(1a) First, note
\[ E(x) = \int_0^1 ax^a \, dx = \frac{a}{a + 1}. \]
Similarly,
\[ E(x^2) = \int_0^1 ax^{(a+1)} \, dx = \frac{a}{a + 2}. \]
Using the fact that
\[ \text{Var}(x) = E(x^2) - E^2(x) \]
and simplifying gives
\[ \text{Var}(x) = \frac{a}{(a + 2)(a + 1)^2}. \]

(1b) Note that
\[ E(x) = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n + 1)}{2} = \frac{n + 1}{2}. \]
Similarly,
\[ E(x^2) = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1}{n} \frac{n(n + 1)(2n + 1)}{6} = \frac{(n + 1)(2n + 1)}{6}. \]
Thus,
\[ \text{Var}(x) = \frac{1}{12} (n + 1)(n - 1). \]

(2) First, the density is non-negative since \( f \) is a valid p.d.f and \( F(x_0) < 1 \). Second, note
\[
\int_{-\infty}^{\infty} g(x) = \int_{x_0}^{\infty} g(x) \\
= \int_{x_0}^{\infty} f(x)/[1 - F(x_0)] \, dx \\
= [1 - F(x_0)]^{-1} \int_{x_0}^{\infty} f(x) \, dx \\
= [1 - F(x_0)]^{-1} [1 - F(x_0)] \\
= 1
\]

(3) Note
\[ E(\exp[tx]) = \int_0^\infty x \exp[x(t - 1)] \, dx. \]
Now, let’s use the integration by parts formula with 

\[ u = x, \quad dv = \exp[x(t - 1)]. \]

Thus, we have, for \( t < 1 \)

\[
E(\exp[tx]) = \left. \frac{x}{t-1} \exp(x[t-1]) \right|_0^\infty - \int_0^\infty \frac{1}{t-1} \exp(x[t-1])dx \\
= -\int_0^\infty \frac{1}{t-1} \exp(x[t-1])dx \\
= -\frac{1}{(t-1)^2} \exp(x[t-1]) \bigg|_0^\infty \\
= \frac{1}{(t-1)^2}
\]

It follows that the first derivative of this function is \(-2(t-1)^{-3}\) and the second derivative is \(6(t-1)^{-4}\). Evaluating this at \( t = 0 \) gives a mean equal to 2 and a second moment equal to 6.

(4) First, note (provided one can differentiate under the integral sign):

\[
\frac{\partial}{\partial t} S(t) = \frac{E(x \exp[tx])}{E(\exp[tx])}.
\]

Evaluating this at \( t = 0 \) gives

\[
\frac{\partial}{\partial t} S(t) = E(X).
\]

Differentiating again, we obtain:

\[
\frac{\partial^2}{\partial t^2} S(t) \bigg|_{t=0} = \frac{E(\exp[tx])E(x^2 \exp[tx]) - E^2(x \exp[tx])}{E^2(\exp[tx])}.
\]

Evaluating this at \( t = 0 \) gives

\[
\frac{\partial^2}{\partial t^2} S(t) = E(x^2) - E^2(x) = \text{Var}(x).
\]

(5) Let \( X \) be the event associated with having a girl on the \( j^{th} \) pregnancy, but not before. Thus, \( \Pr(X = 1) = 1/2, \Pr(X = 2) = 1/4, \cdots, \Pr(X = j) = 1/2^j \). The expected number of children is therefore \( E(\text{kids}) = \sum_{j=1}^\infty j/2^j \). To evaluate this sum, note that \( E(\text{kids}) - 1/2E(\text{kids}) = 1/2E(\text{kids}) = (1/2) + (1/4) + (1/8) + (1/16) + \cdots \). (To see this, write out the sum term-by-term, multiply it by \(1/2\) and then take the difference). The right-hand side is simply a geometric series whose sum equals one. Thus, \(1/2E(\text{kids}) = 1\) or \(E(\text{kids}) = 2\).

(6) MATLAB code is also provided for providing the plots.