \frac{1 + \rho^w}{\sqrt{3\rho^w - \sqrt{\bar{U}^I}}} \right)^2 \rightarrow \sqrt{\bar{U}^I} + \sqrt{\bar{U}^H} \leq \frac{1 + \rho^w}{\sqrt{3\rho^w}} \]

Insuring both gain from trade entails: \( \bar{U}^I \geq 3 - 3 \rightarrow \bar{w}_1 \geq \sqrt{\rho^w} \); thus, both can gain from trade by a wage subsidy to person I even though there is no production substitutability. Note that if lump sum transfers were feasible the optimal policy would be to set domestic prices at world prices and use these transfers, so that higher utility levels would be achievable.

(d)The problem here is essentially the same, except production substitutability is feasible; \textit{letting C be good 1, and M be good 2}; and \( L_j \) be the amount of input \( j \) used to produce good \( i \) we have:

\[ Q_1 = \sqrt{L_{11} \cdot L_{12}}; \quad Q_2 = L_{22}; \quad L_{11} \leq L_1; \quad L_{12} + L_{22} \leq L_2 \]

where \( \{ L_j \} \) is, as before, agent \( i \)'s labor supply.

The dual cost curves are: 

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\[ TC(Q_1, \bar{w}_1, \bar{w}_2) = 2Q_1 \sqrt{\bar{w}_1 \bar{w}_2}; \quad TC(Q_2) = Q_2 \bar{w}_2 \]
where, as above, \((\bar{w}_1, \bar{w}_2)\) denotes input prices paid by firms, and \((\bar{p}_1, \bar{p}_2)\) denotes output prices received by firms. Assuming both goods are produced output prices and total factor demands are given by:

\[ p_1' = 2\sqrt{\bar{w}_1 \bar{w}_2}; \quad p_2' = \bar{w}_2; \quad L^D_1 = Q_1 \sqrt{\bar{w}_2 \bar{w}_1}; \quad L^D_2 = L_{12} + L_{22} = Q_1 \sqrt{\bar{w}_1 \bar{w}_2} + Q_2 \]

Preferences, and hence goods demands/factor supply functions are the same as above in part (c); thus:

\[ L^*_i = \frac{2w_i - T_i}{3w_i}; \quad U^i = \left(\frac{w_i + T_i}{3p_i} \right)^3; \quad C^D_i = \left(\frac{w_i + T_i}{3p_i} \right) + \left(\frac{w_2 + T_2}{3p_2} \right) = \left(\frac{w_i + w_2}{3p_i} \right) \]

where: \(\sum T_i = 0\), and \(T_i\) denotes the transfer to agent \(i\). The fact transfers do not affect aggregate demand (given the aggregate budget constraint) follows from identical and homothetic preferences. As earlier, \((p_1, p_2, \bar{w}_1, \bar{w}_2)\) denotes prices facing consumers (notice that factor supplies are affected by transfers since each person supplies a different type of labor).

(i) Autarky equilibrium: Assuming no taxes or transfers, and using good 1 as numeraire, we have:

\[ L_1^* = L_2^* = \frac{2}{3}; \quad \bar{w}_2 = p_2; \quad p_1 = 2\sqrt{\bar{w}_1 \bar{w}_2} \rightarrow \bar{w}_1 = (4p_2)^{-1}; \quad \bar{p}_1 = 1; \]

\[ Q_1 = C^D_1 = \frac{w_1 + w_2}{3p_1} = \frac{1 + 4(p_2)^2}{12p_2}; \quad Q_2 = C^D_2 = \frac{1 + 4(p_2)^2}{12(p_2)^2}; \quad L^D_1 = Q_1 \sqrt{\frac{\bar{w}_2}{w_1}} = \frac{1 + 4(p_2)^2}{6} \]

Finally, substituting into factor market equilibrium \((L^D_1 = L^*_1)\) yields:

\[ p_2 = w_2 = \frac{\sqrt{3}}{2}; \quad \bar{w}_1 = \frac{1}{2\sqrt{3}}; \quad \left( U^i \right) = \frac{1}{3^3 \cdot 6 \cdot \sqrt{3}}; \quad U^H = \frac{\sqrt{3}}{2 \cdot 3^3} \]

(ii) Trading at \(p_1^\ast = 1, p_2^\ast = 2\) implies that the domestic price of good two rises (the country exports two), and hence the wage earned by agent I declines. Given exogenously determined prices, demand is not needed to determine wages or utility; thus, under trade:

\[ w_2 = p_2^\ast = 2; \quad \bar{w}_1 = (1/8); \quad U^i = \frac{1}{128 \cdot 3^3}; \quad U^H = \frac{2}{3^3}. \]

As is apparent, agent II gains and agent I loses from the move to free trade. It is also apparent that, with lump sum transfers, domestic factor (or commodity) prices will not be affected, but both agents will be made better off.

(iii) Now only distortionary taxes are allowed; the analysis is as in (c): output supplies are given by:
The government budget constraint remains: 
\[ \left[ p_1^w \left( Q_1 - C_1 \right) + p_2^w \left( Q_2 - C_2 \right) \right] \geq 0 \] where demands depend upon prices facing consumers, while supplies depend on producer prices. It is intuitive that output should be chosen to maximize the value of output, at world prices, which implies setting **domestic producer prices equal to world prices**. Then consumer prices are adjusted to reach the utility target. As in (c), this is shown as:

\[
N(\ldots) = \delta_2 \left( \frac{\overline{w}_2}{3\rho} \right)^2 + \delta_1 \left( \frac{\overline{w}_1}{3\rho} - \bar{U}_I \right) + \lambda \left( Q_1 - C_1 + \rho^w (Q_2 - C_2) \right); \quad \rho^w = \frac{p_2^w}{p_1^w}; \quad \delta_2 = 1
\]

The supply rules are given above; demands are:

\[ C_i = \frac{w_{i1} + w_{i2}}{3\rho_i} = \frac{\overline{w}_i + \overline{w}_{i2}}{3\rho_i} \cdot p_i \]

where, as earlier, \( \overline{w}_i \) is the real (take-home) wage of agent \( i \), measured in units of good 1, and \( \rho \) is the relative price of good 2 to consumers. Substituting for outputs and demands into the Lagrangean:

\[
N(\ldots) = \delta_2 \left( \frac{\overline{w}_2}{3\rho} \right)^2 + \delta_1 \left( \frac{\overline{w}_1}{3\rho} - \bar{U}_I \right) + \lambda \left[ \frac{1}{3\rho_2} \left( \frac{\overline{w}_1 + \overline{w}_2}{3\rho} \right) \right] + \rho^w \left( \frac{2}{3} \frac{1}{6\left(p_2^\\prime\right)^2} - \frac{\overline{w}_1 + \overline{w}_2}{3\rho} \right);
\]

Optimizing over \( p_2^\\prime \) yields:

\[
N_{p_2^\\prime}(\ldots) = \lambda \left[ \frac{2\rho^w}{6\left(p_2^\\prime\right)^3} - \frac{1}{3\left(p_2^\\prime\right)^2} \right] = 0 \rightarrow p_2^\\prime = \rho^w
\]

Thus, firms should face world prices (no tariffs or domestic production taxes) in order to maximize the value of domestic output (given inputs); if the goal is to redistribute income it should be done through prices faced by consumers. Finishing the optimization over \( \overline{w}_1, \overline{w}_2 \) and \( \rho \) yields the following solution (method is the same as earlier):

\[
\rho = \rho^w; \quad \left( \overline{w}_1 + \overline{w}_2 \right) = \frac{4\left(\rho^w\right)^2 + 1}{4\rho^w} \quad \text{where the latter comes from the budget constraint. Using the utility constraint yields:}
\]

\[
\overline{w}_1 = \sqrt{3\cdot\rho^w \cdot \bar{U}_I}; \quad \text{hence:} \quad \sqrt{\bar{U}_I} = \frac{4\left(\rho^w\right)^2 + 1}{4\left(3\cdot\rho^w\right)^{1/2}} - \sqrt{\bar{U}_I}
\]

3. Let \( R(\tilde{p}^\\prime, V) \) denote the GNP function and \( e(\tilde{p}^\\prime, U) \) the expenditure function. We assume a single consumer; the previous problem shows how to deal with many consumers. We will implicitly assume lump sum transfers are possible if there are many consumers, so that it will be optimal to choose \( \tilde{p}^\\prime = \tilde{p}^\circ \).
a & b. Assuming $\bar{p}^f = \bar{p}^w$, from balance of trade equilibrium we have: $$\sum_i p_i^w (c_i' - q_i') = 0$$ where $c_i', q_i'$ represent the consumption and production decisions in the trade environment. Now, we have:

$$\sum_i p_i^a q_i^a \geq \sum_i p_i^a q_i$$

from the definition of revenue maximization and the fact that feasible production activities have not changed. Similarly, assuming there are gains from trade (proven for tariff revenue non-negative) we have:

$$\sum_i p_i^a c_i^a \leq \sum_i p_i^a c_i$$

since $\bar{c}^a$ minimizes expenditures to reach $U^a$ at $\bar{p}^a$, and since the expenditure function is increasing in utility: $$(U^f = U(\bar{c}^f) \geq U^a = U(\bar{c}^a))$$ from gains from trade. Thus:

$$\sum_i p_i^a c_i^a \geq \sum_i p_i^a c_i = \sum_i p_i^a q_i^a \geq \sum_i p_i^a q_i \rightarrow \sum_i p_i^a (c_i' - q_i') \geq 0$$ and $$\sum_i p_i^w (c_i' - q_i') = 0$$ imply:

$$\sum_i (p_i^a - p_i^w)(c_i' - q_i') \equiv \sum_i ((p_i^a - p_i^w)m_i') \geq 0; \quad m_i' \equiv (c_i' - q_i')$$

In the case of many goods, this just asserts a correlation between the difference between autarky and world prices, and trade flows: that we will tend to import goods that had high domestic autarky prices (and export goods will low autarky prices). Also, note that there is no presumption of free trade – only that there are gains from trade.

However, in the case of two goods, choose good one to be the numeraire and the above becomes $$(p_2^a - p_2^w)m_1' \geq 0,$$ which states that the country will import good 2 (and export good 1) if the autarky price of good 2 is above the world price – this is the standard partial equilibrium result.

c. Assume a small country pursuing world trade so that $\bar{p}^f = \bar{p}^w$ and $\bar{p}^w$ is exogenous. Then domestic utility is given from the budget constraint:

$$e(\bar{p}^w, U) = R(\bar{p}^w, V_1 + \Delta V_1, V_2, ..., V_m) - W_1(\Delta V_1)$$

where $(V_1, ..., V_m)$ is the domestically owned resource vector, $\Delta V_1$ is the amount of input one imported (exported if it is negative) and $W_1$ is the price paid to foreigners (received from foreigners) for this factor flow. Notice this factor flow is not treated as individuals who remain in the economy in the sense that their utility is not of importance. Thus, this is probably best thought of as temporary workers, imported capital, etc. Totally differentiating the above, given world prices and free trade, yields:

$$e_d U = \left[ \left( \frac{\partial R}{\partial V_1} \right) - W_1 \right] (d \Delta V_1).$$

Thus, the impact on domestic utility just depends on the relationship between the domestic factor price (or shadow price) and the world price. If the domestic factor price exceeds the world price (when $\Delta V_1 = 0$) then the country will gain by importing the factor; if the domestic factor price is less than the world price, then the country will gain by exporting the factor; and if domestic factor price equals world price (factor price equalization) then factor flows will not affect utility. This result holds because the domestic factor price equals the shadow price of the input.
d. With tariffs, this need no longer be true since the tariffs distort domestic factor prices, and hence
the factor price will not equal the shadow price. Using the specification in the problem, where there is only a
tariff on good one, we have:

\[ e(\bar{\bar{p}}^c, U) = R(\bar{\bar{p}}^c, V_i + \Delta V_i, V_{2,\ldots,m}) + \left[ t_i \left( e_{p_i} - R_{p_i} \right) \right] - W_i(\Delta V_i); \quad p_i^c = p_i^w + t_i; \quad p_i^c = p_i^w, \ i \neq 1 \]

Differentiating with respect to \( \Delta V_i \) yields:

\[ e_u \left( \frac{\partial U}{\partial \Delta V_i} \right) = \left( R_{v_i} - W_i \right) + t_i \left( \frac{\partial m_i}{\partial \Delta V_i} \right); \quad m_i \equiv \left( e_{p_i} - R_{p_i} \right) \]

Here \( m_i \) denotes imports of good 1, \( R_{v_i} \) is the domestic factor price of input one, but \( \left\{ R_{v_i} + t_i \left( \frac{\partial m_i}{\partial \Delta V_i} \right) \right\} \)
is the shadow price (true value, given the distortion) of factor 1. Note that if the import of the factor increases
imports of good 1, then the shadow price exceeds the market price, whereas if importing the input reduces
imports, the shadow price is less than the market price. This is because the import tariff creates a distortion by
reducing imports of good one, so that another policy that increases (decreases) these imports has a side-benefit
(side-cost) in addition to its direct impact. The impact on imports is given by:

\[ \left( \frac{\partial m_i}{\partial \Delta V_i} \right) = e_{p_U} \left( \frac{\partial U}{\partial \Delta V_i} \right) - R_{p_{v_i}} \]

where \( R_{p_{v_i}} \) is the change in output of the importable good, at given prices, due to the increased
availability of input one. Substituting and rearranging yields:

\[ e_u \left( \frac{\partial U}{\partial \Delta V_i} \right) = \left( R_{v_i} - W_i \right) + t_i \left\{ e_{p_U} \left( \frac{\partial U}{\partial \Delta V_i} \right) - R_{p_{v_i}} \right\} \rightarrow \]
\[ \left( e_u - t_i e_{p_U} \right) \left( \frac{\partial U}{\partial \Delta V_i} \right) = \left( R_{v_i} - W_i \right) - t_i R_{p_{v_i}} \]

where the term \( \left( e_u - t_i e_{p_U} \right) \) must be positive if both goods are normal. Note that if \( R_{v_i} = W_i \), then imports of
the factor will lower welfare if they reduce production of good one. By continuity, even if \( R_{v_i} > W_i \), factor
imports can lower utility if they lower domestic production of importables. **Of course, if the import of inputs
increases domestic production of importables, then the gains from input imports are positive and larger
than if there were no tariffs.**

The moral – if there are no other distortions, factor movements are beneficial. If there are distortions (tariffs, here) then factor movements may be bad or good, partly depending on whether they worsen or reduce the
market failure that already exists. This is the standard lesson of the theory of the second best.