(1) (a) First, note that the density is always positive over its support. Moreover,
\[ \int_0^\infty \int_x^\infty 2 \exp[-(x + y)] dy dx = 1 \]
(which you can verify through some rather simple integration). Thus, this is indeed a valid density function.

(b) Note:
\[ f_Y(y) = \int_0^y 2 \exp[-(x + y)] dx = 2 \exp(-y) [1 - \exp(-y)] \]
Similarly,
\[ f_X(x) = \int_x^\infty 2 \exp[-(x + y)] dy = 2 \exp(-2x) \]

(c) The variables are not independent; the product of the marginal densities does not equal the joint density. That the variables must be dependent is rather obvious since the (conditional) support of \( X \) depends on \( Y \).

(2) No, in general, zero correlation does not imply independence, though this is true for normal random variables, (and the converse of this statement is always true). To see this, consider two random variables: \( X \sim N(0, 1) \) and \( Y = X^2 \sim \chi^2(1) \) (That is, \( Y \) has a Chi-square distribution with one degree of freedom). Note that \( \text{Cov}(X, Y) = E(XY) - E(X)E(Y) = E(XY) = E(X^3) = 0 \), since all odd-order moments of the normal density is zero. So, \( \text{Corr}(X, Y) = 0 \). However, these random variables are clearly not independent. The conditional distribution \( Y|X \) is not independent of \( X \) - in fact, \( Y|X \) is degenerate around \( X = x^2 \), so the value of \( X \) is quite informative of the value of \( Y \)!!!

(3) She should not necessarily be concerned. Let \( X = 1 \) denote the event that the person has the less serious disease, and \( Y = 1 \) denote the same event for the more serious disease. Suppose that the joint probability distribution for \( X \) and \( Y \) is given by

<table>
<thead>
<tr>
<th></th>
<th>( Y = 0 )</th>
<th>( Y = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X = 0 )</td>
<td>.075</td>
<td>.005</td>
</tr>
<tr>
<td>( X = 1 )</td>
<td>.9</td>
<td>.02</td>
</tr>
</tbody>
</table>
From this table, it is clear that $\Pr(X = 1|Y = 1) = 0.02/0.025 = 0.8$, so this is consistent with the information given. However, $\Pr(Y = 1|X = 1) = 0.02/0.92 \approx 0.022$. So, the person is still quite unlikely to have the serious disease $Y$. The answer depends on the marginal probabilities associated with both $Y$ and $X$.

(4) Let $\hat{C}$ denote the box that is chosen by the contestant and let $C$ denote the actual location of the prize. Let $M$ denote the box that the game show host (whom we will call “Monte”, given that this is the actual situation faced by real game show contestants in the TV show Let’s make a deal, hosted by Monte Hall) actually reveals as being empty. Without loss of generality, let us suppose that $\hat{C} = 1$ and $M = 2$. We also assume

$$\Pr(\hat{C} = j) = 1/3, \quad j = 1, 2, 3.$$ and $C$ and $\hat{C}$ are independent.

These assumptions are reasonable - the contestant randomly chooses among the three doors, and the choice made is independent of the actual location of the prize. The second of these assumptions implies

$$\Pr(\hat{C} = j|C) = \Pr(\hat{C} = j) = 1/3, \quad j = 1, 2, 3.$$ To evaluate if the contestant should switch her choice, we need to consider and quantify the following two probabilities:

$$\Pr(C = 3|M = 2, \hat{C} = 1) \quad \text{and} \quad \Pr(C = 1|M = 2, \hat{C} = 1).$$ Since

$$\Pr(C = 3|M = 2, \hat{C} = 1) = \frac{\Pr(M = 2, \hat{C} = 1|C = 3)\Pr(C = 3)}{\Pr(M = 2, \hat{C} = 1)},$$

and

$$\Pr(C = 1|M = 2, \hat{C} = 1) = \frac{\Pr(M = 2, \hat{C} = 1|C = 1)\Pr(C = 1)}{\Pr(M = 2, \hat{C} = 1)}.$$ it follows that

$$\frac{\Pr(C = 3|M = 2, \hat{C} = 1)}{\Pr(C = 1|M = 2, \hat{C} = 1)} = \frac{\Pr(M = 2, \hat{C} = 1|C = 3)}{\Pr(M = 2, \hat{C} = 1|C = 1)} = \frac{\Pr(M = 2|\hat{C} = 1, C = 3)\Pr(\hat{C} = 1|C = 3)}{\Pr(M = 2|\hat{C} = 1, C = 1)\Pr(\hat{C} = 1|C = 1)} = \frac{\Pr(M = 2|\hat{C} = 1, C = 3)}{\Pr(M = 2|\hat{C} = 1, C = 1)}.$$ The numerator of this last expression must be 1, since Monty has no choice but to reveal box 2 if the contestant actually selected box 1 and the location of the prize is in box 2. We are not certain
of the value of the denominator, but we know it must be less than or equal to one. It is reasonable to assume that Monty randomly selects between the two empty boxes if the contestant actually correctly selects the location of the prize. If so, the probability in the denominator of the above is $(1/2)$ and thus the entire ratio reduces to $2!$

This means that the contestant is twice as likely to win the prize by switching her choice when given the chance!

(5) (a) is clearly not true since it provides no information on how $X$ and $Y$ might move together. This would, of course, be true if the r.v.’s are independent. (b) is clearly true. (c) is not true. To see this, suppose that $X$ and $Y$ are independent. In this case, $p(x|y)$ simply reduces to $p(x)$ so that both of the “givens” simply convey the marginal for $x$. However, no information is provided for the $y$ marginal, so the information provided is insufficient.

The last one is tricky, and turns out to be true (under some restrictions). That is, the conditionals of the model are sufficient to define the joint distribution. (This may not seem like a big deal, but this result provides a cornerstone for modern Bayesian econometrics and simulation methods like Gibbs sampling, but more on that in a later course.)

To see why this is the case, note (dropping unnecessary notation)

$$p(x, y) = p(x)p(y|x) = p(y)p(x|y).$$

Thus, it follows that

$$p(y) = \frac{p(y|x)}{p(x|y)}p(x).$$

Since $p(y)$ is a valid density, we can integrate the equation above over $Y$ to obtain:

$$1 = p(x) \int_Y \frac{p(y|x)}{p(x|y)} dy,$$

so that $p(x)$ can be completely defined in terms of the two conditional distributions. Solving for $p(x)$ and subbing this into our first equation gives:

$$p(x, y) = p(y|x) \left[ \int_Y \frac{p(t|x)}{p(x|t)} dt \right]^{-1}.$$

So, the joint distribution can be completely determined by the two conditional distributions.

Note, of course, that this expression requires that the integral

$$\left[ \int_Y \frac{p(t|x)}{p(x|t)} dt \right]$$

exists (i.e., is finite), and there are some cases, as pointed out by a few of you, where this integral diverges. In most interesting cases, however, this integral exists.
An extension of this result is known as the *Hammersley-Clifford* Theorem in statistics, which states that an arbitrary joint density can be represented in terms of its set of conditional densities. This requires an additional condition known as *positivity*, which is related to the existence of the integral above and is satisfied in most cases. (Positivity basically says that whenever the marginals place positive probability over outcomes $x_0$ and $y_0$ then the joint probability of the pair $(x_0, y_0)$, also has positive probability. This rules out cases like $x = y$, for example, as some of you mentioned to me in class).

Since I was not at all clear regarding what conditions were needed in the derivation, you should get full credit for any reasonable attempt at this question. (If you still have questions or concerns regarding this, please come and talk with me). The bottom line is that under typical conditions, the conditionals are sufficient to define the joint, though notable exceptions do exist.