1. Let $R(p^f, V)$ denote the GNP (revenue) function and $e(p^c, U)$ denote the expenditure function for a representative agent, where $p^f$ denotes the price vector producers face, and $p^c$ denotes the consumer price vector (the two are equal in the absence of taxes). For simplicity, assume a single-agent economy.

i) Show that, absent domestic taxes, trade increases welfare providing net tariff revenue is non-negative (same proof as in class). Does this proof imply free trade is optimal? Explain why trade, in the presence of subsidies, can be welfare deteriorating.

ii) Explain how the proof in (i) uses the assumption of no domestic distortions and no domestic taxes.

iii) Suppose there are $N$ goods, and let $p^A$ denote the autarky price vector and $p^T$ the world price vector with trade (assume good $N$ is the numeraire in both cases). Given these prices vectors, what can you conclude about the pattern of trade? If $\left(\frac{p^T_i}{p^T_j}\right) > \left(\frac{p^A_i}{p^A_j}\right)$, can we conclude $j^*$ will be exported? If $\left(\frac{p^T_i}{p^T_k}\right) < \left(\frac{p^A_i}{p^A_k}\right)$ can we conclude that $k^*$ will be imported? Contrast with the Ricardian model.

iv) Assume a two good economy; let good 1 be the numeraire. Further, assume that production of good 2 generates a negative externality that harms consumers (the utility function is given by: $U = f(C_1, C_2) - g(Q_2)$ where $C_i$ is consumption of good $i$, $Q_i$ is production of good $i$ and $g(Q_2) > 0$, $g' > 0$ represents the negative externality). Assuming no government intervention (e.g., to offset pollution), show how the autarky equilibrium is determined and then show how opening the economy up to trade will affect welfare (for the last part, you may assume a “small” economy - i.e., treat the world price of good 2 as exogenous). Under what conditions can the movement from autarky to trade lower welfare? Explain.

2. Consider a model with two goods $(Q_1, Q_2)$ and two inputs $(L_1, L_2)$ (let $L_1$ denote “unskilled” labor and $L_2$ denote “skilled” labor); and assume input supply is determined by utility maximization. Further, suppose there are two households (or an equal number of two types of households); household I is endowed with $L^*_1$ units of unskilled labor, which can be consumed or sold to firms; and household II is endowed with $L^*_2$ units of skilled labor, which can be consumed or sold to firms. Preferences for each are:

$U^I = \theta(C_1, M_1, \ell_1)$; $\ell_1 + L_1 \leq L^*_1$; 
$U^U = \phi(C_2, M_2, \ell_2)$; $\ell_2 + L_2 \leq L^*_2$

where $(C_1, M_1, \ell_1)$ is the consumption vector of household I and $(C_2, M_2, \ell_2)$ is the consumption vector of household II, and $(L_1, L_2)$ denote the inputs (V) available for production. Let $R(p, V)$ denote the GNP function (as in problem 1); assume that it exhibits constant returns to scale (in V).
(a) Show how the autarky equilibrium is derived (i.e., write down the set of equations characterizing that equilibrium).

(b) Show that, if lump sum transfers are feasible, the movement from autarky to trade must be potentially beneficial. If lump sum transfers are not feasible, must both households gain from trade?

For the remaining parts assume households have the following preferences, endowments and budget constraint:

\[ U^I = C_1 \cdot M_1 \cdot \ell_1; \quad L_1 = 1; \quad W_1 (1 - \ell_1) + T_1 - P_c C_1 - P_m M_1 \geq 0 \]
\[ U^II = C_2 \cdot M_2 \cdot \ell_2; \quad L_2 = 1; \quad W_2 (1 - \ell_2) + T_2 - P_c C_2 - P_m M_2 \geq 0 \]

where \((T_1, T_2)\) denotes lump sum transfers to households, \((W_1, W_2)\) denotes factor prices received by households and \((P_c, P_m)\) denotes goods prices paid by consumers; note that due to constant returns, there are no profits to distribute to households.

(c) Using the above preferences, assume technology is given by: \(Q_c = L_1; \quad Q_m = L_2\)

i) Find the autarky equilibrium.

ii) Assume the country can trade at the given price vector: \(P_c^w = 1; \quad P_m^w = 2\); show how free trade affects consumption, utility and factor prices. Do both gain from trade? If lump sum transfers are available, can both be made better off?

iii) Assume no lump sum transfers are possible. Assuming only commodity and factor taxes can be used, can both agents be made better off with trade? Explain/prove. (Be sure to indicate the government budget constraint).

(d) Repeat part (c) using the following technology:

\[ Q_c = \sqrt{L_{c1} \cdot L_{c2}}; \quad Q_m = L_{m2}; \quad L_{c1} \leq L_1; \quad L_{c2} + L_{m2} \leq L_2 \]

3. Using the general duality approach with GNP function \(R(pf, V)\) and expenditure function \(e(pc, U)\):

a) Assuming no domestic taxes, show that in a two good world the country will export the good that is relatively cheaper at home than abroad.

b) Show how this result can be extended, if at all, to a \(N\) good world.

c) Suppose that the country can import a fixed amount of input one \((\Delta V')\) at a per unit price \(W'_1\). How will this affect the net income (welfare) of its domestic residents if the country pursues free trade and world prices are not affected by this factor movement?

d) Same as part (c), but suppose the country has an import tariff on good 1. Might that change your result? Explain.
HINT: With an import tariff on good 1, \( p_i^c = p_i^f = (p_i^w + t_i) \), where \( p_i^w \) is the world price. Net domestic income is: 

\[
Y = R(\bar{p}, V') - W_i(\Delta V') + t_i(e_{p_i} - R_{p_i})
\]

where \( W_i(\Delta V') \) is the payment for the import inputs, and \( t_i(e_{p_i} - R_{p_i}) \) is the tariff revenue on imports of good 1, and \( V' = (V + \Delta V') \). Given world prices and the tariff, domestic utility is determined by: 

\[
\epsilon(p^c, U) = Y.
\]

Given this, show how utility changes with \( \Delta V' \).