Problemset 3

1. Let $X$ and $Y$ be normally distributed. Show that if $\text{COV}(x, y) = 0$ then $X$ and $Y$ are independent.

2. If the probability density of $y$ is $\alpha y^2 (1 - y^3)$ for $y$ between 0 and 1, then what is $\alpha$? What is the probability that $y$ is between 0.25 and 0.75.

3. The random variable $x$ has a continuous distribution $f(x)$ and cumulative distribution function $F(x)$. A sample is drawn as independent draws from $x$. What is the probability distribution of the sample maximum? (Hint: In a random sample of $n$ observations, if $z$ is the maximum, then every observation in the sample is less than or equal to $z$. Use the cdf.)

4. Assume that the distribution of $x$ is $f(\theta) = \frac{1}{\theta}$, $0 \leq x \leq \theta$. In random sampling from this distribution, prove that the sample maximum, $x_{\text{max}}$ satisfies that $x_{\text{max}} \xrightarrow{p} \theta$.

5. Given the current technology, the production of active matrix color screens for notebook computers is a difficult process that results in a significant proportion of defective screens being produced. At one company, the daily proportion of defective 9.5” and 10.4” screens is the outcome of a bivariate random variable, $X$, with joint density function $f(x_1, x_2; \alpha) = \left[ \alpha x_1 + (2 - \alpha) x_2 \right] I_{[0,1]}(x_1) I_{[0,1]}(x_2)$, $\alpha \in (0, 2)$. The daily proportion of defectives are independent from day to day. A collection of $n$ i.i.d outcomes of $X$ will be used to generate an estimate of the $(2 \times 1)$ vector of mean daily proportion of defectives, $\mu$, for the two types of screens being produced, as

$$ x_{\text{max}} = \frac{X_1 \cdots X_n}{n} $$

where

$$ X_i = \begin{bmatrix} X_{1i} \\ X_{2i} \end{bmatrix}. $$

(a) Does $\bar{X}_n \xrightarrow{as} \mu$? Does $\bar{X}_n \xrightarrow{p} \mu$? Does $\bar{X}_n \xrightarrow{d} \mu$?
(b) Define an asymptotic distribution for the bivariate random variable $\bar{X}_n$. If $\alpha = 1$, $n = 200$, what is the approximate probability that $\bar{X}_n[1] > 0.70$ given that $\bar{X}_n[2] = 0.60$?

(c) Consider using an outcome of the function $g (\bar{X}_n) = \bar{X}_n [1] / \bar{X}_n [2]$ to generate an estimate of the relative expected proportions of defective 9.5” and 10.4” screens, $\mu_1/\mu_2$. Does $g (\bar{X}_n) \overset{as}{\to} \mu_1/\mu_2$? Does $g (\bar{X}_n) \overset{p}{\to} \mu_1/\mu_2$? Does $g (\bar{X}_n) \overset{d}{\to} \mu_1/\mu_2$?

(d) Define an asymptotic distribution for $g (\bar{X}_n)$. If $\alpha = 1$, $n = 200$, what is the approximate probability that the outcome of $g (\bar{X}_n)$ will exceed 1?