International Trade, Factor Market Distortions, and the Optimal Dynamic Subsidy

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International trade theorists have long favored free trade, arguing that the only proper basis for a tariff is in response to some monopoly power in international trade. Pragmatic politicians, on the other hand, wary of the wrath of their constituents, have often resorted to the use of tariffs, arguing that the protection afforded by tariffs was necessary to prevent unemployment. Recent papers on the theory of domestic distortions and optimal policy interventions can be interpreted as an attempt to reconcile these two divergent views. For example, if factors are immobile and if distortions exist in the factor markets (due to factor price rigidities), then it is argued that the optimal policy intervention is not a tariff, which destroys the equality between the marginal rate of substitution (MRS) and foreign rate of transformation (FRT), but rather a wage subsidy. This policy prescription, which recognizes the pragmatic difficulties politicians must face, offers a feasible remedy that is proved superior to the use of tariffs.

However, one can argue that the theory of optimal policy intervention goes a bit too far. While it may well be true that factors are not instantaneously mobile, they certainly do become mobile through time; capital depreciates, new workers enter the labor force, and factories can be moved intertemporally. The policies that eliminate unemployment at the same time destroy the incentives for resource reallocation (particularly if the institutional constraint stipulates that factor prices must be the same in all sectors). Thus, while the standard wage subsidy that would promote full employment may be efficient in the short run, it may prove inefficient in a dynamic setting.

Nevertheless, this does not imply that the efficient long-run policy entails no intervention; rather, it is likely that it will call for some wage subsidy which is smaller than the static optimum subsidy. It is the purpose of this paper to discuss how this dynamic optimum subsidy is determined and to specify its implications for the economic system.

I. The Problem

For the analysis, I shall use a standard two-sector, one-factor trade model.¹ Let the two commodities be $M$ and $C$, and let $N_m$ and $N_c$ represent the labor employed in each sector. Then:

$$ M = F_m(N_m), \quad C = F_c(N_c) $$

$$ F_i' > 0, \quad F_i'' < 0; \quad i = c, m $$

Furthermore, choose $C$ as the numeraire and let $P$ represent the relative price of $M$.

In addition, assume that the number of potential workers in each sector ($L_i$) is fixed at any moment, and that the total

¹ If desired, it can be assumed that other factors are used in each sector, but that these are fixed for the period under study. John Harris and Michael Todaro use this model.
supply of available workers is fixed for the period in question:

\[ N_i \leq L_i, \quad i = c, m \]
\[ L_c + L_m = \bar{L} \]

Institutionally, it is assumed that firms behave competitively in factor (and product) markets, so that workers are paid their marginal value product (and there cannot be excess demand for factors in either sector). However, it is also assumed that because of distortions in the factor market, the wage rate must be the same in each sector.\(^2\) Thus, in the absence of any wage subsidies, we have:

\[ F'_c(N_c) = PF'_m(N_m) \]
\[ N_c = L_c \quad \text{or} \quad N_m = L_m \]

The above three equations constitute a long-run equilibrium if commodity markets clear and if full employment occurs in each sector.

To simplify the analysis, let us assume that the economy in question is open and "small," so that it can trade at unchanging terms of trade.\(^3\) Assuming the economy is in long-run equilibrium it will be producing at some point (such as B in Figure 1) on the long-run production possibility frontier and will consume somewhere on the price line tangent to the production possibility frontier at B.\(^4\)

\(^2\) A common alternative assumption is that "real" wages are rigid downward; the implications of this assumption depend on the definition of real wages. If they are defined in terms of C, then (assuming we start from full employment) a fall in P will create unemployment, but a rise in P will have no adverse effect on employment; opposite results hold if real wages are defined in terms of M. Finally, if they are defined in terms of some weighted price index, any change in P will cause unemployment; this latter assumption gives results that are qualitatively the same as assuming wages must be equalized across sectors.

\(^3\) This assumption, which is commonly used (see Jagdish Bhagwati and T. N. Srinivasan) allows us to ignore the utility function and it implies the optimum tariff is zero. The results would not be significantly altered by dropping this assumption.

\(^4\) The long-run production possibility frontier represents the efficient production locus when labor is assumed perfectly mobile.

Now suppose that due to external conditions, the terms of trade shift; for example, assume the country was exporting C and let its terms of trade improve (P falls). If factors are immobile, the short-run production possibility frontier is given by \(ABD\); and if no distortions exist in the factor markets, the country will continue to produce at B and will benefit from the improved terms of trade. However, if factors are immobile and if factor prices must be the same in each sector, an excess supply of labor will develop in M and output will occur somewhere along the open segment \(AB\). Thus, it is conceivable that the improved terms of trade might leave the country worse off.

As has been demonstrated in the literature,\(^5\) the optimal policy under these circumstances is a wage subsidy to M that restores production at point B. Thus, if we let \(P^*\) be the old price at which resources were fully employed, and \(P^*\) the new world price (\(P^* < P\)), and if \(S\) represents the optimum wage subsidy to sector M, then:

\(^5\) For example, see the articles by Harry Johnson and Stephen Magee.
This subsidy restores full employment and enables the country to benefit from its improved terms of trade.

However, this equilibrium is still sub-optimum in a long-run context since the change in the terms of trade moves the long-run optimum production point from \( B \) to \( E \) (in Figure 1). The obvious question is what determines intertemporal factor migration and what is the best policy to be pursued in a dynamic context? If, for example, factor migration between sectors is determined by economic factors, such as different wages or unemployment rates in each sector, then the static optimum subsidy \( S \) destroys any economic incentives for migration since, by assumption, wages are equalized between sectors while, by choice, unemployment is reduced to zero in each sector. Thus, while \( S \) may be efficient in a short-run context, it may not be optimal when considered in a long-run context.\(^6\)

Clearly, the government may have other tools at its disposal that will enable it to maintain full employment in the short run while encouraging migration in the direction of long-run equilibrium. One policy tool, suggested in Harris and Todaro, would be simply to force (or restrict) migration in a particular direction while maintaining the optimal subsidy \( S \) needed to guarantee full employment of resources if there were factor market distortions (naturally, \( S \) changes through time as labor migrates). However, this particular policy is most likely to be opposed because it infringes on personal liberty (and it is not likely to be very efficient if all workers are not identical).

The economic tools the government should use to maintain full employment and encourage migration depend upon the causes of migration and the institutional constraints.\(^7\) If migration rates between sectors depend upon differences in real incomes between the sectors, the government should attempt to enlarge these differences, while at the same time pursuing a policy that maintains full employment in all sectors. For example, if we do not assume that wages must be equalized across sectors, then the government could apply large wage subsidies to sector \( C \) (where labor is more productive) to encourage migration toward \( C \), while at the same time use the minimum wage subsidy necessary to promote full employment in \( M \). Even if wages must be equalized across sectors, the government could encourage migration to \( C \) by imposing a lump sum tax on workers in \( M \) (or a lump sum transfer to workers in \( C \)), while at the same time subsidizing wages in \( M \) in order to promote full employment.

However, the above analysis is subject to criticism on several grounds. It assumes that somehow labor (effectively) demands a minimum real wage but apparently cannot effectively demand a minimum real standard of living. Specifically, the above policy can work because it assumes that a wage subsidy can be applied to sector \( M \) to meet minimum wage demands and promote full employment, while (some of) the income from this wage subsidy can be removed by lump sum taxation (to promote real income differences between the sectors) without altering the wage demands of laborers in this sector.

Moreover, it can be argued that the

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\(^6\) Even if no distortions exist in factor markets, so that full employment (at \( B \)) occurs without subsidization, it does not follow that this solution is optimal in a long-run context when we are interested in encouraging migration between sectors.

\(^7\) In the following analysis it is assumed that labor's marginal value product is higher in \( C \), so that the government wishes to encourage migration toward \( C \) while maintaining full employment in \( M \) (by means of an optimum subsidy).
economic efficiency of the wage subsidy to C (to promote migration) depends crucially upon the assumption that the return to labor is a pure economic rent. If each individual’s labor supply decision were responsive to the wage rate, then the subsidy in C would create inefficiencies by causing a discrepancy between the marginal value product of labor and that worker’s marginal rate of substitution between consumption and leisure. Thus, while the wage subsidy (to the more efficient sector) attempts to promote migration, and hence dynamic efficiency, it produces static inefficiency.

Given the preceding discussion, we are skeptical about the ability of the government to costlessly reallocate resources in the economy. Thus for the remainder of the paper we explicitly assume this reallocation cannot be achieved costlessly, and that it is the unemployment rate (in M) that determines the rate of out-migration. One might rationalize this assumption by arguing that workers are risk averse and hence are unwilling to move to an unfamiliar situation (a new industry) if there is a reasonable chance of finding a job in their own sector. In addition, if it is assumed that wages must be the same in all sectors, the assumption that unemployment rates determine migration is equivalent to the assertion that migration is determined by the differences in expected wages between the sectors.⁸

In the analysis that follows, no explicit assumption will be made about wage rigidity in either sector; rather, the key relation will be the effect of unemployment rates on migration. The central planner’s task is to determine the time path of the unemployment rate that balances the costs of unemployment (due to lost output) against the gains from migration (due to a more efficient labor allocation). However, given the institutional constraints (shown in fn. 8), the optimal wage subsidy can be inferred from the time path of the unemployment rate.

In the following section we develop the formal model and discuss the properties of an optimal solution; attention is focused on determination of the optimal unemployment rate. The principal conclusion of the section is that for long planning horizons (or small discount rates), some unemployment is desirable during early stages of the plan. Since this implies that the static subsidy is not in general dynamically efficient, the properties of the optimal dynamic subsidy are discussed in Section III.

II. Optimal Labor Transfer

Suppose the economy is out of long-run equilibrium in the sense that labor’s marginal value product is larger at full employment in sector C than in M.⁹ Furthermore, assume that labor is not instantaneously mobile, but that it moves through time in response to unemployment. We seek to determine the optimum time path of unemployment, assuming that the central planner wishes to maximize the present discounted value of the stream of income over the interval (0, T).¹⁰ ¹¹

Let production and factor endowments be as described in equations (1) and (2).

⁸ For further discussion of the role of expected wages, see Harris and Todaro. The particular institutional assumptions we make for the analysis of Sections II and III are: (i) Wages to workers must be the same in each sector and there cannot be an excess demand for labor in either sector: \( W_r = W_w = \max(P^e_r(L_r), P^e_w(L_w)) \); (ii) Migration between sectors occurs due to differences in the expected wage rate, and hence responds to differences in the unemployment rate between sectors; (iii) The government controls the unemployment rate through wage subsidies. Consequently, there is a tradeoff between current output and future output in that large subsidies that promote full employment do so at the expense of migration that would ultimately raise the value of output.

⁹ The difference in marginal value products may have arisen because of some exogenous change in the terms of trade. Also, the analysis could readily be generalized to handle the case in which labor’s marginal value product is larger in M.

¹⁰ Given the assumption the economy is open and “small,” maximization of the present value of the in-
Since we wish to encourage migration towards \( C \), it will always be optimal to maintain full employment there. Thus, labor migration towards \( C \) can be postulated as a function of the unemployment rate \( (u) \) in \( M \):

\[
\frac{dL_c}{dt} = -\frac{dL_m}{dt} = \Phi(u)L_m, \quad u \in [0, 1]
\]

\[
\Phi(0) = 0; \quad \Phi' > 0, \; \Phi'' \leq 0
\]

From the definitions and the initial conditions:

\[
N_c = L_c, \quad N_m = L_m(1 - u)
\]

\[
F'_c(L_c(0)) > PF'_m(L_m(0))
\]

Given (1), (2), (5), (6), and (7), we seek to maximize:

\[
V = \int_0^T [F_c(L_c) + PF_m(N_m)] e^{-rt} dt
\]

The central planner must determine the time path of the unemployment rate that maximizes the present discounted value of GNP over the interval. Since the costs of unemployment depend upon labor's marginal value product in \( M \) (and are immediate), while the benefits due to migration depend upon the differences in marginal value products between sectors and are deferred, it follows that the optimal policy (at each period) depends upon the current labor allocation, the discount rate, and the amount of time left in the planning horizon. Intuitively, we expect that unemployment is undesirable near the end of the planning horizon, but (relatively) more desirable during the early stages of the plan.\(^{12}\)

Formally, we seek to determine the time path of the unemployment rate \( u^*(t) \) that maximizes (8) subject to (1), (2), (5), (6), and (7). The problem as stated is an optimal control problem in one state variable \( L_c(t) \) and one control variable \( (u) \). The Hamiltonian is:

\[
H = [F_c(L_c) + PF_m(N_m)] e^{-rt} + \lambda[L_m \Phi(u)]
\]

In (9), \( \lambda(t) \) represents the value, discounted to time 0, of transferring a worker at time \( t \) to sector \( C \). It is convenient to define \( q(t) \) to represent the value at time \( t \) of such a transfer:

\[
q(t) = \lambda(t)e^{rt}
\]

Optimizing the Hamiltonian over \( u \) yields:\(^{14}\)

\[
H_u = L_m e^{-rt}[q\Phi'(u) - PF'_m(N_m)] \leq 0
\]

\[
u \in [0, 1]
\]

In addition, for an optimum path we derive the canonical equations and the transversality condition:

\[
\frac{dq}{dt} = [r + \Phi(u)]q - [F'_c - P(1 - u)F'_m]
\]

\(^{12}\) The unemployment rate plays a role similar to that of savings in the one-sector growth models. Savings is undesirable during the latter stages of those models unless a value is placed on the terminal capital stock.

\(^{13}\) The model (1)-(8) can be used to analyze optimal labor transfer even if no factor price rigidities exist (or if it is not unemployment that induces \( ml \) ration), provided this transfer is costly due to time lost from work.

\(^{14}\) We assume \( F'_m(0) = \infty \); thus some employment is always desirable in \( M \). Also, we assume \( \Phi'(0) \) is finite; if it is not, some unemployment will always be optimal (if marginal value products differ), and the static subsidy will always be inefficient.
(13) \[ \frac{dL_c}{dt} = L_m \Phi(u) \]

(14) \[ q(T) = 0, \quad L_c(0) \text{ given} \]

By the assumptions on \( F''_i \) and \( \Phi'' \), any path that satisfies (11)–(14) constitutes an optimum solution to the problem. In order to characterize the properties of this solution, it is helpful to construct the phase diagram for the system of differential equations in \((L_c, q)\) space.\(^{15}\)

First, consider (11) which serves to determine (for given \( q, L_c \)) the optimal unemployment rate; it states that if some unemployment is desirable, then the social value attributable to an additional unemployed worker (due to induced migration and equal to \( q \Phi'(u) \)) should equal the value of output lost due to his unemployment. However, if \( H_u < 0 \) for all \( u \), then full employment is desirable. Thus, the boundary between full employment and some unemployment is defined by the locus such that \( H_u = 0 \) for \( u = 0 \):

(15) \[ g(q, L_c) = [q \Phi'(0) - PF'_m(L_m)] = 0 \]

As shown in the Appendix, this curve (and all iso-unemployment curves) is positively sloped in \((L_c, q)\) space and is asymptotic to the line \( L_c = \bar{L} \). Larger values of \( q \) are associated with higher unemployment rates (given \( L_c \)); thus, points above the \( g(q, L_c) \) locus correspond to \( u > 0 \), \( (dL_c/dt) > 0 \), whereas points below it correspond to \( u = 0 \), \( (dL_c/dt) = 0 \). The curve \( AB \) in Figure 2 depicts this locus.

Next, consider how \( q(t) \) changes along an optimum path. Since \( q(t) \) is the value of labor transfer at time \( t \), it must fall through time since: (i) the differences in marginal value products between sectors is a nonincreasing function of time, and (ii) as time passes, less time (for finite \( T \)) remains to reap the benefits of further labor transfers. To determine the optimal region in \((L_c, q)\) space it is useful to sketch the curve \((dq/dt) = 0\). From (11), \( u \) can be written as an implicit function of \( L_c \) and \( q \); substitution in (12) yields:

(16) \[ dq/dt = h(q, L_c) = (r + \Phi)q \]

\[ - [F'_c - P(1 - u)F'_m] = 0 \]

In the Appendix it is shown this locus is negatively sloped and intersects the \( q = 0 \) axis at \( \bar{L}_c \), the labor allocation that equalizes marginal value products across sectors.\(^{16}\) Points above the locus correspond to \((dq/dt) > 0\), whereas those below correspond to \((dq/dt) < 0\); this locus is sketched in Figure 2 as the curve \( DE \).

Given this phase diagram, \( q(0) \) must be chosen to satisfy (14), the transversality condition; from the previous arguments, \((dq/dt) < 0\) and thus \( q(0) \) lies below \( DE \). Once \( q(0) \) is chosen, the unemployment rate (and the subsidy rate) can be de-

\(^{15}\) In the following, we assume \( r > 0 \); even if the social discount rate is zero, it is appropriate to choose \( r > 0 \) if borrowing and lending can occur (at a positive rate) on world capital markets. When it is appropriate, we shall indicate the implications of assuming \( r = 0 \). Note that if \( r = 0 \), (8) will not converge as \( T \to \infty \). However, an equivalent problem would be to minimize the deviation of actual output from maximum output: \[ [F_c(L_c) + PF_m(L_m) - F_c(L_c) - PF_m(N_m)], \] where \( F_c(L_c) = PF_m(L_m). \) Since this integral converges, and since the two problems give equivalent conditions, we can proceed with the problem as stated.

\(^{16}\) For \( r = 0 \), the locus \((dq/dt) = 0\) intersects the \( g(q, L_c) = 0 \) locus at \( \bar{L}_c \), and it becomes vertical at this point, intersecting the \( q = 0 \) locus at \( \bar{L}_c \). However, for \( u > 0 \), the locus is negatively sloped.
duced from Figure 2. To further clarify the nature of the solution, define \((L^*_c, q^*)\) as the point \((G\) in Figure 2) at which (15) and (16) intersect. Solving these equations simultaneously yields:

\[
(17) \quad F'_c(L^*_c) - PF'_m(L_m^*) = PF_m(r/\Phi'(0)) > 0; \\
q^* = PF'_m/\Phi'(0), \quad L_c < L_c^* \quad \text{for } r > 0
\]

If \(L_c(0) \geq L_c^*\), then full employment is always optimal, regardless of the length of the planning horizon.\(^{17}\) Since \(L_c^*\) is a decreasing function of \(r\), it follows that for relatively large discount rates or a relatively small misallocation of labor, it does not pay to incur unemployment, and thus the static subsidy is dynamically efficient.\(^{18}\)

However, if \(L_c(0) < L_c^*\), the optimal solution depends on \(T\). For this case, it can be shown that the solution to the differential equations is a saddlepoint, and that the unique path which converges to \((L_c^*, q^*)\) is the turnpike—the optimum solution for the infinite time horizon problem.\(^{19}\)

The infinite time horizon problem is characterized by decreasing \(q(t)\) and increasing \(L_c(t)\), and thus falling unemployment rates. Note that even in this case, it does not pay to equalize labor’s marginal value product across sectors.\(^{20}\)

For any finite \(T\), it is clear that the choice of \(q(0)\) lies below the turnpike—and thus less labor will be transferred during the planning period. Moreover, it is readily seen that the shorter the time horizon, the less desirable unemployment becomes, so that the initial unemployment rate (and \(q(0)\)) decreases, and the total labor transfer decreases, as \(T\) decreases. Since \(q(t)\) falls along an optimum path, the unemployment rate falls (for \(u > 0\)) along this path.

Moreover, the terminal stage of any finite horizon path must be characterized by full employment. To show this, assume an optimal path has been followed during the interval \((0, T - \tau)\), and let \(L_c(T - \tau)\) represent the corresponding labor allocation. The cost of incurring unemployment during the interval \((dt)\) is \(A:\)

\[
(18) \quad A \approx PL_m u F_m'dt
\]

The increased flow of output due to this labor transfer, assuming full employment subsequently, is:

\[
(19) \quad B = (F'_c - PF'_m)\Phi(u)L_md\tau
\]

where \(dL_c = \Phi(u)L_md\tau\)

Discounting future benefits and integrating over the remaining time yields:

\[
(20) \quad B' \approx (F'_c - PF'_m)\Phi(u)L_m\left[(1 - e^{-\tau})/\tau\right]dt
\]

If \((B' - A)\) is nonpositive for all \(u\), then full employment is optimal for the whole interval. Thus, \(u = 0\) if:\(^{21}\)

\[
(21) \quad e^{-\tau r} \geq \left[\frac{F'_c - PF'_m(1 + [r/\Phi'(0)])}{(F'_c - PF'_m)}\right]^{r > 0}
\]

If the numerator on the right-hand side is nonpositive \((L_c \geq L_c^*)\), then full employment is optimal, regardless of \(\tau\). Even for

\(^{17}\) By assumption, \(L_c(0) < L_c^*\); for \(r = 0\), \(L_c^* = L_c > L_c(0)\). For \(r > 0\), \(L_c(0) \geq L_c^*\); \(u = 0\) is optimal; \(q(0)\) is chosen so that \(q(T) = 0\). If the horizon is infinite, \(q(0)\) is chosen so that \((dq/dt) = 0)\).

\(^{18}\) This is analogous to the results of the one-sector growth model when there is no population growth or capital depreciation. In this model, steady-state per capita consumption is maximized at the capital-labor ratio \(k^*\) such that \(f'(k^*) = 0\). However, the modified golden rule determines \(k\) such that \(f'(k) = r\). If \(k^* > k(0)\) > \(k\), then the optimal policy is to do nothing; if \(k(0) < k\), some savings will be optimal for a sufficiently long planning horizon.

\(^{19}\) Naturally, if \(T\) is unbounded, the transversality condition given by (14) must be suitably modified.

\(^{20}\) For \(r = 0\), the marginal value products are equalized asymptotically. It can be shown that for the infinite horizon case, \((L_c^*, q^*)\) is not reached in finite time; thus some unemployment is always desirable, though \(u(t) \rightarrow 0\). The proof is omitted to save space.

\(^{21}\) If \(r = 0\), (21) becomes \(\tau \leq PF''_m/\Phi'(0)(F'_c - PF'_m)\). Thus, for \(r = 0\), large \(\tau\) implies \(u > 0\). Nevertheless, it is clear that it does not pay to equalize labor’s marginal value product for any finite horizon.
$L_e(0) < L_e^*$, full employment is optimal for small $\tau$ (since the right-hand side of (21) is less than one). Thus, for short planning periods, and for the terminal stage of any finite plan, it is optimal to maintain full employment.

From (21), it is apparent that given $r$ and $L_e(0) < L_e^*$, there exists a $T^*$ such that for $T < T^*$, $u(t) = 0$; for $T > T^*$, the initial stages of the optimal solution are characterized by positive but decreasing unemployment rates, and the terminal stage by full employment. As noted earlier, $L_e(T)$ is an increasing function of $T$; thus (21) implies that the length of the period of full employment increases as $T$ increases.

To recapitulate, I have shown that (for $r > 0$) it is never optimal to fully reallocate labor between the sectors. Also, we have seen full employment is always optimal either if the discount rate is large or the planning horizon is short; in these cases, the static subsidy that maintains full employment is optimal. However, if $r$ is not large (or initial misallocations of labor are large) and if the planning horizon is sufficiently long, unemployment is desirable during the initial stages of the plan; for these cases, the static subsidy is inefficient. Therefore, I conclude that the static subsidy represents myopic behavior because it discourages resource reallocation.

III. The Economic Properties of the Optimal Path

The preceding section discussed the qualitative properties of an optimal solution; we now derive some of the quantitative properties of this solution. Specifically, I shall discuss how: (i) $GNP$, (ii) employment in $M$, and (iii) the optimal subsidy change through time.

If full employment is always optimal, then all economic variables are constant, and the static subsidy is dynamically inefficient. However, we have argued that this solution represents myopic behavior; in the following analysis it is assumed that some unemployment is optimal for the initial stage of the plan.

First, consider the marginal value product of labor in each sector. Given the assumptions, $F'_e(L_e) > PF'_m(L_m)$; however, since there is unemployment in $M$, it does not immediately follow that $F'_e(L_e) > PF'_m(N_m)$. From (11) and (12) we have, for $u > 0$:

$$
(22) \quad \frac{dq}{dt} = [r + \Phi - u\Phi']q 
- [F'_e - PF'_m(N_m)] < 0
$$

since $q(t)$ falls along the optimum path. But $[\Phi - u\Phi'] \geq 0$ for all $u$, since $\Phi' \leq 0$; thus (22) implies $F'_e(L_e) > PF'_m(N_m)$ along the optimum path.

Consider how $GNP$ changes through time; I have shown that $L_e(t)$ increases and $u(t)$ decreases through time $(u > 0)$. From (22) it immediately follows that $GNP$ is increasing through time since total employment is rising and labor is being reallocated to its more productive use.

Less apparent is how total employment in $M$ changes through time—$u(t)$ is falling, as is $L_m(t)$, so the net result could be ambiguous. Without further assumptions on $\Phi''$ or $F''_m$, it does not appear possible to describe how $N_m$ changes through time. However, if $\Phi'' = 0$, the answer is apparent. From (11), for $u > 0$:

$$(11') \quad \Phi'(u) \cdot q = \Phi'(0) \cdot q = PF'_m(N_m)$$

$$
\frac{dq}{dt} < 0
$$

Since $q$ decreases if $\Phi'' = 0$, then $PF'_m(N_m)$ must also decrease through time. Thus, for $\Phi'' = 0$, total employment in $M$ rises along the optimum path $(u > 0)$.

Furthermore, this same result holds during the final portion (of the period of unemployment) for any finite horizon path.
During this portion $u$ must be small, and since $dq/dt$ does not tend to zero as $u$ tends to zero (for finite $T$), it follows that $\Phi'(u) \cdot q$ and hence $PF_m'(N_m)$ must be falling.\textsuperscript{23} Therefore, towards the end of the period of unemployment, $N_m$ must rise.

Finally, we consider the behavior of the optimum dynamic subsidy $S^*(t)$, assuming wages must be equalized across sectors. Given this assumption, the static subsidy needed to maintain full employment is:\textsuperscript{24}

\begin{equation}
S(t) = 1 - \left[ PF_m'(L_m)/F_e'(L_e) \right] > 0 \quad L_e < L_e
\end{equation}

The dynamic subsidy depends upon the unemployment rate:

\begin{equation}
S^*(t) = 1 - \left[ PF_m'(N_m)/F_e'(L_e) \right] \quad \quad S^* < S, \quad u > 0
\end{equation}

Note that (22) implies $S^*(t) > 0$. Since $S^* < S$ for $u > 0$, and $S^* = S$ for $u = 0$, the gap between these subsidies must eventually be eliminated for any finite horizon. However, without further assumptions, it does not appear possible to conclude that the gap monotonically decreases through time.

If we compare the fraction of wages paid by employers under each subsidy, we can determine how this ratio changes through time if the production function in $M$ is isoelastic. Specifically, let:

\begin{equation}
F_m(N_m) = N_m^\alpha, \quad \alpha \in (0, 1)
\end{equation}

From (23) and (24) we have, given (25):

\begin{equation}
[(1 - S^*)/(1 - S)] = \left[ PF_m'(N_m)/F_e'(L_m) \right] = (1 - u)^{-1} > 1, \quad u > 0
\end{equation}

Since $du/dt < 0$, $(1 - S^*)/(1 - S)$ decreases through time, approaching one. Therefore, the ratio of the percent of wages paid by employers in the dynamic case to the static case decreases through time. However, even this does not imply $(S - S^*)$ monotonically decreases through time; without (25) it does not appear possible to reach specific conclusions.

The final issue to be resolved concerns the time path of the optimum subsidy. From (24) it is apparent that if $N_m(t)$ decreases through time, then $S^*$ also decreases. However, we have shown that $dN_m/dt > 0$ for $\Phi'' = 0$ and for latter portions of the optimal path. Thus, the time path of $S^*(t)$ is not immediately apparent.

Further information can be obtained by considering the time derivative of (24). Letting $PF_m'' = q\Phi'$ in (24) and differentiating yields:

\begin{equation}
dS^*/dt = \frac{\Phi''q(du/dt)}{F_e'} - \frac{\Phi'(dq/dt)}{F_e'} + \frac{F_e'\Phi'q(dL_e/dt)}{(F_e')^2}
\end{equation}

While the sign of (27) is not known for all $t$, it is clear that $dS^*/dt > 0$ near the end of the period of unemployment (for finite $T$). Therefore, the optimum subsidy must be rising during the final stages of unemployment.\textsuperscript{25}

Rewriting (27), assuming $\Phi'' = 0$, yields:

\begin{equation}
dS^*/dt = - \left[ q\Phi'/F_e' \right] \left[ r - \frac{S^*\Phi'}{(1 - S^*)} + \frac{\beta\Phi L_m}{L_e} \right]
\end{equation}

where $\beta \equiv (L_e F_e''/F_e') > 0$. Assuming $\beta$ is constant, differentiating (27') with respect to $t$ and evaluating it at $[dS^*/dt] = 0$ yields:

\begin{equation}
(27') \quad \text{for finite } T \text{ since } u \text{ is a continuous function of time, and thus } (dL_e/dt) \text{ and } (du/dt) \text{ tend to zero as } u \to 0. \text{ However, } (dq/dt) < 0 \text{ and } \Phi' > 0 \text{ implies } (dS^*/dt) > 0 \text{ as } u \to 0.
\end{equation}
\[
\begin{align*}
\left[ \frac{d^2 S^*}{dt^2} \right]_{dS^*/dt = 0} &= \left[ \frac{\Phi' L_m (du/dt)}{F_c} \beta \Phi' \frac{u L_m}{F_c} \beta \Phi' L_m \right] > 0 \\
\end{align*}
\]

The sign of (28) follows directly from the result \( du/dt < 0 \) for \( u > 0 \).

Equation (28) says that once \( S^*(t) \) starts to increase, it must continue to do so. It follows that for any finite horizon path, if ever \( dS^*/dt < 0 \), this must occur early in the plan. For infinite horizon paths, (28) is even more informative; if \( r = 0, S^*(t) \rightarrow 0 \) asymptotically. Since \( S^*(t) > 0 \) for \( u > 0 \), it follows that the optimal subsidy falls monotonically for the infinite horizon, zero-discount case (given the constancy of \( \beta \) and \( \Phi' \)).

Similar results hold for \( r > 0 \). Let \( \hat{S} \) be the optimum subsidy corresponding to the stationary point \( (L_c^*, q^*) \); asymptotically \( S^*(t) \to \hat{S} \) for the infinite horizon path \( (L_c(0) < L_c^*) \). From (17) and (24):

\[
\begin{align*}
\hat{S} &= \frac{r}{\Phi' + r} \\
\frac{\hat{S}}{1 - \hat{S}} &= \frac{r}{\Phi'} \\
\end{align*}
\]

From (27'), it is clear that \( dS^*/dt < 0 \) for \( S^* \leq \hat{S}, u > 0 \); but this implies that \( S^*(t) \) tends to \( \hat{S} \) from above, so that \( S^*(t) \) decreases monotonically for the infinite horizon case.\(^{26}\)

Equations (27'), (28), and (29) imply that for any finite \( T, S^*(t) > \hat{S} \), since \( S^*(t) \) rises during the terminal stages of the finite horizon path. Moreover, these equations imply that for large \( T, S^*(t) \) must be decreasing during early stages of the plan. This result holds since, given \( L_c(0) \), the larger is \( T \), the larger is \( u(0) \), and the smaller is \( S^*(0) \); further, as \( T \) increases, these values approach the rates for the infinite horizon solution, and I have already shown \( S^*(t) \) decreases monotonically in that case. Thus, given \( L_c(0) < L_c^* \), there exists a \( \bar{T} \) such that for \( T < \bar{T}, S^*(t) \) rises monotonically along the optimum path, whereas for \( T > \bar{T}, S^*(t) \) first decreases, reaches a minimum (which exceeds \( \hat{S} \)), and then starts increasing (for \( u > 0 \)).

Thus, the time path of \( S^*(t) \) for any finite horizon can be characterized as follows:

(i) for small \( T, S^* \) is constant and equal to the static subsidy;

(ii) for intermediate values of \( T, S^*(t) \) rises for \( u > 0 \), and then remains constant for \( u = 0 \);

(iii) for larger values of \( T, S^*(t) \) initially decreases, reaches a minimum, then increases until \( u = 0 \).\(^{27}\)

Figures 3 and 4 depict these results (the dashed curves in Figure 3 represent iso-

\(^{26}\) I have shown \( S^*(t) > \hat{S} \); thus \( S^*(t) \) approaches \( \hat{S} \) monotonically from above since once \( S^* \) starts to increase, it must continue to do so.

\(^{27}\) For initially large misallocations of labor, it is likely that \( dS^*/dt < 0 \) during early stages of the plan. To see this, rewrite (27') as:

\[
\begin{align*}
\frac{dS^*}{dt} &= - \left( \frac{\Phi' + r}{F_c} \right) \left[ \beta u \frac{F_c}{L_c} - \frac{\Phi' F_c}{F_c} \right] \\
&= - \left( \frac{\Phi' + r}{F_c} \right) \left[ \beta u \frac{F_c}{L_c} - \left( \frac{F_c}{F_c} - \frac{F_c}{F_c} \right) \right] \\
&= - \left( \frac{\Phi' + r}{F_c} \right) \left( \beta u \frac{F_c}{L_c} - \frac{F_c}{F_c} \right) \\
\end{align*}
\]

where \( \Phi = \Phi'(0) \cdot u \) since \( \Phi' = 0 \). Given \( T \), let \( L_c(0) \to 0 \); then, if \( F_c(1 - \beta) \to 0 \) as \( L_c(0) \to 0 \) and \( (L_c F_c/F_c) = (1 - \beta) \); also \( \beta u \) increases as \( L_c(0) \) decreases and \( PF_c(1 - \beta) \) remains nonzero, given finite \( T \). Thus, there exists \( L_c \) such that for \( L_c(0) < L_c \), \( dS^*/dt(0) < 0 \), indicating \( S^* \) decreases during early stages of the plan.
subsidy lines). Figure 4 is drawn under the assumption that \( L_e(0) \) is constant; all that varies is the length of the planning horizon.\(^{28}\) Finally, note that regardless of the assumptions on \( \Phi'' \) and \( [L_e F_c''/F_c'] \), \( S^*(t) \) must increase during the final stages of unemployment for any finite horizon path.

### IV. Conclusion

We have seen that if resources cannot be transferred costlessly, the static subsidy is inefficient in a dynamic context unless the planning horizon is short, or discount rates are large. Thus in some sense the static subsidy represents myopic behavior. Moreover, I have shown how the optimum path can be determined and have characterized the properties of the path. In particular, we have seen that for long time horizons and low discount rates, it will always be optimal to have some unemployment initially; and the initial level of unemployment increases with the time horizon. Nevertheless, if wages must be equalized across sectors, some subsidy will always be needed; and we have discussed how this subsidy changes through time. Thus, a realistic policy must recognize that resources are not instantaneously mobile, but it must also recognize that too large a subsidy removes the incentives for inter-temporal reallocation of resources.

### Appendix

To show iso-unemployment loci are positively sloped, consider (11), which defines an implicit function in \( u, L_e, \) and \( q \). Differentiating (11) yields:

\[
\begin{align*}
(A1) \quad \partial u/\partial q &= -\Phi/[q \Phi'' + PL_m F''_m] > 0 \\
(A2) \quad \partial u/\partial L_e &= -[(1 - u) PF''_m] / [q \Phi'' + PL_m F''_m] < 0
\end{align*}
\]

Thus, all iso-unemployment loci are positively sloped; also from (11) it is clear that as \( L_e \rightarrow L, q \rightarrow \infty \) for \( H_u = 0 \). From (A1), it is seen that points above the \( g(q, L_e) = 0 \) locus represent positive unemployment.

For the locus \( h(q, L_e) = 0 \), defined by (16), we find by differentiating:

\[
\begin{align*}
(A3) \quad \partial h/\partial q &= [r + \Phi] - PN_m F''_m (\partial u/\partial q) > 0 \\
(A4) \quad \partial h/\partial L_e &= -F''_c - [P(1 - u)^2 q \Phi'' F''_m] / [q \Phi'' + PL_m F''_m] > 0
\end{align*}
\]

Equations (A1) and (A2) are used in deriving (A3) and (A4) since \( u \) is defined as an implicit function of \( L_e \) and \( q \) by (11). Thus, the \((dq/dt) = 0 \) locus is negatively sloped. However, for \( r = 0, u = 0, (dh/\partial q) = 0 \), indicating that the locus becomes vertical at the point where it crosses the \( g(q, L_e) = 0 \) locus. For \( r > 0, u = 0 \), it is readily seen from (11) that \((dq/dt) = 0 \) at \( q = 0, F'_c = PF_c' \). Finally, (A3) asserts that points above the \( h(q, L_e) = 0 \) locus correspond to \((dq/dt) > 0 \).

### References


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\(^{28}\) In Figure 4, the vertical broken lines are meant to indicate the end of the planning horizon, at which time the subsidy program ends.

