SAMPLE MOMENTS

1. Population Moments

1.1. **Moments about the origin (raw moments).** The rth moment about the origin of a random variable X, denoted by μ'_r, is the expected value of X^r; symbolically,

$$\mu'_r = E(X^r) \tag{1}$$

$$= \sum_x x^r f(x) \tag{2}$$

for r = 0, 1, 2, . . . when X is discrete and

$$\mu'_r = E(X^r) \atop v \tag{3}$$

when X is continuous. The rth moment about the origin is only defined if E[X^r] exists. A moment about the origin is sometimes called a raw moment. Note that $\mu'_1 = E(X) = \mu_X$, the mean of the distribution of X, or simply the mean of X. The rth moment is sometimes written as function of θ where θ is a vector of parameters that characterize the distribution of X.

If there is a sequence of random variables, $X_1, X_2, \ldots X_n$, we will call the rth population moment of the ith random variable $\mu'_{i,\,r}$ and define it as

$$\mu'_{i\,,\,r} = E\,(X^r_i) \tag{4}$$

1.2. **Central moments.** The rth moment about the mean of a random variable X, denoted by μ_r, is the expected value of $(\,X - \mu_X)^r$ symbolically,

$$\begin{aligned}\mu_r &= E[\,(\,X \;-\; \mu_X\,)^r\,] \\ &= \sum_x \,(\,x - \mu_X)^r\, f(x)\end{aligned} \tag{5}$$

for r = 0, 1, 2, . . . when X is discrete and

$$\begin{aligned}\mu_r &= E[\,(X \;-\; \mu_X)^r\,] \\ &= \textstyle\int_{-\infty}^{\infty}\,(x \;-\; \mu_X)^r f(x)\,dx\end{aligned} \tag{6}$$

when X is continuous. The rth moment about the mean is only defined if E[(X - μ_X)^r] exists. The rth moment about the mean of a random variable X is sometimes called the rth central moment of X. The rth central moment of X about a is defined as E[(X - a)^r]. If a = μ_X, we have the rth central moment of X about μ_X.

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Note that μ_1 = E[(X - μ_X)] = 0 and μ_2 = E[(X - μ_X)^2] = Var[X]. Also note that all odd moments of X around its mean are zero for symmetrical distributions, provided such moments exist.

If there is a sequence of random variables, X_1, X_2, . . . X_n, we will call the rth central population moment of the ith random variable $\mu_{i\,,\,r}$ and define it as

$$\mu_{i,\,r} = E\,\left(X^r_i - \mu'_{i,\,1}\right)^r \tag{7}$$

When the variables are identically distributed, we will drop the i subscript and write μ'_r and $\mu_{\,r}$.

2. Sample Moments

2.1. **Definitions.** Assume there is a sequence of random variables, X_1, X_2, . . . X_n. The first sample moment, usually called the average is defined by

$$\bar{X}_n = \frac{1}{n} \sum_{i\,=\,1}^{n} \; X_i \tag{8}$$

Corresponding to this statistic is its numerical value, $\bar{x}_n$, which is defined by

$$\bar{x}_n = \frac{1}{n} \sum_{i\,=\,1}^{n} \; x_i \tag{9}$$

where x_i represents the observed value of X_i. The rth sample moment for any t is defined by

$$\bar{X}^r_n = \frac{1}{n} \sum_{i\,=\,1}^{n} \; X^r_i \tag{10}$$

This too has a numerical counterpart given by

$$\bar{x}^r_n = \frac{1}{n} \sum_{i\,=\,1}^{n} \; x^r_i \tag{11}$$

2.2. **Properties of Sample Moments.**

2.2.1. *Expected value of $\bar{X}_n{}^r$.* Taking the expected value of equation 10 we obtain

$$E\left[\bar{X}^r_n\right] = E\bar{X}^r_n = \frac{1}{n} \sum_{i\,=\,1}^{n} \; E\;X^r_i = \frac{1}{n} \sum_{i\,=\,1}^{n} \; \mu'_{i\,,\,r} \tag{12}$$

If the X's are identically distributed, then

$$E\left[\bar{X}^r_n\right] = E\bar{X}^r_n = \frac{1}{n} \sum_{i\,=\,1}^{n} \; \mu'_{\,r} = \mu'_r \tag{13}$$

2.2.2. *Variance of $\bar{X}^r_n$.*

$$E\left[\bar{X}^r_n\right] = E\bar{\;}X^r_n = \frac{1}{n} \sum_{i\,=\,1}^{n} \; \mu'_{\,r} = \mu'_r \tag{14}$$

2.2.3. *Variance of $\bar{X}_n^r$.* First consider the case where we have a sample $X_1, X_2, \ldots X_n$.

$$Var\left(\bar{X}_n^r\right) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i^r\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i^r\right) \tag{15}$$

If the X's are independent, then

$$Var\left(\bar{X}_n^r\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var\left(X_i^r\right) \tag{16}$$

If the X's are independent and identically distributed, then

$$Var\left(\bar{X}_n^r\right) = \frac{1}{n} Var\left(X^r\right) \tag{17}$$

where X denotes any one of the random variables (because they are all identical). In the case where r =1, we obtain

$$Var\left(\bar{X}_n\right) = \frac{1}{n} Var\left(X\right) = \frac{\sigma^2}{n} \tag{18}$$

3. Sample Central Moments

3.1. **Definitions.** Assume there is a sequence of random variables, $X_1, X_2, \ldots X_n$. We define the sample central moments as

$$\begin{aligned} C_n^r &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\left(X_i - \mu'_{i,1}\right)^r, \; r = 1, 2, 3, \cdots, \\ \Rightarrow C_n^1 &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\left(X_i - \mu'_{i,1}\right) \\ \Rightarrow C_n^2 &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\left(X_i - \mu'_{i,1}\right)^2 \end{aligned} \tag{19}$$

These are only defined if $\mu'_{i,1}$ is known.

3.2. **Properties of Sample Moments.**

3.2.1. *Expected value of C_n^r.* The expected value of C_n^r is given by

$$E\left(C_n^r\right) = \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\left(X_i - \mu'_{i,1}\right)^r = \tfrac{1}{n}\sum_{i=1}^{n}\mu_{i,r} \tag{20}$$

The last equality follows from equation 7.

If the X_i are identically distributed, then

$$\begin{aligned} E\left(C_n^r\right) &= \mu_r \\ E\left(C_n^1\right) &= 0 \end{aligned} \tag{21}$$

3.2.2. *Variance of C_n^r.* First consider the case where we have a sample $X_1, X_2, \ldots X_n$.

$$Var\,(\,C_n^r\,) = Var\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \mu'_{i,1}\right)^r\right) = \frac{1}{n^2}\;Var\left(\sum_{i=1}^{n}\left(X_i - \mu'_{i,1}\right)^r\right) \tag{22}$$

If the X's are independently distributed, then

$$Var\,(C_n^r\,) = \frac{1}{n^2}\;\sum_{i=1}^{n}\;Var\left[\left(\,X_i - \mu'_{i,1}\,\right)^r\right] \tag{23}$$

If the X's are independent and identically distributed, then

$$Var\,(C_n^r\,) = \frac{1}{n}\;Var\left[\,(X - \mu'_1\,)^r\,\right] \tag{24}$$

where X denotes any one of the random variables (because they are all identical). In the case where r =1, we obtain

$$\begin{aligned} Var\,(\,C_n^r\,) &= \frac{1}{n}\;Var\,[\,X - \mu'_1\,] \\ &= \frac{1}{n}\;Var\,[\,X - \mu\,] \\ &= \frac{1}{n}\sigma^2 - 2\,Cov\,[\,X\,,\;\mu\,] + Var\,[\,\mu\,] \\ &= \frac{1}{n}\sigma^2 \end{aligned} \tag{25}$$

4. Sample About the Average

4.1. **Definitions.** Assume there is a sequence of random variables, $X_1, X_2, \ldots X_n$. Define the rth sample moment about the average as

$$M_n^r = \frac{1}{n}\;\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^r\;, r\;=\;1\,,\;2\,,\;3\,,\;\cdots\,, \tag{26}$$

This is clearly a statistic of which we can compute a numerical value. We denote the numerical value by, m_n^r, and define it as

$$m_n^r = \frac{1}{n}\sum_{i=1}^{n}\,(\,x_i - \bar{x}_n)^r \tag{27}$$

In the special case where r = 1 we have

$$\begin{aligned} M_n^1 &= \frac{1}{n}\;\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right) \\ &= \frac{1}{n}\;\sum_{i=1}^{n}\;X_i - \bar{X}_n \\ &= \bar{X}_n - \bar{X}_n = 0 \end{aligned} \tag{28}$$

4.2. **Properties of Sample Moments about the Average when r = 2.**

4.2.1. *Alternative ways to write M_n^r.* We can write M_n^r in an alternative useful way by expanding the squared term and then simplifying as follows

$$\begin{aligned} M_n^r &= \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^r \\ \Rightarrow M_n^2 &= \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^2 \\ &= \frac{1}{n} \left(\sum_{i=1}^{n} \left[X_i^2 - 2\, X_i \bar{X}_n + \bar{X}_n^{\,2}\right]\right) \\ &= \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \frac{2\bar{X}_n}{n} \sum_{i=1}^{n} X_i + \frac{1}{n} \sum_{i=1}^{n} \bar{X}_n^2 \\ &= \frac{1}{n} \sum_{i=1}^{n} X_i^2 - 2\bar{X}_n^2 + \bar{X}_n^2 \\ &= \frac{1}{n} \left(\sum_{i=1}^{n} X_i^2 \right) - \bar{X}_n^2 \end{aligned} \tag{29}$$

4.2.2. *Expected value of M_n^r.* The expected value of M_n^r is then given by

$$\begin{aligned} E\left(M_n^2\right) &= \frac{1}{n} E\left[\sum_{i=1}^{n} X_i^2\right] - E\left[\bar{X}_n^2\right] \\ &= \frac{1}{n} \sum_{i=1}^{n} E\left[X_i^2\right] - \left(E\left[\bar{X}_n\right]\right)^2 - Var(\bar{X}_n) \\ &= \frac{1}{n} \sum_{i=1}^{n} \mu'_{i,2} - \left(\frac{1}{n}\sum_{i=1}^{n} \mu'_{i,1}\right)^2 - Var(\bar{X}_n) \end{aligned} \tag{30}$$

The second line follows from the alternative definition of variance

$$\begin{aligned} Var(X) &= E\left(X^2\right) - \left[E(X)\right]^2 \\ \Rightarrow E\left(X^2\right) &= \left[E(X)\right]^2 + Var(X) \\ \Rightarrow E\left(\bar{X}_n^{\,2}\right) &= \left[E\left(\bar{X}_n\right)\right]^2 + Var(\bar{X}_n) \end{aligned} \tag{31}$$

and the third line follows from equation 12. If the X_i are independent and identically distributed, then

$$\begin{aligned} E\left(M_n^2\right) &= \frac{1}{n} E\left[\sum_{i=1}^{n} X_i^2\right] - E\left[\bar{X}_n^{\,2}\right] \\ &= \frac{1}{n} \sum_{i=1}^{n} \mu'_{i,2} - \left(\frac{1}{n} \sum_{i=1}^{n} \mu'_{i,1}\right)^2 - Var(\bar{X}_n) \\ &= \mu'_2 - \left(\mu'_1\right)^2 - \frac{\sigma^2}{n} \\ &= \sigma^2 - \frac{1}{n}\sigma^2 \\ &= \frac{n-1}{n}\sigma^2 \end{aligned} \tag{32}$$

where μ'_1 and μ'_2 are the first and second population moments, and μ_2 is the second central population moment for the identically distributed variables. Note that this obviously implies

$$\begin{aligned} E\left[\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2\right] &= n\, E\left(M_n^2\right) \\ &= n\left(\frac{n-1}{n}\right)\sigma^2 \\ &= (n-1)\,\sigma^2 \end{aligned} \tag{33}$$

4.2.3. *Variance of M_n^2.* By definition,

$$Var\left(M_n^2\right) = E\left[\left(M_n^2\right)^2\right] - \left(E\, M_n^2\right)^2 \tag{34}$$

The second term on the right on equation 34 is easily obtained by squaring the result in equation 32.

$$\begin{aligned} E\left(M_n^2\right) &= \frac{n-1}{n}\sigma^2 \\ \Rightarrow \left(E\left(M_n^2\right)\right)^2 &= \left(E\, M_n^2\right)^2 = \frac{(n-1)^2}{n^2}\sigma^4 \end{aligned} \tag{35}$$

Now consider the first term on the right hand side of equation 34. Write it as

$$E\left[\left(M_n^2\right)^2\right] = E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2\right)^2\right] \tag{36}$$

Now consider writing $\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2$ as follows

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 &= \frac{1}{n}\sum_{i=1}^{n}\left((X_i - \mu) - (\bar{X} - \mu)\right)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2 \\ where\ Y_i &= X_i - \mu \\ \bar{Y} &= \bar{X} - \mu \end{aligned} \tag{37}$$

Obviously,

$$\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 = \sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2,\ where\ Y_i = X_i - \mu\,, \bar{Y} = \bar{X} - \mu \tag{38}$$

Now consider the properties of the random variable Y_i which is a transformation of X_i. First the expected value.

$$\begin{aligned} Y_i &= X_i - \mu \\ E\,(\,Y_i\,) &= E\,(X_i\,) - E\,(\,\mu) \\ &= \mu - \mu \\ &= 0 \end{aligned} \tag{39}$$

The variance of Y_i is

$$\begin{aligned} Y_i &= X_i - \mu \\ Var\,(Y_i) &= Var\,(X_i) \\ &= \sigma^2\ if\ X_i\ are\ independently\ and\ identically\ distributed \end{aligned} \tag{40}$$

Also consider $E(Y_i^4)$. We can write this as

$$\begin{aligned} E(\,Y^4) &= \int_{-\infty}^{\infty} y^4\,f\,(\,x\,)\;d\,x \\ &= \int_{-\infty}^{\infty} (\,x - \mu)^4\;f(x)\,d\,x \\ &= \mu_4 \end{aligned} \tag{41}$$

Now write equation 36 as follows

$$\begin{aligned} E\left[\left(M_n^2\right)^2\right] &= E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\,\right)^2\right)^2\right] \\ &= E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right)^2\right] \\ &= E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2\right)^2\right] \\ &= \frac{1}{n^2}\,E\left[\left(\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2\right)^2\right] \end{aligned} \tag{42}$$

Ignoring $\frac{1}{n^2}$ for now, expand equation 42 as follows

$$
\begin{aligned}
E\left[\left(\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right)^2\right)^2\right] &= E\left[\left(\sum_{i=1}^{n}\left(Y_i^2-2\,Y_i\bar{Y}+\bar{Y}^2\right)\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^{n} Y_i^2-2\,\bar{Y}\sum_{i=1}^{n} Y_i+\sum_{i=1}^{n}\bar{Y}^2\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^{n} Y_i^2-2\,n\,\bar{Y}^2+n\,\bar{Y}^2\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^{n} Y_i^2-2n\bar{Y}^2\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^{n} Y_i^2\right)^2-2\,n\,\bar{Y}^2\sum_{i=1}^{n} Y_i^2+n^2\bar{Y}^4\right] \\
&= E\left[\left(\sum_{i=1}^{n} Y_i^2\right)^2\right]-2\,n\,E\left[\bar{Y}^2\sum_{i=1}^{n} Y_i^2\right]+n^2\,E\left(\bar{Y}^4\right)
\end{aligned}
\tag{43}
$$

Now consider the first term on the right of 43 which we can write as

$$
\begin{aligned}
E\left[\left(\sum_{i=1}^{n} Y_i^2\right)^2\right] &= E\left[\sum_{i=1}^{n} Y_i^2\sum_{j=1}^{n} Y_j^2\right] \\
&= E\left[\sum_{i=1}^{n} Y_i^4+\sum\sum\nolimits_{i\neq j} Y_i^2\,Y_j^2\right] \\
&= \sum_{i=1}^{n} E\,Y_i^4+\sum\sum\nolimits_{i\neq j} E\,Y_i^2\,E\,Y_j^2 \\
&= n\,\mu_4+n\,(n-1)\,\mu_2^2 \\
&= n\,\mu_4+n\,(n-1)\,\sigma^4
\end{aligned}
\tag{44}
$$

Now consider the second term on the right of 43 (ignoring 2n for now) which we can write as

$$
\begin{aligned}
E\left[\bar{Y}^2 \sum_{i=1}^{n} Y_i^2\right] &= \frac{1}{n^2} E\left[\sum_{j=1}^{n} Y_j \sum_{k=1}^{n} Y_k \sum_{i=1}^{n} Y_i^2\right] \\
&= \frac{1}{n^2} E\left[\sum_{i=1}^{n} Y_i^4 + \sum\sum_{i \neq j} Y_i^2\, Y_j^2 + \sum\sum_{j \neq k} Y_j Y_k \sum_{\substack{i \neq j \\ i \neq k}} Y_i^2\right] \\
&= \frac{1}{n^2}\left[\sum_{i=1}^{n} E\, Y_i^4 + \sum\sum_{i \neq j} E\, Y_i^2\, E\, Y_j^2 + \sum\sum_{j \neq k} E\, Y_j E\, Y_k \sum_{\substack{i \neq j \\ i \neq k}} E\, Y_i^2\right] \\
&= \frac{1}{n^2}\left[n\,\mu_4 + n\,(n-1)\,\mu_2^2 + 0\right] \\
&= \frac{1}{n}\left[\mu_4 + (n-1)\sigma^4\right]
\end{aligned}
\tag{45}
$$

The last term on the penultimate line is zero because $E(Y_j) = E(Y_k) = E(Y_i) = 0$.

Now consider the third term on the right side of 43 (ignoring n^2 for now) which we can write as

$$
\begin{aligned}
E\left[\bar{Y}^4\right] &= \frac{1}{n^4} E\left[\sum_{i=1}^{n} Y_i \sum_{j=1}^{n} Y_j \sum_{k=1}^{n} Y_k \sum_{\ell=1}^{n} Y_\ell\right] \\
&= \frac{1}{n^2} E\left[\sum_{i=1}^{n} Y_i^4 + \sum\sum_{i \neq k} Y_i^2\, Y_k^2 + \sum\sum_{i \neq j} Y_i^2\, Y_j^2 + \sum_{i \neq j} Y_i^2\, Y_j^2 + \cdots\right]
\end{aligned}
\tag{46}
$$

where for the first double sum ($i = j \neq k = \ell$), for the second ($i = k \neq j = \ell$), and for the last ($i = \ell \neq j = k$) and ... indicates that all other terms include Y_i in a non-squared form, the expected value of which will be zero. Given that the Y_i are independently and identically distributed, the expected value of each of the double sums is the same, which gives

$$
\begin{aligned}
E\left[\bar{Y}^4\right] &= \tfrac{1}{n^4} E\left[\sum_{i=1}^{n} Y_i^4 + \sum\sum_{i \neq k} Y_i^2\, Y_k^2 + \sum\sum_{i \neq j} Y_i^2\, Y_j^2 + \sum_{i \neq j} Y_i^2\, Y_j^2 + \cdots\right] \\
&= \tfrac{1}{n^4}\left[\sum_{i=1}^{n} E\, Y_i^4 + 3\sum\sum_{i \neq j} Y_i^2\, Y_j^2 + terms\ containing\ E\ X_i\right] \\
&= \tfrac{1}{n^4}\left[\sum_{i=1}^{n} E\, Y_i^4 + 3\sum\sum_{i \neq j} Y_i^2\, Y_j^2\right] \\
&= \tfrac{1}{n^4}\left[n\,\mu_4 + 3\,n\,(n-1)\,(\mu_2)^2\right] \\
&= \tfrac{1}{n^4}\left[n\,\mu_4 + 3\,n\,(n-1)\,\sigma^4\right] \\
&= \tfrac{1}{n^3}\left[\mu_4 + 3\,(n-1)\,\sigma^4\right]
\end{aligned}
\tag{47}
$$

Now combining the information in equations 45, 46, and 47 we obtain

$$
\begin{aligned}
E\left[\left(\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right)^2\right)^2\right] =& E\left[\left(\sum_{i=1}^{n}\left(Y_i^2-2\,Y_i\bar{Y}+\bar{Y}^2\right)\right)^2\right] \qquad (48)\\
=& E\left[\left(\sum_{i=1}^{n} {Y_i}^2\right)^2\right]-2\,n\,E\left[\bar{Y}^2\sum_{i=1}^{n}Y_i^2\right]+n^2\,E\left(\bar{Y}^4\right)\\
=& n\mu_4+n(n-1)\mu_2^2-2n\left[\frac{1}{n}\left[\mu_4+(n-1)\mu_2^2\right]\right]+n^2\left[\frac{1}{n^3}\left[\mu_4+3(n-1)\mu_2^2\right]\right]\\
=& n\,\mu_4+n(n-1)\mu_2^2-2\left[\mu_4+(n-1)\mu_2^2\right]+\left[\frac{1}{n}\left[\mu_4+3(n-1)\mu_2^2\right]\right]\\
=& \frac{n^2}{n}\mu_4-\frac{2\,n}{n}\mu_4+\frac{1}{n}\mu_4+\frac{n^2(n-1)}{n}\mu_2^2-\frac{2\,n\,(n-1)}{n}\mu_2^2+\frac{3(n-1)}{n}\mu_2^2\\
=& \frac{n^2-2\,n+1}{n}\mu_4+\frac{(n-1)\,(n^2-2\,n+3)}{n}\mu_2^2\\
=& \frac{n^2-2\,n+1}{n}\mu_4+\frac{(n-1)\,(n^2-2\,n+3)}{n}\sigma^4
\end{aligned}
$$

Now rewrite equation 42 including $\frac{1}{n^2}$ as follows

$$
\begin{aligned}
E\left[\left(M_n^2\right)^2\right] =& \frac{1}{n^2}E\left[\left(\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right)^2\right)^2\right]\\
=& \frac{1}{n^2}\left(\frac{n^2-2\,n+1}{n}\mu_4+\frac{(n-1)\,(n^2-2\,n+3)}{n}\sigma_4\right)\\
=& \frac{n^2-2\,n+1}{n^3}\,\mu_4+\frac{(n-1)\,(n^2-2\,n+3)}{n^3}\sigma_4 \qquad (49)\\
=& \frac{(n-1)^2}{n^3}\,\mu_4+\frac{(n-1)\,(n^2-2\,n+3)}{n^3}\sigma_4
\end{aligned}
$$

Now substitute equations 35 and 49 into equation 34 to obtain

$$
\begin{aligned}
Var\left(M_n^2\right) =& E\left[\left(M_n^2\right)^2\right]-\left(E\,M_n^2\right)^2 \qquad (50)\\
=& \frac{(n-1)^2}{n^3}\,\mu_4+\frac{(n-1)\,(n^2-2\,n+3)}{n^3}\sigma^4-\frac{(n-1)^2}{n^2}\,\sigma^4
\end{aligned}
$$

We can simplify this as

$$\begin{aligned} Var\left(M_n^2\right) =& E\left[\left(M_n^2\right)^2\right] - \left(E\ M_n^2\right)^2 \\ =& \frac{(n-1)^2}{n^3}\ \mu_4 + \frac{(n-1)(n^2-2n+3)}{n^3}\ \sigma^4 - \frac{n(n-1)^2}{n^3}\ \sigma^4 \\ =& \frac{\mu_4\ (n-1)^2 + \left[(n-1)\sigma^4\right]\ \left(n^2 - 2n + 3 - n(n-1)\right)}{n^3} \\ =& \frac{\mu_4\ (n-1)^2 + \left[(n-1)\sigma^4\right]\ \left(n^2 - 2n + 3 - n^2 + n\right)}{n^3} \\ =& \frac{\mu_4\ (n-1)^2 + \left[(n-1)\sigma^4\right]\ (3-n)}{n^3} \\ =& \frac{\mu_4\ (n-1)^2 - \left[(n-1)\sigma^4\right]\ (n-3)}{n^3} \\ =& \frac{(n-1)^2\ \mu_4}{n^3} - \frac{(n-1)(n-3)\ \sigma^4}{n^3} \end{aligned} \tag{51}$$

5. Sample Variance

5.1. **Definition of sample variance.** The sample variance is defined as

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^n \left(X_i - \bar{X}_n\right)^2 \tag{52}$$

We can write this in terms of moments about the mean as

$$\begin{aligned} S_n^2 =& \frac{1}{n-1}\sum_{i=1}^n \left(X_i - \bar{X}_n\right)^2 \\ =& \frac{n}{n-1} M_n^2\ where\ M_n^2 = \frac{1}{n}\ \sum_{i=1}^n \left(X_i - \bar{X}_n\right)^2 \end{aligned} \tag{53}$$

5.2. **Expected value of S^2.** We can compute the expected value of S^2 by substituting in from equation 32 as follows

$$\begin{aligned} E\left(S_n^2\right) =& \frac{n}{n-1}\, E\left(M_n^2\right) \\ =& \frac{n}{n-1}\,\frac{n-1}{n}\ \sigma^2 \\ =& \sigma^2 \end{aligned} \tag{54}$$

5.3. **Variance of S^2.** We can compute the variance of S^2 by substituting in from equation 51 as follows

$$\begin{aligned} Var\left(\S_n^2\right) =& \frac{n^2}{(n-1)^2}\, Var\left(M_n^2\right) \\ =& \frac{n^2}{(n-1)^2}\left(\frac{(n-1)^2\ \mu_4}{n^3} - \frac{(n-1)\ (n-3)\ \sigma^4}{n^3}\right) \\ =& \frac{\mu_4}{n} - \frac{(n-3)\ \sigma^4}{n\ (n-1)} \end{aligned} \tag{55}$$

5.4. **Definition of $\hat{\sigma}^2$.** One possible estimate of the population variance is $\hat{\sigma}^2$ which is given by

$$\begin{aligned} \hat{\sigma}^2 =& \frac{1}{n}\sum_{i=1}^n \left(X_i - \bar{X}_n\right)^2 \\ =& M_n^2 \end{aligned} \tag{56}$$

5.5. **Expected value of $\hat{\sigma}^2$.** We can compute the expected value of $\hat{\sigma}^2$ by substituting in from equation 32 as follows

$$\begin{aligned} E\left(\hat{\sigma}^2\right) &= E\left(M_n^2\right) \\ &= \frac{n-1}{n}\,\sigma^2 \end{aligned} \tag{57}$$

5.6. **Variance of $\hat{\sigma}^2$.** We can compute the variance of $\hat{\sigma}^2$ by substituting in from equation 51 as follows

$$\begin{aligned} Var\left(\hat{\sigma}^2\right) &= Var\left(M_n^2\right) \\ &= \frac{(n-1)^2\,\mu_4}{n^3} - \frac{(n-1)\,(n-3)\,\sigma^4}{n^3} \\ &= \frac{\mu_4 - \mu_2^2}{n} - \frac{2\,(\mu_4 - 2\,\mu_2^2)}{n^2} + \frac{\mu_4 - 3\,\mu_2^2}{n^3} \end{aligned} \tag{58}$$

We can also write this in an alternative fashion

$$\begin{aligned} Var\left(\hat{\sigma}^2\right) &= Var\left(M_n^2\right) \\ &= \frac{(n-1)^2\,\mu_4}{n^3} - \frac{(n-1)\,(n-3)\,\sigma^4}{n^3} \\ &= \frac{(n-1)^2\,\mu_4}{n^3} - \frac{(n-1)\,(n-3)\,\mu_2^2}{n^3} \\ &= \frac{n^2\,\mu_4 - 2\,n\,\mu_4 + \mu_4}{n^3} - \frac{n^2\,\mu_2^2 - 4\,n\,\mu_2^2 + 3\,\mu_2^2}{n^3} \\ &= \frac{n^2\,(\mu_4 - \mu_2^2) - 2\,n\,(\mu_4 - 2\,\mu_2^2) + \mu_4 - 3\,\mu_2^2}{n^3} \\ &= \frac{\mu_4 - \mu_2^2}{n} - \frac{2\,(\mu_4 - 2\,\mu_2^2)}{n^2} + \frac{\mu_4 - 3\,\mu_2^2}{n^3} \end{aligned} \tag{59}$$

6. Normal populations

6.1. **Central moments of the normal distribution.** For a normal population we can obtain the central moments by differentiating the moment generating function. The moment generating function for the central moments is as follows

$$M_X(t) = e^{\frac{t^2\,\sigma^2}{2}}\,. \tag{60}$$

The moments are then as follows. The first central moment is

$$\begin{aligned} E\,(X - \mu) &= \frac{d}{dt}\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\Big|_{t=0} \\ &= t\,\sigma^2\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\Big|_{t=0} \\ &= 0 \end{aligned} \tag{61}$$

The second central moment is

$$\begin{aligned} E\,(X - \mu)^2 &= \frac{d^2}{dt^2}\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(t\,\sigma^2\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\right)\Big|_{t=0} \\ &= \left(t^2\,\sigma^4\left(e^{\frac{t^2\,\sigma^2}{2}}\right) + \sigma^2\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\right)\Big|_{t=0} \\ &= \sigma^2 \end{aligned} \tag{62}$$

The third central moment is

$$
\begin{aligned}
E\,(X\;-\;\mu\,)^3 =& \tfrac{d^3}{dt^3}\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\;|_{t\,=\,0} \\
=& \tfrac{d}{dt}\left(t^2\,\sigma^4\;\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+\sigma^2\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\right)\;|_{t\,=\,0} \\
=& \left(t^3\,\sigma^6\;\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+2\,t\,\sigma^4\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+t\,\sigma^4\;\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\;\right)\;|_{t\,=\,0} \\
=& \left(t^3\,\sigma^6\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+3\,t\,\sigma^4\;\;\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\right)\;|_{t\,=\,0} \\
=& 0
\end{aligned}
\tag{63}
$$

The fourth central moment is

$$
\begin{aligned}
E\,(X\;-\;\mu\,)^4 =& \tfrac{d^4}{dt^4}\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\;|_{t\,=\,0} \\
=& \tfrac{d}{dt}\left(t^3\,\sigma^6\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+3\,t\,\sigma^4\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\right)\;|_{t\,=\,0} \\
=& \left(t^4\,\sigma^8\;\;\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+3\,t^2\,\sigma^6\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+3\,t^2\,\sigma^6\;\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\;+3\,\sigma^4\;\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\right)\;|_{t\,=\,0} \\
=& \left(t^4\,\sigma^8\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+6\,t^2\,\sigma^6\;\;\left(e^{\frac{t^2\,\sigma^2}{2}}\right)+3\,\sigma^4\left(e^{\frac{t^2\,\sigma^2}{2}}\right)\right)\;|_{t\,=\,0} \\
=& 3\,\sigma^4
\end{aligned}
\tag{64}
$$

6.2. **Variance of $\mathbf{S}^2$.** Let $X_1, X_2, \ldots X_n$ be a random sample from a normal population with mean μ and variance $\sigma^2 < \infty$.

We know from equation 55 that

$$
\begin{aligned}
Var\;\;(\;S_n^2\;) =& \tfrac{n^2}{(\;n\;-\;1)^2}\;Var\left(M_n^2\right) \\
=& \tfrac{n^2}{(\;n\;-\;1)^2}\left(\tfrac{(\;n\;-\;1)^2\;\mu_4}{n^3}-\tfrac{(\;n\;-\;1)\;(n-3)\;\sigma^4}{n^3}\right) \\
=& \tfrac{\mu_4}{n}-\tfrac{(n-3\;)\;\sigma^4}{n\;(\;n\;-\;1)}
\end{aligned}
\tag{65}
$$

If we substitute in for μ_4 from equation 64 we obtain

$$
\begin{aligned}
Var\left(S_n^2\;\right) =& \tfrac{\mu_4}{n}-\tfrac{(n-3\;)\;\sigma^4}{n\;(n\;-\;1)} \\
=& \tfrac{3\,\sigma^4}{n}-\tfrac{(n-3\;)\;\sigma^4}{n\;(\;n\;-\;1)} \\
=& \tfrac{(\,3\,(\;n\;-\;1\,)-(n-3\;))\;\sigma^4}{n\;(\;n\;-\;1)} \\
=& \tfrac{(\,3\,n-3-n+3\,))\;\sigma^4}{n\;(\;n\;-\;1)} \\
=& \tfrac{2\,n\,\sigma^4}{n\;(\;n\;-\;1)} \\
=& \tfrac{2\;\sigma^4}{(\;n\;-\;1)}
\end{aligned}
\tag{66}
$$

6.3. **Variance of $\hat{\sigma}^2$.** It is easy to show that

$$
Var\,(\hat{\sigma}^2\,) = \frac{2\,\sigma^4\;(\;n\;\;-\;\;1\,)}{n^2}
$$