1. In random sampling from any population with \( E(X) = \mu \) and \( \text{Var}(X) = \sigma^2 \), show (using Chebyshev's inequality) that sample mean converges in probability to \( \mu \).

2. In random sampling from any population with \( E(X) = \mu \) and \( \text{Var}(X) = \sigma^2 \), show without using Chebyshev's inequality:
   a. The sample mean converges in mean square to \( \mu \).
   b. The sample mean converges in probability to the population mean.

3. Prove the following theorem:

   If \( T_n \) is a sequence of random variables with \( \lim E(T_n) = c \) and \( \lim \text{Var}(T_n) = 0 \) then \( T_n \) converges in mean square to \( c \).

4. Show that \( \mathbb{E}(X^2) \rightarrow 0 \Rightarrow X \overset{p}{\rightarrow} 0 \)

5. Consider a variable \( T \), which is distributed as a t with \( v \) degrees of freedom. The t distribution has mean \( v \) and variance \( 2v \). The variable \( T \) can be written as

   \[
   T = \frac{Z}{\sqrt{\chi^2/v}}
   \]

   Using the Chebyshev inequality and Slutsky's theorem show that

   \[
   T \overset{d}{\rightarrow} Z \sim \mathcal{N}(0,1)
   \]

   as \( v \) goes to infinity. The Chebyshev inequality is given by

   \[
   P(\lvert X - \mu \rvert \geq \delta \sigma) \leq \frac{1}{\delta^2}
   \]

   \[
   P(\lvert X - \mu \rvert \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}
   \]

   \[
   P(\lvert X - \mu \rvert < \varepsilon) \geq 1 - \frac{\sigma^2}{\varepsilon^2}
   \]
6. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with mean $\mu$ and variance $\sigma^2 < \infty$. Define $S_n^2$ as

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

Using Chebyshev’s inequality, find a sufficient condition that $S_n^2$ converges in probability to $\sigma^2$.

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with mean $\mu$ and variance $\sigma^2 < \infty$. It can be shown that $\text{Var}(S^2)$ can be written as

$$\text{Var}(S^2) = \frac{\mu_4 - \mu_2^2}{n} - \frac{2(\mu_4 - 2\mu_2^2)}{s^2} + \frac{\mu_4 - 3\mu_2^2}{s^3}$$

where $S^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$ and $\mu_k$ denotes the kth population moment about the population mean.

a. Show that for a normal population,

$$\text{Var}(S^2) = \frac{2\mu_2^2(n-1)}{s^2}$$

The moment generating function for a normal distribution will be useful here.

b. Show that for a normal population,

$$\text{Var}(S^2) = \frac{2\sigma^4(n-1)}{s^2}$$

c. Show that for a normal population

$$\text{Var}(S^2) = \frac{2\sigma^4}{(n-1)}$$
8. Let \( x_1, x_2, \ldots, x_n \) be the realizations that result from a random sample from a population with mean \( \mu \) and variance \( \sigma^2 < \infty \). Also assume that \( X_i \) is distributed normally. Consider the following matrix \( M \), which is \( n \times n \)

\[
M = \begin{pmatrix}
1 & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\
-\frac{1}{n} & 1 & -\frac{1}{n} & \cdots & -\frac{1}{n} \\
-\frac{1}{n} & -\frac{1}{n} & 1 & \cdots & -\frac{1}{n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & 1 + \frac{1}{n}
\end{pmatrix}
\]  

(a) Show that \( M \) is idempotent.

(b) What is the trace of \( M \)?

(c) Show the general form of the row vector \( x'M \).

\[
x'M = [x_1, x_2, x_3, \ldots, x_n]
\]

\[
\begin{pmatrix}
1 & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\
-\frac{1}{n} & 1 & -\frac{1}{n} & \cdots & -\frac{1}{n} \\
-\frac{1}{n} & -\frac{1}{n} & 1 & \cdots & -\frac{1}{n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & 1 + \frac{1}{n}
\end{pmatrix}
\]  

Hint:

\[
[x_1, x_2, x_3, \ldots, x_n] \begin{pmatrix}
1 - \frac{1}{n} \\
-\frac{1}{n} \\
\vdots \\
-\frac{1}{n}
\end{pmatrix} = x_1 - \frac{1}{n}x_1 - \frac{1}{n}x_2 - \frac{1}{n}x_3 + \cdots - \frac{1}{n}x_n
\]  

(16)
d. Show the general form of the scalar $x'Mx$.

e. Show that $\sum_{j=1}^{n} (x_j - \bar{x})^2 = \sum_{j=1}^{n} x_j^2 - n\bar{x}^2$

f. Using the information in parts a-e and appropriate theorems show that
$$\frac{\sum_{j=1}^{n} (x_j - \bar{x})^2}{\sigma^2} = \chi^2(n - 1)$$

9. Let $X_1, X_2, \ldots X_n$ be a random sample from a population with mean $\mu$ and variance $\sigma^2 < \infty$. Now define the rth moment of $X_i$, usually denoted by $\mu_i^r$, as

$$\mu_i^r = E[X_i^r]$$

$$= \int x_i^r f(x_i; \theta_{1}, \ldots, \theta_{k}) \, dx_i$$

(20)

The rth central moment of $X_i$ about a is defined as $E[(X_i - a)^r]$. If $a = \mu$, we have the rth central moment of $X_i$ about $\mu$, denoted by $\mu_i^r$, which is

$$\mu_i^r = E[(X_i - \mu)^r]$$

$$= \int (x_i - \mu)^r f(x_i; \theta_{1}, \ldots, \theta_{k}) \, dx_i$$

(21)

The rth sample moment is defined as

$$\bar{\mu}_i^r = \frac{1}{n} \sum_{i=1}^{n} x_i^r$$

(22)

The rth central sample moment is defined as

$$\bar{c}_i^r = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_i^r)^r$$

$$= c_i^r = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_i^r) = \bar{X} - \mu$$

$$= c_i^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_i^r)^2 = \frac{1}{n} \left( \sum_{i=1}^{n} x_i^2 \right) - \mu^2$$

where $\mu_i^1$ is the first raw moment of the distribution which is $E(X_i) = \mu$. The rth central sample moment about the sample mean is defined as
\[ M'_2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 \]

\[-M'_2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}) = \bar{X} - \bar{X} = 0 \]

\[-M'_2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{1}{n} \left( \sum_{i=1}^{n} X_i^2 \right) - \bar{X}^2 \]

a. Show that

\[ E(M'_2) = \frac{1}{n} \sum_{i=1}^{n} E(X_i^2) - \text{Var}(\bar{X}) - (E\bar{X})^2 \]

\[ = \mu_2 - (\mu_1)^2 - \frac{\mu_2}{n} \]

\[ = \frac{n-1}{n} \sigma^2 \]

where \( \mu_1 \) and \( \mu_2 \) are the first and second population moments, and \( \mu_2 \) is the second central population moment. Note that this obviously implies

\[ E \left[ \sum_{i=1}^{n} (X_i - \bar{X})^2 \right] = E \left[ \left( \sum_{i=1}^{n} X_i^2 \right) - n\bar{X}^2 \right] \]

\[ = (n-1) \mu_2 \]

b. Now write \( \sum_{i=1}^{n} (X_i - \bar{X})^2 \) as follows

\[ \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu) - (\bar{X} - \mu)^2 \]

\[ Y_i = X_i - \mu \]

\[ \bar{Y} = \bar{X} - \mu \]

Show the following

i) \( E(Y_i) = 0 \)

ii) \( \text{Var}(Y_i) = \sigma^2 \)

iii) \( E(Y_i^4) = \mu_4 \) (fourth central moment of \( X_i \))
c. Now consider computing $\text{Var}\left( \sum_{j=1}^{n} (x_j - \bar{x})^2 \right) = \text{Var}\left( \sum_{i=1}^{n} (y_i - \bar{y})^2 \right)$. This can be written as follows

$$\text{Var}\left( \sum_{i=1}^{n} (y_i - \bar{y})^2 \right) = E \left[ \left( \sum_{i=1}^{n} (y_i - \bar{y})^2 \right)^2 \right] - \left[ E \left( \sum_{i=1}^{n} (y_i - \bar{y})^2 \right) \right]^2$$

Show that

$$E \left[ \left( \sum_{i=1}^{n} (y_i - \bar{y})^2 \right)^2 \right] = E \left[ \sum_{i=1}^{n} y_i^2 \right] - 2nE \left[ \bar{y}^2 \sum_{i=1}^{n} y_i^2 \right] + n^2 E(\bar{y}^4)$$

d. Show that

$$E \left[ \left( \sum_{i=1}^{n} y_i^2 \right)^2 \right] = E \left[ \sum_{i=1}^{n} y_i^2 \sum_{j=1}^{n} y_j^2 \right] = E \left[ \sum_{i=1}^{n} y_i^2 + \sum_{i} \sum_{j} y_i^2 y_j^2 \right]$$

$$= n\mu_4 + n(n-1)\mu_2^2$$

$$= n\mu_4 + n(n-1)\sigma^4$$

e. Now show that

$$E \left[ \bar{y}^2 \sum_{i=1}^{n} y_i^2 \right] = \frac{1}{n^2} E \left[ \sum_{i=1}^{n} y_i \sum_{j=1}^{n} y_j \sum_{i=1}^{n} y_i^2 \right]$$

$$= \frac{1}{n^2} \left[ \sum_{i=1}^{n} y_i^4 + \sum_{i=1}^{n} \sum_{j=1}^{n} y_i y_j y_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} y_i y_j y_k y_i^2 \right]$$

$$= \frac{1}{n^2} \left[ n\mu_4 + n(n-1)\mu_2^2 \right]$$

$$= \frac{1}{n^2} \left[ \mu_4 + (n-1)\sigma^4 \right]$$

f. Now show that

$$E[\bar{y}^4] = \frac{1}{n^4} E \left[ \sum_{i=1}^{n} y_i \sum_{j=1}^{n} y_j \sum_{k=1}^{n} y_k \sum_{l=1}^{n} y_l \right]$$

$$= \frac{1}{n^2} \left[ \sum_{i=1}^{n} y_i^4 + \sum_{i=1}^{n} \sum_{j=1}^{n} y_i^2 y_j^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} y_i y_j y_i^2 y_j^2 + \cdots \right]$$

where for the first double sum (i = j ≠ k = l), for the second (i = k ≠ j = l), and for the last (i = l ≠ j = k) and ... indicates that all other terms include $y_i$. 
g. Now show that
\[ B\left[ \bar{Y}^4 \right] = \frac{1}{\mu_4} \left[ n \mu_4 + 3n(n-1)\mu_2^2 \right] \]
\[ = \frac{1}{\mu_3} \left[ \mu_4 + 3(n-1)\sigma^4 \right] \]

h. Now combining the information in all the parts we can show that
\[ V_X \left( \sum_{i=1}^{n} (X_i - \bar{X})^2 \right) = V_X \left( \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right) \]
is as follows

\[ V_X \left( \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right) = B \left[ \left( \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right)^2 \right] - \left[ B \left( \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right) \right]^2 \]
\[ = B \left[ \left( \sum_{i=1}^{n} Y_i^2 \right)^2 \right] - 2nB \left[ \sum_{i=1}^{n} Y_i^2 \right] + n^2B \left( \bar{Y}^4 \right) - \left[ B \left( \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right) \right]^3 \]
\[ = n\mu_4 + n(n-1)\mu_2^2 - 2n \left[ \frac{1}{\mu_4} \left( \mu_4 + (n-1)\mu_2^2 \right) \right] - n^2 \left[ \frac{1}{\mu_4} \left( \mu_4 + 3(n-1)\mu_2^2 \right) \right] - (n-1)^2\mu_2^2 \]
\[ = n\mu_4 + n(n-1)\mu_2^2 - 2\left( \mu_4 + (n-1)\mu_2^2 \right) + \left[ \frac{1}{\mu_4} \left( \mu_4 + 3(n-1)\mu_2^2 \right) \right] - (n-1)^2\mu_2^2 \]

Now show that
\[ V_X \left[ \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right] = \frac{(n-1)^2\mu_4}{n} - \frac{(n-1)(n-3)\mu_2^2}{n} \]
\[ = \frac{(n-1)^2\mu_4}{n} - \frac{(n-1)(n-3)\sigma^4}{n} \]

i. Now show that
\[ V_X[\bar{X}^2] = \frac{\mu_4}{n} - \frac{(n-3)\mu_2^2}{n(n-1)} \]

j. Now show that
\[ V_X[\bar{Y}^2] = \frac{(n-1)^2\mu_4}{n} - \frac{(n-1)(n-3)\mu_2^2}{n^2} \]
k. Now show that

\[ \mathbb{V} \Phi(\theta^2) = \frac{\mu_4 - \mu_2^2}{\sigma} - \frac{2(\mu_4 - 2\mu_2^2)}{\sigma^2} + \frac{(\mu_4 - 3\mu_2^2)}{\sigma^3} \]

10. Let the density function be given by

\[ f(x; \theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0 \]  \hspace{1cm} (44)

a. Find the expected value of x. Also show that

\[ \int_0^1 x^{\theta-1} \, dx = \frac{1}{\theta} \]  \hspace{1cm} (45)

b. Show that

\[ \int_0^1 x^{\theta-1} \ln x \, dx = \frac{-1}{\theta^2} \]  \hspace{1cm} (46)

and then find the E(ln x).

c. By differentiating under the integral sign find

\[ \int_0^1 x^{\theta-1} (\ln x)^2 \, dx = \frac{2}{\theta^3} \]  \hspace{1cm} (47)

and then find the Var(ln x).

d. Now consider a random sample \( x_1, x_2, \ldots, x_n \). Given the above density function, what is the likelihood function? What is a sufficient statistic for \( \theta \)?

e. What is the MLE estimator of \( \theta \)? You can use the log likelihood or remember that

\[ \frac{d}{dy} \Phi(y) = \Phi' \log \Phi \]  \hspace{1cm} (48)

and

\[ \frac{d}{dy} \Phi(\Phi') = \Phi' \log \Phi \frac{d\Phi}{dy} \]  \hspace{1cm} (49)

f. Now consider the parameter \( \alpha \) which is given by \( \alpha = 1/\theta \). What is the MLE estimator of \( \alpha \), call it \( \hat{\alpha} \). Is it unbiased?

g. What is the variance of \( \hat{\alpha} \)?
h. What is the Fisher information number of $\mathbf{A}$?

i. What is the Cramer-Rao lower bound for $\mathbf{A}$?

11. This problem will analyze an idempotent matrix using MATLAB.

a. Set the length of the vector $y$ equal to $n$, i.e. $n = 4$.
b. Create an $n$ vector of ones, call it $\mathbf{i}$.
c. Create an $n \times n$ identity matrix.
d. Create the matrix $\mathbf{M} = \mathbf{I} - \frac{1}{n} \mathbf{i} \cdot \mathbf{i}'$
e. Check that $\mathbf{M}$ is idempotent.
f. Input the vector $y$

\[
\begin{pmatrix}
-2 \\
5 \\
3 \\
-2
\end{pmatrix}
\] (55)

g. Find the mean of the elements of the vector $y$.
h. Find $\mathbf{y}'\mathbf{M}'$ and comment on what it is.
i. Show that $\mathbf{y}'\mathbf{M}' = \mathbf{y}'\mathbf{M}\mathbf{y}$
j. Find the eigenvectors and values of the matrix $\mathbf{M}$, call them $\mathbf{Q}$ and $\mathbf{D}$ respectively.
k. Check to make sure MATLAB computes a $\mathbf{Q}$ that is orthogonal.
l. Find the trace and rank of $\mathbf{M}$.
m. Define $\mathbf{v} = \mathbf{Q}'\mathbf{y}$.

\[
\begin{pmatrix}
-2 \\
5 \\
3 \\
-2
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -1 \\
1 & 4 \\
1 & 2 \\
1 & -2
\end{pmatrix}
\] (56)

q. Show that $\mathbf{v}'\mathbf{Q}'\mathbf{M}\mathbf{Q}\mathbf{v} = \mathbf{v}'\mathbf{D}\mathbf{v} = \mathbf{y}'\mathbf{M}\mathbf{y}$
r. Which element of $\mathbf{v}$ is not included in the summation of squares implied in $q$?
s. Create the matrix $\mathbf{L} = \text{ones}(3,4)$
t. Compute $\mathbf{LM}$.
u. Compute $\mathbf{C} = \mathbf{L}\mathbf{Q}$.
v. What is the structure of $\mathbf{C}$?
w. Compute $\mathbf{Ly}$
x. Show that $\mathbf{C}\mathbf{v} = \mathbf{Ly}$

12. This problem will analyze a projection problem using MATLAB

a. Set the length of the vector $y$ equal to $n$, i.e. $n = 4$.
b. Create an $n \times n$ identity matrix.
c. Input the vector $y$ and the matrix $\mathbf{X}$

\[
\mathbf{y} = \begin{pmatrix}
-2 \\
5 \\
3 \\
-2
\end{pmatrix}
\]

\[
\mathbf{X} = \begin{pmatrix}
1 & -1 \\
1 & 4 \\
1 & 2 \\
1 & -2
\end{pmatrix}
\]

d. Create the projection matrix $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$
e. Check that $P$ is idempotent.
f. Check that $PX = X$.
g. Compute the projection of $y$ on the column space of $X$, $p = Py$.
h. Compute the error $e_1 = y - p$.
i. Check that $e_1$ is orthogonal to $X$.
j. Find the vector $c$ which forms the linear combination of $X$ closest to $y$, $c = (X'X)^{-1}X'y$
k. Check that $Xc = p$
l. Compute the least square estimate of $c_1$ in the equation $y = Xc_1$ using $c_1 = X'y$
m. Create the matrix $M = I - X(X'X)^{-1}X'$
n. Check that $M$ is idempotent.
o. Check that $PM = 0$.
p. Show that $y'M'y = y'M'My$
q. Define $e = My$.
r. Check that $e = e_1$.
s. Show that $e'e = y'My$
t. Show that $MX = 0$.

13. Assume that the birth weight in grams of a baby born in the United States is normal with a mean of 3315 and a standard deviation of 525, boys and girls combined. Let $X$ equal the weight of a baby girl who is born at home in Dallas County and assume that the distribution of $X$ is $N(\mu_x, \sigma^2_x)$.

a. Assuming 11 observations of $X$, give the test statistic and critical region for testing $H_0: \mu_x = 3315$ against the alternative hypothesis $H_1: \mu_x > 3315$ (home-born babies are heavier) if $\alpha = 0.01$.
b. Calculate the value of the test statistic and give your conclusion using the following weights:

<table>
<thead>
<tr>
<th>3119</th>
<th>2657</th>
<th>3459</th>
<th>3629</th>
<th>3345</th>
<th>3629</th>
</tr>
</thead>
<tbody>
<tr>
<td>3515</td>
<td>3856</td>
<td>3629</td>
<td>3345</td>
<td>3062</td>
<td></td>
</tr>
</tbody>
</table>

c. What is the p-value of the test?
d. Give the test statistic and critical region for testing $H_0: \sigma^2_x = 525^2$ against the alternative hypothesis $H_1: \sigma^2_x < 525^2$ (less variation in the weights of home-born babies) if $\alpha = 0.05$.
e. Calculate the value of your test statistic and state your conclusion.
f. What is the p-value of this test.

14. Let $Y$ equal the weight in grams of a baby boy who is born at home in Dallas County and assume that the distribution of $Y$ is $N(\mu_y, \sigma^2_y)$. Using the following weights:

<table>
<thead>
<tr>
<th>4082</th>
<th>3686</th>
<th>4111</th>
<th>3686</th>
<th>3175</th>
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<td>3430</td>
<td>3289</td>
<td>3657</td>
<td>4082</td>
<td></td>
</tr>
</tbody>
</table>

answer the same questions as in problem 13.
15. Consider generating multivariate normal random variables with the following mean vector and covariance matrix.

\[
\mu = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad \text{COV} = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}
\]  

(61)

MATLAB has a function for generating random draws from a normal population, \( R = \text{normrnd}(\mu, \Sigma) \). We can use this to generate multivariate random numbers by factoring the covariance matrix and using a transformation. The following code is helpful where \( cv \) is the covariance matrix and \( n \) is the number of random numbers we want to create.

```matlab
n=500
mu = [-2;3]
k = length(mu)
cv = [1 0.7; 0.7 1]
R = chol(cv)
Z = normrnd(0,1,[n,k]);
X = Z*R + ones(n,1)*mu';
```

Compare the mean and covariance of the sample to that of the posited underlying distribution.
Extra on problem 8.

\[
\begin{bmatrix}
\bar{x}_1 - \bar{x}, \bar{x}_2 - \bar{x}, \bar{x}_3 - \bar{x}, \ldots, \bar{x}_n - \bar{x}
\end{bmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
= x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 - n(\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \ldots + \bar{x}_n)
\]

\[= \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \tag{62}\]

\[
\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - 2\bar{x} \sum_{i=1}^{n} x_i + n\bar{x}^2
\]

\[= \sum_{i=1}^{n} x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 \tag{63}\]

\[= \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \]

Alternative form of expectation.

\[
E\left[M_n^2\right] = \frac{1}{n^2} \left[ E\left[\left(\sum_{i=1}^{n} X_i^2\right)^2\right] - 2n E\left[\bar{X}^2 \sum_{i=1}^{n} X_i\right] - n^2 E\left[\bar{X}^4\right]\right]
\]
Derivation of variance

\[
V_{\bar{Y}} \left[ \sum_{i=1}^{n} (Y_i - \overline{Y})^2 \right] = n\mu_4 + n(n-1)\mu_2^2 - 2n\mu_4 + (n-1)\mu_2^2 + \left( \frac{1}{n} \right)^2 \mu_4 + 3(n-1)\mu_2^2 - (n-1)^2 \mu_2^4 \\
= \frac{n^2\mu_4 + n^2(n-1)\mu_2^2 - 2n\mu_4 + 2n(n-1)\mu_2^2 + \mu_4 + 3(n-1)\mu_2^2 - n(n-1)^2 \mu_2^4}{n} \\
= \frac{\mu_4(n^2 - 2n + 1) + (n-1)\mu_2^2}{n} \left( n^2 - 2n + 3 - n(n-1) \right) \\
= \frac{\mu_4(n-1)^2 - (n-1)\mu_2^2}{n} (n - 3) \\
= \frac{(n-1)^2\mu_4}{n} - \frac{(n-1)(n-3)\mu_2^2}{n} \\
= \frac{(n-1)^2\mu_4}{n} - \frac{(n-1)(n-3)s^4}{n}
\]

j. Now show that

\[
V_{\bar{Y}}[\sigma^2] = \frac{(n-1)^2\mu_4}{n^3} - \frac{(n-1)(n-3)\mu_2^2}{n^3} \\
= \frac{n^2\mu_4 - 2n\mu_4 + \mu_4 - 4n\mu_2^2 + 3\mu_2^2}{n^3} \\
= \frac{\mu_4 - \mu_2^2}{n} - \frac{2(\mu_4 - 2\mu_2^2)}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3}
\]

k. Now show that

\[
V_{\bar{Y}}[\sigma^2] = \frac{\mu_4 - \mu_2^2}{n} - \frac{2(\mu_4 - 2\mu_2^2)}{n^2} + \frac{(\mu_4 - 3\mu_2^2)}{n^3}
\]