**Economics 573**
**Examination 1**

1. Consider the following density function.

$$f(X) = X e^{-X} \quad 0 \leq X \leq \infty$$
$$= 0 \quad \text{otherwise}$$

a. Find the expected value of x directly by integration.

b. What is the moment generating function for this random variable? It will be a function of t.

c. Use the derived moment generating function to find the mean and variance of this distribution.

2. Given that

$$f(X, Y) = 2e^{-(X+Y)} \quad 0 \le X \le Y, \quad 0 \le Y$$
$$= 0 \quad \textit{otherwise} \qquad \textbf{(2)}$$

a. What is the marginal density of X?

b. What is the marginal density of Y?

c. Show that the density integrates to one (be careful on the order in which you integrate).

d. Find $P(Y < 3X)$.

3. The random variable Y has the following pdf

$$F_Y(y;\, \theta) = \left(\theta^2 + \theta\right) y^{\theta - 1}(1 - y), \quad 0 < y < 1 \qquad \textbf{(3)}$$

a. Find the first theoretical moment of Y.

b. What is the method of moments estimator of $\theta$?

4. Let the random variable Y have the following density function

$$f_Y(y;\theta) = \frac{\theta}{2\sqrt{y}} \cdot e^{-\theta\sqrt{y}}, \quad y > 0 \qquad \textbf{(4)}$$

a. Find the maximum likelihood estimator of $\theta$ in a random sample of size n.

b. What is the MLE estimate of $\theta$ if there are five observations as follows: [1, 4 , 9, 4, 16 ]?

5. Chebyshev's inequality says that if X is a random variable and g(x) be a non-negative function, then for r > 0,

$$P\left[g(X) \geq r\right] \leq \frac{Eg(X)}{r} \qquad \textbf{(5)}$$

a. Prove this theorem

b. Let X be a random variable with mean $\mu$ and variance $\sigma^2$. Prove that for $\delta > 0$

$$P\left[|X-\mu| \geq \delta\sigma\right] \leq \frac{1}{\delta^2} \qquad \textbf{(6)}$$

6. Consider a variable $T_v$ which is distributed as a t with v degrees of freedom. The $\chi^2$ distribution has mean v and variance 2v. The variable $T_v$ can be written as

$$T_v = \frac{Z}{\sqrt{\chi^2_v / v}}$$

Using the Chebyshev inequality and Slutsky's theorem show that

$$T_v = \frac{Z}{\sqrt{\chi^2_v / v}} \xrightarrow{d} Z \sim N(0,1)$$

as v goes to infinity. The Chebyshev inequality is given by

$$P[|X-\mu| \geq \delta\sigma] \leq \frac{1}{\delta^2}$$

$$P[|X-\mu| \geq \varepsilon] \leq \frac{\sigma^2}{\varepsilon^2}$$

$$P[|X-\mu| < \varepsilon] \geq 1 - \frac{\sigma^2}{\varepsilon^2}$$

7. Define the following terms

a. An <u>unbiased estimator</u> of a parameter $\theta$.

b. A best linear unbiased estimator (BLUE) of $\theta$.

c. The asymptotic expectation of a random variable $\theta_n$.

d. Convergence in probability of a random variable $\theta_n$ to $\mu$.

e. Consistency of an estimator of the parameter $\lambda$. The estimator is denoted $\theta$.

f. Mean square error consistency of an estimator $\theta$.

g. A maximum likelihood estimator (MLE) of $\theta$.

h. A sufficient statistic

i. Characteristic root of a matrix

j. Student's t distribution

8. Prove the following theorem.

Let A be a square idempotent matrix of order n. Then its characteristic roots are either zero or one.

9. The random variable X has the following pdf

$$f(X;\theta,\kappa) = \frac{\kappa}{\theta\left(1 + \frac{X}{\theta}\right)^{\kappa+1}}, \quad X > 0,\ \theta > 0,\ \kappa > 0 \tag{10}$$

Now consider the random variable Z = ln X.

a. What are the bounds on Z?

b. What is the inverse function X = g(Z)?

c. What is the pdf of Z?

d. Now consider an additional transformation $Y = \alpha + \beta Z$ ($\beta > 0$). What is the inverse function Z = h(Y)?

e. Now set $\theta = \kappa = 1$. What is the pdf of Y?

10. Let X equal the exam score for students in Economics 573 at Iowa State University. Assume that the distribution of X is $N(\mu_X, \sigma_X^2)$. Assume you collect a random sample of 30 scores as follows.

X = [72 74 71 62 73 70 55 77 74 67 77 59 82 65 78 78 70 58 80 60 80 57 65 60 63 86 99 63 51 74].

You also compute the following sample statistics.

$$\bar{x} = 70$$

$$s^2 = 109.3103$$

a. Assuming 30 observations of X, give the test statistic and critical region for testing $H_0$: $\mu_X \leq 80$ against the alternative hypothesis $H_1$: $\mu_X > 80$ at the 2.5% level. The following numbers may be useful for a t distribution with 29 degrees of freedom.

$$0.025 = \int_{-\infty}^{-2.045} f(t;\ 29)\, dt \qquad 0.975 = \int_{-\infty}^{2.045} f(t;\ 29)\, dt$$

$$0.05 = \int_{-\infty}^{-1.699} f(t;\ 29)\, dt \qquad 0.95 = \int_{-\infty}^{1.699} f(t;\ 29)\, dt$$

b. Form a confidence interval for the mean of the distribution at the 95% level.

c. Calculate the value of the test statistic from part a and give your conclusion regarding the hypothesis suggested there.

d. Give the test statistic and critical region for testing $H_0$: $\sigma_X^2 = 100$ against the alternative hypothesis $H_1$: $\sigma_X^2 < 100$ if $\alpha = 0.05$. The following numbers may be useful for a chi squared distribution with 29 degrees of freedom.

$$.05 = \int_0^{17.71} f(\chi;\ 29)\, d\chi \qquad 0.95 = \int_0^{42.56} f(\chi;\ 29)\, d\chi$$

e. Calculate the value of your test statistic and state your conclusion.