INTERVAL ESTIMATION AND HYPOTHESES TESTING

1. Idea

An interval rather than a point estimate is often of interest. Confidence intervals are thus important in empirical work. To construct interval estimates, standardized normal random variables are often used.

2. Standardized Normal Variables and Confidence Intervals for the Mean with σ Known

If Y is a normal random variable with mean β and variance σ^2 then

$$Z = \left(\frac{Y - \beta}{\sigma}\right) \tag{1}$$

is a standard normal variable with mean zero and variance one. An estimate of β, $\hat{\beta}$ is given by the sample mean y. This then implies

$$\begin{aligned} \hat{\beta} &= \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y} \sim N\left(\beta,\, \frac{\sigma^2}{n}\right) \\ z &= \frac{\hat{\beta} - \beta}{\left[\frac{\sigma}{\sqrt{n}}\right]} \sim N(0,\, 1) \end{aligned} \tag{2}$$

So if γ_1 is the upper $\alpha/2$ percent critical value of a standard normal variable, i.e.

$$\begin{aligned} &\int_{\gamma_1}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z^2}\, dz = \frac{\alpha}{2} \quad \text{then} \\ 1 - \alpha = F(\gamma_1) - F(-\gamma_1) &= \Pr\left[-\gamma_1 \leq \frac{\bar{y} - \beta}{\frac{\sigma}{\sqrt{n}}} \leq \gamma_1\right] \\ &= \Pr\left[-\gamma_1 \frac{\sigma}{\sqrt{n}} \leq \bar{y} - \beta \leq \gamma_1 \frac{\sigma}{\sqrt{n}}\right] \\ &= \Pr\left[\gamma_1 \frac{\sigma}{\sqrt{n}} \geq -\bar{y} + \beta \geq -\gamma_1 \frac{\sigma}{\sqrt{n}}\right] \\ &= \Pr\left[\bar{y} - \gamma_1 \frac{\sigma}{\sqrt{n}} \leq \beta \leq \bar{y} + \gamma_1 \frac{\sigma}{\sqrt{n}}\right] \end{aligned} \tag{3}$$

Date: August 13, 2004.

Therefore, with σ known,

$$\left[\bar{y} - \gamma_1 \frac{\sigma}{\sqrt{n}} \leq \beta \leq \bar{y} + \gamma_1 \frac{\sigma}{\sqrt{n}}\right] \tag{4}$$

is said to be the $(1 - \alpha)$ 100% confidence interval for β.

3. Confidence Intervals for the Mean with σ Unknown

The previous section gave an interval estimate for the mean of a population when σ was known. When σ is unknown, another method must be used. Recall from the section on probability distributions (equation 20) that the t random variable is defined as

$$t = \frac{z}{\sqrt{\frac{\chi^2(\nu)}{\nu}}} \tag{5}$$

where z is a standard normal and $\chi^2(\nu)$ is a χ^2 random variable with ν degrees of freedom and z and $\chi^2(\nu)$ are independent.

Also note from the same section (equation 9) that if $X_i \sim N(\mu,\, \sigma^2)$ then

$$\begin{aligned} &\sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2 \sim \chi^2(n) \\ \text{and} \quad &\sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma}\right)^2 \sim \chi^2(n-1) \end{aligned} \tag{6}$$

So if $Y \sim N(\beta,\, \sigma^2)$ and $\bar{Y} = \hat{\beta}$, then we have the following

$$\sum_{i=1}^{n} \left(\frac{y_i - \hat{\beta}}{\sigma}\right)^2 \sim \chi^2(n-1) \tag{7}$$

We can use this information to find the distribution of $\frac{(n-1)S^2}{\sigma^2}$ where S^2 is equal to

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \hat{\beta})^2 \tag{8}$$

Now substitute for S^2 and simplify as follows.

$$\begin{aligned} \frac{(n-1)S^2}{\sigma^2} &= \frac{(n-1)}{\sigma^2} \cdot \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \hat{\beta})^2 \\ &= \frac{1}{\sigma^2} \cdot \sum_{i=1}^{n} (y_i - \hat{\beta})^2 \\ &= \sum_{i=1}^{n} \left(\frac{y_i - \hat{\beta}}{\sigma}\right)^2 \sim \chi^2(n-1) \end{aligned} \tag{9}$$

The last line then indicates that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \tag{10}$$

It can be shown that these two variables are independent so that

$$\begin{aligned} \frac{\frac{\frac{\hat{\beta}-\beta}{\sigma}}{\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2(n-1)}}} &= \frac{\frac{\frac{\hat{\beta}-\beta}{\sigma}}{\sqrt{n}}}{\frac{1}{\sigma}\sqrt{S^2}} \\ &= \frac{\hat{\beta}-\beta}{\frac{S}{\sqrt{n}}} \sim t(n-1) \end{aligned} \tag{11}$$

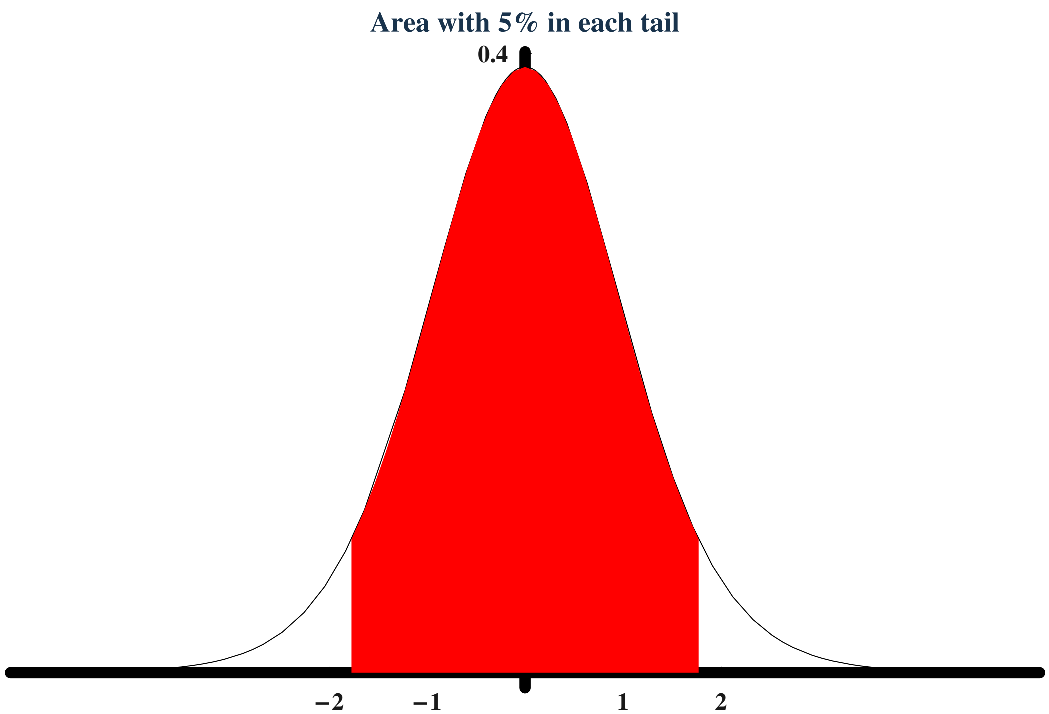
If γ_1 is the upper $\alpha/2$ percent critical value of a t random variable then

$$\int_{\gamma_1}^{\infty} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1+\frac{t^2}{\nu}\right)^{-\frac{(\nu+1)}{2}} dt = \frac{\alpha}{2} \tag{12}$$

and

$$\begin{aligned} 1-\alpha = F(\gamma_1) - F(-\gamma_1) &= \Pr\left[-\gamma_1 \leq \frac{\frac{\frac{\hat{\beta}-\beta}{\sigma}}{\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2(n-1)}}} \leq \gamma_1\right] \\ &= \Pr\left[-\gamma_1 \leq \frac{\sqrt{n}(\hat{\beta}-\beta)}{S} \leq \gamma_1\right] \\ &= \Pr\left[\frac{-\gamma_1 S}{\sqrt{n}} \leq \hat{\beta}-\beta \leq \frac{\gamma_1 S}{\sqrt{n}}\right] \\ &= \Pr\left[\hat{\beta} - \frac{\gamma_1 S}{\sqrt{n}} \leq \beta \leq \hat{\beta} + \frac{\gamma_1 S}{\sqrt{n}}\right] \end{aligned} \tag{13}$$

This is referred to as $(1-\alpha)(100\%)$ confidence interval for β. The following figure gives the area such that 5% of the distribution is the left of the shaded area and 5% of the distribution is the right of the shaded area.



4. Confidence Intervals for the Variance

Remember from equation 10 that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \tag{14}$$

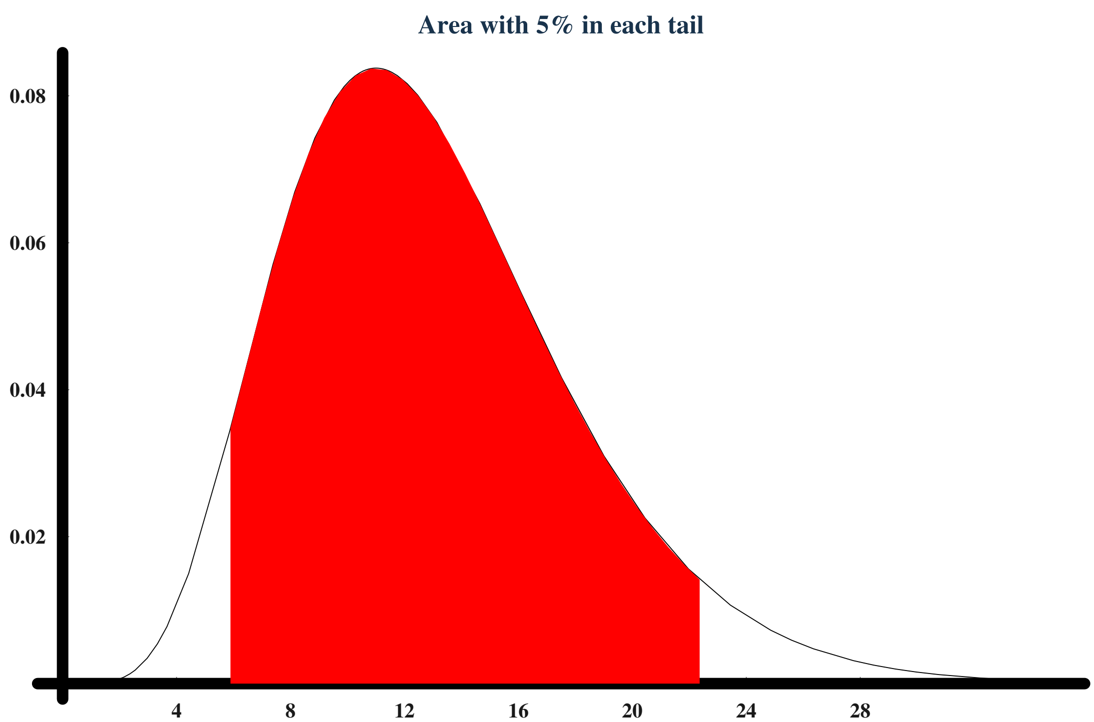
Now if γ_1 and γ_2 are such that

$$\begin{aligned} \Pr\big(\chi^2(\nu) \leq \gamma_1\big) &= \frac{\alpha}{2} \\ \Pr\big(\chi^2(\nu) \geq \gamma_2\big) &= \frac{\alpha}{2} \end{aligned} \tag{15}$$

Then

$$\begin{aligned} 1-\alpha = F(\gamma_2,\,\nu) - F(\gamma_1,\,\nu) &= \Pr\left[\gamma_1 \le \frac{(n-1)S^2}{\sigma^2} \le \gamma_2\right] \\ &= \Pr\left[\frac{\gamma_1\sigma^2}{n-1} \le S^2 \le \frac{\gamma_2\sigma^2}{n-1}\right] \\ &= \Pr\left[\frac{(n-1)}{\sigma^2}\gamma_1 \ge \frac{1}{S^2} \ge \frac{(n-1)}{\sigma^2\gamma_2}\right] \\ &= \Pr\left[\frac{(n-1)S^2}{\gamma_2} \le \sigma^2 \le \frac{(n-1)S^2}{\gamma_1}\right] \end{aligned} \tag{16}$$

is the $(1-\alpha)$ 100% confidence interval for the variance σ^2. The following figure gives the area such that 5% of the distribution is the left of the shaded area and 5% of the distribution is the right of the shaded area.



5. Two Samples and a Confidence Interval for the Variances

Suppose that $y_1^1,\, y_2^1,\, \ldots\,,\, y_n^1$ is a random sample from a distribution $Y_1 \sim N(\beta_1,\, \sigma_1^2)$ and $y_1^2,\, y_2^2,\, \ldots\,,\, y_n^2$ is a random sample from a distribution $Y_2 \sim N(\beta_2,\, \sigma_2^2)$.

Now remember that the ratio of two chi-squared variables divided by their degrees of freedom is distributed as an F variable. For this example

$$\begin{aligned} \frac{(n_1-1)S_1^2}{\sigma_1^2} &\sim \chi^2(n_1-1) \\ \frac{(n_2-1)S_2^2}{\sigma_2^2} &\sim \chi^2(n_2-1) \end{aligned} \tag{17}$$

Now divide each of the chi-squared variables by its degrees of freedom and then take the ratio as follows

$$\frac{S_1^2}{S_2^2} \cdot \frac{\sigma_2^2}{\sigma_1^2} \sim F\left(n_1-1,\, n_2-1\right) \tag{18}$$

F distributions are normally tabled giving the area in the upper tail, i.e.

$$1-\alpha = \Pr\left(F_{n_1-1,\,n_2-1} \leq \gamma\right) \quad \text{or} \quad \alpha = \Pr\left(F_{n_1-1,\,n_2-1} \geq \gamma\right) \tag{19}$$

Now let γ_1 and γ_2 be such that

$$\begin{aligned} \Pr\left(F_{n_1-1,\,n_2-1} \leq \gamma_1\right) &= \frac{\alpha}{2} \\ \Pr\left(F_{n_1-1,\,n_2-1} \geq \gamma_2\right) &= \frac{\alpha}{2} \\ \Pr\left(F_{n_1-1,\,n_2-1} \leq \gamma_2\right) &= 1-\frac{\alpha}{2} \\ \Pr\left(F_{n_1-1,\,n_2-1} \leq \gamma_2\right) - \Pr\left(F_{n_1-1,\,n_2-1} \leq \gamma_1\right) &= 1-\frac{\alpha}{2}-\frac{\alpha}{2} = 1-\alpha \end{aligned} \tag{20}$$

Remembering that

$$F_{1-\alpha/2,\,n_1-1,\,n_2-1} = \frac{1}{F_{\alpha/2,\,n_2-1,\,n_1-1}} \tag{21}$$

then if

$$\Pr\left(F_{n_1-1,\,n_2-1} \leq \gamma_2\right) = 1-\frac{\alpha}{2}$$

then

$$\Pr(F_{n_2-1,\,n_1-1} \leq \frac{1}{\gamma_2}) = \frac{\alpha}{2} \tag{22}$$

We can now construct a confidence interval as follows:

$$\begin{aligned} 1-\alpha &= \Pr\left[\gamma_1 \leq \frac{S_1^2\sigma_2^2}{S_2^2\sigma_1^2} \leq \gamma_2\right] \\ &= \Pr\left[\frac{S_2^2}{S_1^2}\gamma_1 \leq \frac{\sigma_2^2}{\sigma_1^2} \leq \frac{S_2^2}{S_1^2}\gamma_2\right] \\ &= \Pr\left[\frac{S_1^2}{S_2^2}\frac{1}{\gamma_1} \geq \frac{\sigma_1^2}{\sigma_2^2} \geq \frac{S_1^2}{S_2^2}\frac{1}{\gamma_2}\right] \\ &= \Pr\left[\frac{S_1^2}{S_2^2}\frac{1}{\gamma_2} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{S_1^2}{S_2^2}\frac{1}{\gamma_1}\right] \end{aligned} \tag{23}$$

This is then the $(1-\alpha)$ 100% confidence interval for ratio of the variances.

6. Hypothesis Testing

6.1. A statistical hypothesis is an assertion or conjecture about the distribution of one or more random variables.

6.2. To test statistical hypotheses it is necessary to formulate alternative hypotheses.

For example if the hypothesis is that the mean of a population β is greater than or equal to β_0 the hypothesis is

$$\beta \geq \beta_0$$

and an alternative would be

$$\beta < \beta_0$$

Frequently statisticians state their hypotheses as the opposite of what they believe to be true with the hope that the test procedures will lead to their rejection.

6.3. The procedure to test hypotheses is to compute a test statistic $\hat{\theta}$ which will tell whether the hypothesis should be accepted or rejected. The sample space is partitioned into two regions: the acceptance region and the rejection region. If $\hat{\theta}$ is in the rejection region the hypothesis is rejected, and not rejected otherwise.

6.4. Suppose that the null hypothesis is $\theta = \theta_0$ and the alternative is $\theta = \theta_1$. The statistician can make two possible types of errors. If $\theta = \theta_0$ and the test rejects $\theta = \theta_0$ and concludes $\theta = \theta_1$ then the error is of type I.

Rejection of the null hypothesis when it is true is a type I error.

If $\theta = \theta_1$ and the test does not reject $\theta = \theta_0$ but accepts that $\theta = \theta_0$ then the error is of type II.

Acceptance of the null hypothesis when it is false is a type II error.

6.5. The rejection region for a hypothesis H_0 is called the **critical region** of the test and **the probability of obtaining a value of the test statistic inside the critical region when H_0 is true is called the size of the critical region**.

Thus the size of the critical region is just the probability of committing a type I error denoted by α. This probability is also called the level of significance of the test. The probability of committing a type II error is denoted by β.

6.6. Tests where H_0 is of the form $\theta = \theta_0$ and the alternative H_1 is of the two-sided form $\theta \neq \theta_0$ are called two-tailed tests. When H_0 is of the form $\theta = \theta_0$ but H_1 $\theta < \theta_0$ or $\theta > \theta_0$, the test is called a one-tailed test.

6.7. The probabilities of committing the two types of error can be summarized as follows.

		Decision	
		Do not reject H_0	Reject H_0
$\boldsymbol{H_0}$	True	Correct Decision	Type I Error
	False	Type II Error	Correct Decision

6.8. **Probability values (p-values) for statistical tests.** The p-value associated with a statistical test is the probability that we obtain the observed value of the test statistic or a value that is more extreme in the direction of the alternative hypothesis calculated when H_0 is true. Rather than the critical region ahead of time, the p-value of a test can be reported and the reader make a decision based on it.

If T is a test statistic, the p-value, or *attained significance level*, is the smallest level of significance α for which the observed data indicate that the null hypothesis should be rejected.

The smaller the p-value becomes, the more compelling is the evidence that the null hypothesis should be rejected. If the desired level of significance for a statistical test is greater than or equal to the p-value, the null hypothesis is rejected for that value of α. The null hypothesis should be rejected for any value of α down to and including the p-value.

6.9. **Example.** Consider the following scores from a graduate economics class which has eighteen students.

Scores $= \{46,\ 58,\ 87,\ 97.5,\ 82.5,\ 68,\ 83.25,\ 99.5,\ 66.5,\ 75.5,\ 62.5,\ 67,\ 78,\ 32,\ 74.5,\ 47,\ 99.5,\ 26\}$

The mean of the data is 69.4583. The variance is 466.899 and the standard deviation is 21.6078. We are interested in the null hypothesis $\mu = 80$. We can compute the t-statistic as follows.

$$
\begin{aligned}
t &= \frac{\bar{x} - \mu}{\frac{S}{\sqrt{n}}} \\
&= \frac{69.4583 - 80}{\frac{21.6078}{\sqrt{18}}} \\
&= -2.069\,83
\end{aligned}
\tag{24}
$$

If we look at the tabled distribution of the t distribution with 17 degrees of freedom and with 0.025 in each tail we see that $\gamma_1 = 2.110$. This means then that

$$
\begin{aligned}
1 - \alpha = 0.95 = F(\gamma_1) - F(-\gamma_1) &= \Pr\left[-\gamma_1 \leq \frac{\sqrt{n}(\bar{y} - \mu)}{S} \leq \gamma_1\right] \\
&= \Pr\left[-2.110 \leq \frac{4.2426(69.4583 - \mu)}{21.6078} \leq 2.110\right] \\
&= \Pr\left[-10.746\,35 \leq 69.4583 - \mu \leq 10.746\,35\right] \\
&= \Pr\left[10.746\,35 \geq -69.4583 + \mu \geq -10.746\,35\right] \\
&= \Pr\left[80.204\,65 \geq 69.4583 + \mu \geq -58.7119\right] \\
&= \Pr\left[58.7119 \leq \mu \leq 80.204\,65\right]
\end{aligned}
\tag{25}
$$

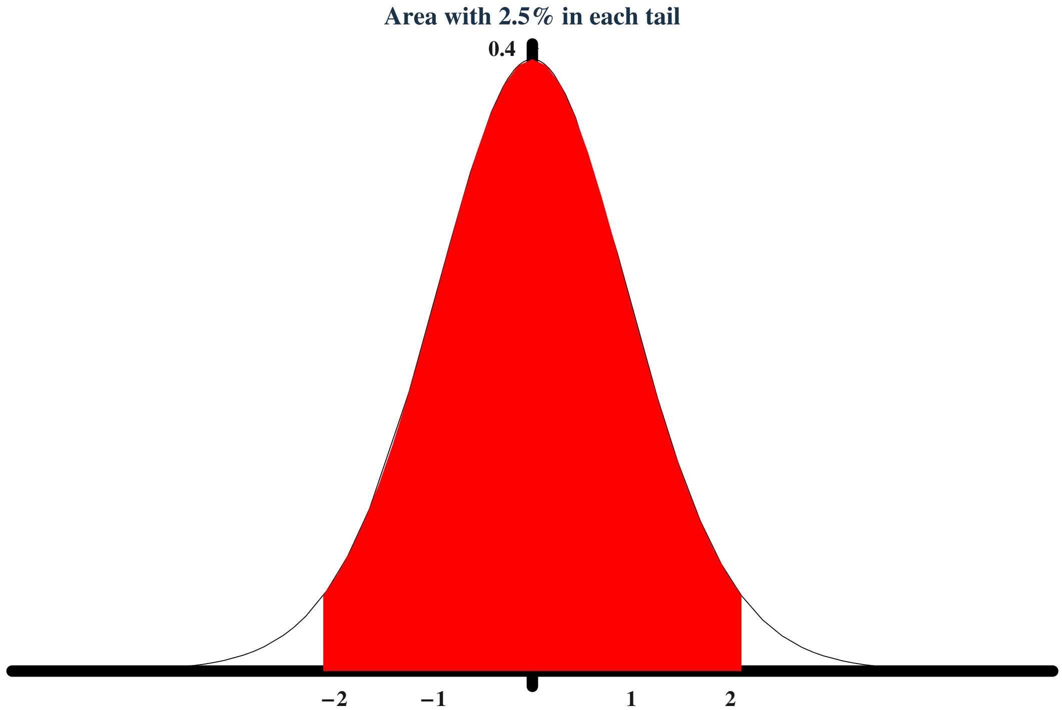
Given that 80 lies within this bound, we cannot reject the hypothesis that $\mu = 80$.

Now consider a two sided test that $\mu = 80$. The value of the t statistic is $-2.069\,83$. We reject the null hypothesis if $|-2.069\,83| > t_{n-1,\,.025}$. From the t-tables $t_{17,\,.025} = 2.110$. Given that 2.069 83 is less than 2.110, we cannot reject the null hypothesis that $\mu = 80$.

Now consider a one sided test that $\mu = 80$ with alternative $\mu < 80$. The value of the t statistic is $-2.069\,83$. We reject the null hypothesis if $-2.069\,83 < -t_{n-1,\,.05}$. From the t-tables $t_{17,\,.05} = 1.74$. Given that $-2.069\,83$ is less than -1.74, we reject the null hypothesis that $\mu = 80$.

To compute the p-value we find the probability that a value of the t-distribution lies to the left of $-2.069\,83$ or the right of 2.069 83. Given that the t-distribution is symmetric we can find just one of these probabilities and multiply by 2. So what we need is the integral of the t-distribution from 2.069 83 to ∞. This is given by

$$
\int_{2.069\,83}^{\infty} \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)} \left(1 + \frac{w^2}{r}\right)^{\frac{-(r+1)}{2}} \, dw \tag{26}
$$



where r is the degrees of freedom which in this case is 17. This integral can be evaluated numerically and will have value 0.027 008 3. If we double this we get 0.054 016 5. So 2.700 83% of the t-distribution with 17 degrees of freedom lies to the right of 2.069 83 and 2.700 83% of the distributionlies to the left of $-2.069\,83$.

7. Summary Tables

Tables 1, 2, 3 and 4 below summarize the information on conducting some of the more common hypothesis tests on simple random samples. We use the notation that

$$z_\alpha = \left[z_\alpha : \int_{z_\alpha}^{\infty} f(z;\, \theta)\, dz = \alpha \right]$$

i.e. z_α is the value such that α of the distribution lies to the right of z_α.

TABLE 1. Level α Tests on μ when σ^2 is Known.
Test Statistic: $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$. ($\Phi(z)$ is the distribution function for a standard normal random variable.)

Testing Problem	Hypotheses	Reject H_0 if	P-Value
Upper One-sided	$H_0 : \ \mu \leq \mu_0$ vs $H_1 : \ \mu > \mu_0$	$z > z_\alpha$ $\Leftrightarrow$ $\bar{x} > \mu_0 + z_\alpha \dfrac{\sigma}{\sqrt{n}}$	$P(Z \geq z \mid H_0)$ $1 - \Phi(z)$
Lower One-sided	$H_0 : \ \mu \geq \mu_0$ vs $H_1 : \ \mu < \mu_0$	$z < -z_a$ $\Leftrightarrow$ $\bar{x} < \mu_0 - z_\alpha \dfrac{\sigma}{\sqrt{n}}$	$P(Z \leq z \mid H_0)$ $=$ $\Phi(z)$
Two-sided	$H_0 : \ \mu = \mu_0$ vs $H_1 : \ \mu \neq \mu_0$	$\lvert z \rvert > z_{\alpha/2}$ $\Leftrightarrow$ $\lvert \bar{x} - \mu_0 \rvert > z_{\alpha/2} \dfrac{\sigma}{\sqrt{n}}$	$P(\lvert Z \rvert \geq \lvert z \rvert \mid H_0)$ $2[1 - \Phi(\lvert z \rvert)]$

TABLE 2. Level α Tests on μ when σ^2 is Unknown.
Test Statistic: $t = \dfrac{\bar{x} - \mu_0}{S/\sqrt{n}}$. ($P(T_{n-1} \geq t)$ is the area in the upper tail of the t-distribution greater than t.)

Testing Problem	Hypotheses	Reject H_0 if	P-Value
Upper One-sided	$H_0 : \ \mu \leq \mu_0$ vs $H_1 : \ \mu > \mu_0$	$t > t_{n-l,\,\alpha}$ $\Leftrightarrow$ $\bar{x} > \mu_0 + t_{n-1,\,\alpha} \dfrac{S}{\sqrt{n}}$	$P(T_{n-1} \geq t)$
Lower One-sided	$H_0 : \ \mu \geq \mu_0$ vs $H_1 : \ \mu < \mu_0$	$t < -t_{n-1,\,\alpha}$ $\Leftrightarrow$ $\bar{x} < \mu_0 - t_{n-1,\,\alpha} \dfrac{S}{\sqrt{n}}$	$P(T_{n-1} \leq t)$
Two-sided	$H_0 : \ \mu = \mu_0$ vs $H_1 : \ \mu \neq \mu_0$	$\lvert t \rvert > t_{n-1,\,\alpha/2}$ $\Leftrightarrow$ $\lvert \bar{x} - \mu_0 \rvert > t_{n-1,\,\alpha/2} \dfrac{S}{\sqrt{n}}$	$P(\lvert T_{n-1} \rvert \geq \lvert t \rvert)$ $2P(T_{n-1} \geq \lvert t \rvert)$

TABLE 3. Level α Tests on σ^2.
Test Statistic: $\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$ ($P_U = P\left(\chi^2_{n-1} \geq \chi^2\right)$ is the area in the upper tail of the χ^2_{n-1} distribution greater than χ^2)

Testing Problem	Hypotheses	Reject H_0 if	P-Value
Upper One-sided	$H_0:\ \sigma^2 \leq \sigma_0^2$ vs $H_1:\ \sigma^2 > \sigma_0^2$	$\chi^2 > \chi^2_{n-1,\,\alpha}$ $\Leftrightarrow$ $s^2 > \dfrac{\sigma_0^2\chi^2_{n-1,\,\alpha}}{n-1}$	$P_U = P\left(\chi^2_{n-1} \geq \chi^2\right)$
Lower One-sided	$H_0:\ \sigma^2 \geq \sigma_0^2$ vs $H_1:\ \sigma^2 < \sigma_0^2$	$\chi^2 < \chi^2_{n-1,\,1-\alpha}$ $\Leftrightarrow$ $s^2 < \dfrac{\sigma_0^2\chi^2_{n-1,\,1-\alpha}}{n-1}$	$P_L = P\left(\chi^2_{n-1} \leq \chi^2\right)$
Two-sided	$H_0:\ \sigma^2 = \sigma_0^2$ vs $H_1:\ \sigma^2 \neq \sigma_0^2$	$\chi^2 > \chi^2_{n-1,\,\alpha/2}$ or $\chi^2 < \chi^2_{n-1,\,1-\alpha/2}$ $\Leftrightarrow$ $s^2 > \dfrac{\sigma_0^2\chi^2_{n-1,\,\alpha/2}}{n-1}$ or $s^2 < \dfrac{\sigma_0^2\chi^2_{n-1,\,1-\alpha/2}}{n-1}$	$2\min\{P_U,\ P_L = 1 - P_U\}$

TABLE 4. Level α Tests for Equality of σ_1^2 and σ_2^2.
Test Statistic: $F = \dfrac{S_1^2}{S_2^2}$. ($f_{n_1-1,\,n_2-1,\,\alpha}$ is the critical value such that α% of the area of the F distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom is greater than this value.)

Testing Problem	Hypotheses	Reject H_0 if
Upper One-sided	$H_0:\ \sigma_1^2 \leq \sigma_2^2$ vs $H_1:\ \sigma_1^2 > \sigma_2^2$	$F > f_{n_1-1,\,n_2-1,\,\alpha}$
Lower One-sided	$H_0:\ \sigma_1^2 \geq \sigma_2^2$ vs $H_1:\ \sigma_1^2 < \sigma_2^2$	$F < f_{n_1-1,\,n_2-1,\,1-\alpha}$
Two-sided	$H_0:\ \sigma_1^2 = \sigma_2^2$ vs $H_1:\ \sigma_1^2 \neq \sigma_2^2$	$F < f_{n_1-1,\,n_2-1,\,1-\alpha/2}$ or $F < f_{n_1-1,\,n_2-1,\,1-\alpha/2}$

8. Example

Consider the following income data for carpenters and house painters.

	carpenters	painters
sample size	$n_1 = 12$	$n_2 = 15$
mean income	$\bar{x} = \$6000$	$\bar{y} = \$5400$
estimated variance	$s_x^2 = \$565\,000$	$s_y^2 = \$362\,500$

8.1. If it is known that $\sigma_x^2 = 600\,000$, test the hypothesis that $\mu_x = \$7000$ with $\alpha = .05$ against the alternative $\mu_x \neq \$7000$. The solution is as follows:

$$\begin{aligned} \bar{x} &\sim N\left(\mu_x,\, \frac{\sigma_x^2}{n_1}\right) \\ z &= \frac{\bar{x} - \mu_x}{\frac{\sigma_x}{\sqrt{n_1}}} \sim N(0,\, 1) \\ &= \frac{6000 - 7000}{\sqrt{\frac{600\,000}{12}}} = \frac{-1000}{223.61} = -4.47 \end{aligned} \tag{27}$$

The critical region is obtained by finding the value of γ such that the area in each tail of the distribution is equal to .025. This is done by consulting a normal table. The value of γ is 1.96. Thus if $|z| \geq 1.96$ we reject the hypothesis. Therefore, we reject $H_0 :\ \mu_x = \$7000$.

8.2. Test the hypothesis that $\mu_x \geq \$7000$ versus the alternative $\mu < \$7000$ assuming that σ^2 is known as before.

We need a critical region such that only 5% of the distribution lies to the left of γ. Specifically we want a critical value for the test such that the probability of a type I error is equal to α. We write this as

$$\begin{aligned} P(\text{type I error}) &= P(\bar{x} < c \mid \mu = 7000) \\ &= P\left(z = \frac{\bar{x} - \mu_x}{\frac{\sigma_x}{\sqrt{n_1}}} < \frac{c - \mu_x}{\frac{\sigma_x}{\sqrt{n_1}}} \,\middle|\, \mu = 7000\right) \\ &= 0.05 \end{aligned} \tag{28}$$

Since the normal is symmetric we use the probability that 5% is the right to γ. This gives a critical value of 1.645.

Computing gives

$$
\begin{aligned}
\frac{c - 7000}{\sqrt{\frac{600\,000}{12}}} &= -1.645 \\
\Rightarrow c - 7000 &= (-1.645)(\sqrt{50\,000}) \\
\Rightarrow c - 7000 &= -367.833 \\
\Rightarrow c &= 6\,632.166
\end{aligned} \tag{29}
$$

So we reject the hypothesis if $\bar{x}$ is less than 6632.166. Alternatively this test can be expressed in terms of the standardized test statistic.

$$
z = \frac{6000 - 7000}{\sqrt{\frac{600\,000}{12}}} = \frac{-1000}{223.61} = -4.47 \tag{30}
$$

We reject the null hypothesis if $z \leq -1.645$. So we reject the null hypothesis.

8.3. Note that with two-tailed tests an equivalent procedure is to construct the $(1 - \alpha)$ level confidence interval and reject the null hypothesis if the hypothesized value does not lie in that interval.

8.4. Test the same hypothesis in (8.1) for the case in which σ_x^2 is not known.
We use a t-test as follows.

$$
\begin{aligned}
\bar{x} &\sim N\left(\mu_x, \frac{\sigma_x^2}{n_1}\right) \\
\frac{\bar{x} - \mu_x}{s_{\bar{x}}} &= \frac{\bar{x} - \mu_x}{\frac{s_x}{\sqrt{n}}} \sim t(n-1) \\
&= \frac{6000 - 7000}{\sqrt{\frac{565\,000}{12}}} = \frac{-1000}{216.99} \\
&= -3.46
\end{aligned} \tag{31}
$$

The critical region is obtained by finding the value of γ such that the area in each tail of the distribution is equal to .025. This is done by consulting a t-table with $(n - 1 = 11)$ degrees of freedom. The value of γ is 2.201. Thus if $|t| \geq 2.201$ we reject the hypothesis. Therefore, we reject $H_0: \ \mu_x = \$7000$.

Using confidence intervals we can construct the 95% confidence interval as follows:

$$\begin{aligned} 1-\alpha &= \Pr\left[6000 - \frac{\sqrt{565\,000}}{\sqrt{12}}(2.201) \leq \mu_x \leq 6000 + \frac{\sqrt{565\,000}}{\sqrt{12}}(2.201)\right] \\ &= \Pr\left[6000 - (216.66)(2.201) \leq \mu_x \leq 6000 + (216.99)(2.201)\right] \\ &= \Pr\left[5522 \leq \mu_x \leq 6477\right] \end{aligned} \tag{32}$$

Since 7000 is not in the interval, we reject the null hypothesis.

8.5. Test the hypothesis that

$$\sigma_y^2 = 400\,000$$

against the alternative

$$\sigma_y^2 \neq 400\,000$$

We will use a two-tailed chi-square test. We must find levels of γ_1 and γ_2 such that the area in each tail is .025. If we consult a χ^2 table with 14 degrees of freedom we obtain 5.63 and 26.12. If the computed value is outside this range we reject the hypothesis. Computing gives

$$\begin{aligned} \frac{(n_2-1)s_y^2}{\sigma_y^2} &\sim \chi^2(15-1) \\ &= \frac{(14)(362\,500)}{400\,000} = 12.687 \end{aligned} \tag{33}$$

Therefore fail to reject $H_0:\ \sigma_y^2 = 400\,000$.

8.6. **Hypotheses Concerning Equality of Two Variances.** Let H_0 be $\sigma_1^2 = \sigma_2^2$. If H_1 is $\sigma_1^2 > \sigma_2^2$, then the appropriate test statistic is the ratio of the sample variances since under the null hypothesis the variances are equal. We reject the hypothesis if

$$\frac{S_1^2}{S_2^2} > F_{\alpha,\, n_1-1,\, n_2-1} \tag{34}$$

If H_1 is $\sigma_1^2 < \sigma_2^2$ then the appropriate test statistic is the ratio of the sample variances since under the null hypothesis the variances are equal. We reject the hypothesis if

$$\frac{S_2^2}{S_1^2} > F_{\alpha,\, n_2-1,\, n_1-1} \tag{35}$$

If H_1 is $\sigma_1^2 \neq \sigma_2^2$ then the appropriate test statistic is the ratio of the sample variances since under the null hypothesis the variances are equal. We now have to decide which variance to put on top of the ratio. If the first is larger compute as in (31) and if the second is larger compute as in (32). Now however use $\alpha/2$ for significance. We reject the hypothesis if

$$\begin{aligned} &\frac{S_1^2}{S_2^2} > F_{\alpha/2,\, n_1-1,\, n_2-1} \\ \text{or} \quad &\frac{S_2^2}{S_1^2} > F_{\alpha/2,\, n_2-1,\, n_1-1} \end{aligned} \tag{36}$$

Consider this last case. Since s_x^2 is larger, put it on top in the ratio. The critical value of F is $F(11,\, 14 :\ .025) = 3.10$. For this case we obtain

$$\frac{s_x^2}{s_y^2} = \frac{565\,000}{362\,500} = 1.5586 \tag{37}$$

Since this is smaller than 3.10 we fail to reject $H_0 :\ \sigma_x^2 = \sigma_y^2$.

Consider also a confidence interval for the ratio of the variances with α = .05. Use equation 23 as follows

$$1 - \alpha = \Pr\left[\frac{S_x^2}{S_y^2}\frac{1}{\gamma_2} \leq \frac{\sigma_x^2}{\sigma_y^2} \leq \frac{S_x^2}{S_y^2}\frac{1}{\gamma_1}\right] \tag{38}$$

The upper critical level is easy to obtain as $F(11,\, 14, :\ .025) = 3.10$ since we want the area to the left of the critical value to be .975. Since the tables don't contain the lower tail, we obtain the critical value $\gamma_1(11,\, 14)$ as $(1/\gamma_1(14,\, 11)$. This critical value is given by $F(14,\, 11 :\ .025) = 3.36$. The reciprocal of this is .297. Notice that the confidence interval is in reciprocal form so we also need the reciprocal of $3.1 = .3225$. The confidence interval is then given by

$$\begin{aligned} 1 - \alpha &= \Pr\left[\frac{565\,000}{362\,500}\frac{1}{3.1} \leq \frac{\sigma_x^2}{\sigma_y^2} \leq \frac{565\,000}{362\,500}\frac{1}{.297}\right] \\ &= \Pr\left[.502\,78 \leq \frac{\sigma_x^2}{\sigma_y^2} \leq 5.247\right] \end{aligned} \tag{39}$$

For problems such as this, it is much easier to use computer programs to obtain the necessary values and probabilities from the F distribution than to use tables.