(1) Suppose that $X_1, X_2, \cdots, X_n$ denote an iid sample from an exponential distribution with density function:

$$f(X_i) = \beta^{-1} \exp(-x_i \beta^{-1}), \quad x_i \geq 0 \quad \forall i.$$ 

(1a) Using the moment generating function approach, derive the exact sampling distribution for the sum

$$Z_n \equiv X_1 + X_2 + \cdots + X_n.$$ 

(Hint: you may need to go back to your lecture notes to find the distribution corresponding to the moment generating function that you obtain).

(1b) Using your solution to (1a) and a change of variables, derive the exact distribution associated with the standardized average:

$$Y_n = \frac{\sqrt{n}(\bar{X}_n - \beta)}{\beta}.$$ 

(Hint: Be sure to correctly note the support of the random variable $Y_n$.)

(1c) The central limit theorem tells us that as $n \to \infty$, the distribution you found in (1b) should converge to the standard normal distribution.

Using MATLAB, document this by plotting the density in (1b) against the standard normal density. In all cases, set $\beta = 1$. Do this for $n = 1$, $n = 5$, $n = 10$ and $n = 100$. How accurate is the normal approximation for small $n$?

Please include your plots and MATLAB code for this question.
(2) One early method of (approximately) generating a standard normal random variable was to generate

\[ X = -6 + \sum_{i=1}^{12} U_i, \]

where

\[ U_i \overset{iid}{\sim} U(0,1), \]

that is, the \( U_i \) are iid draws from a uniform distribution over the unit interval.

(2a) Justify the argument that \( X \) is approximately \( N(0,1) \).

(2b) Is there something obvious that tells you right away that \( X \) can not be exactly normally distributed?
(3) An economist has a set of iid draws from the exponential distribution:

\[ f(x) = \frac{1}{\theta} \exp(-x/\theta), \quad x > 0 \]

denoted \( x_1, x_2, \ldots, x_n \) and seeks to estimate the variance of \( x \), which is \( \theta^2 \).

The economist knows that \( \bar{x}_n \) is a consistent estimator of \( \theta \) and decides to use

\[
\frac{\bar{x}_n^2}{n^2} = \left[ \frac{1}{n} \sum_{i=1}^{n} x_i \right]^2
\]

as an estimator of \( \theta^2 \).

(3a) Is \( \bar{x}_n^2 \) an unbiased estimator of \( \theta^2 \)? Is it asymptotically unbiased?

After completing your theoretical derivation, consider \( n = 2 \) and investigate the bias using MATLAB. That is:

1. Set \( \theta = 1 \).
2. Generate 2 draws from the exponential distribution with \( \theta = 1 \). (See lecture notes for how to do this).
3. Compute the sample average of these two draws, and then take the square of the sample average.
4. Store this squared value, and then repeat this process (in a loop) 10,000 times.
5. Use the program “epanech2” to provide a density estimate associated with the 10,000 values of \( \bar{x}_n^2 \).

Where is the (exact) sampling distribution centered? (To answer this, calculate the sample average of your 10,000 draws). Is this consistent with your theoretical answer in (3a)? Does the sampling distribution with \( n = 2 \) resemble a normal distribution?

(3b) Does \( \text{plim} (\bar{x}_n^2) = \theta^2 \)?

(3c) What is the asymptotic distribution for \( \bar{x}_n \)?
(4) Suppose that $x_i, i = 1, 2, \ldots, n$ is an iid sample from a chi-square distribution with one degree of freedom. In this case, we know that

$$E(X_i) = 1, \quad \text{Var}(X_i) = 2.\]$$

(4a) Derive the asymptotic distribution associated with $X_n$.

(4b) Rearrange the result in (4a) to show

$$\sum_{i=1}^{n} X_i \xrightarrow{d} N(n, 2n).$$

It can also be shown that $\sum_{i=1}^{n} X_i$ has an exact chi-square distribution with $n$ degrees of freedom. (You can show this via moment generating functions, but you do not need to do so for this exercise).

To assess the accuracy of the normal approximation, you can consult a chi-square table (you can find one on the web if needed at http://www.richland.edu/james/lecture/m170/tbl-chi.html) to show that

$$\Pr(\sum_{i=1}^{10} X_i \leq 15.987) = .9$$
$$\Pr(\sum_{i=1}^{25} X_i \leq 34.382) = .9$$
$$\Pr(\sum_{i=1}^{50} X_i \leq 63.167) = .9$$
$$\Pr(\sum_{i=1}^{100} X_i \leq 118.498) = .9$$

Compare these exact probabilities with those associated with the normal approximation to assess the accuracy of the approximation.