Economics 671
Solutions: Problem Set #4
Convergence Concepts

(1) Fix $\epsilon > 0$. We seek to show that
\[ \lim_{n \to \infty} \Pr \{ |X_n^2 - a^2| > \epsilon \} = 0. \]

Choose a $\delta > 0$ such that $\delta^2 + 2|a|\delta < \epsilon$ and write the above probability as
\[ \Pr \{ |X_n^2 - a^2| > \epsilon \} = \Pr \{ |X_n^2 - a^2| > \epsilon, |X_n - a| \leq \delta \} + \Pr \{ |X_n^2 - a^2| > \epsilon, |X_n - a| > \delta \} \]
\[ = \Pr \left\{ |X_n^2 - a^2| > \epsilon \left| |X_n - a| \leq \delta \right. \right\} \Pr \{ |X_n - a| \leq \delta \} \]
\[ + \Pr \left\{ |X_n^2 - a^2| > \epsilon \left| |X_n - a| > \delta \right. \right\} \Pr \{ |X_n - a| > \delta \} \]
\[ \leq \Pr \left\{ |X_n^2 - a^2| > \epsilon \left| |X_n - a| \leq \delta \right. \right\} + \Pr \{ |X_n - a| > \delta \} \]

The limit of this second term is zero since we are given that $X_n \xrightarrow{p} a$. Thus,
\[ \lim_{n \to \infty} \Pr \{ |X_n^2 - a^2| > \epsilon \} \leq \lim_{n \to \infty} \Pr \left\{ |X_n^2 - a^2| > \epsilon \left| |X_n - a| \leq \delta \right. \right\} . \]

To evaluate the term above, note that
\[ (X_n - a)^2 = X_n^2 + a^2 - 2X_na \]
\[ = X_n^2 - a^2 + a^2 + a^2 - 2X_na \]
\[ = (X_n^2 - a^2) + 2a(a - X_n) \]
so that, after rearranging,
\[ X_n^2 - a^2 = (X_n - a)^2 + 2a(X_n - a). \]

Taking absolute values gives
\[ |X_n^2 - a^2| = |(X_n - a)^2 + 2a(X_n - a)| \]
and the triangle inequality gives
\[ |X_n^2 - a^2| \leq |(X_n - a)^2| + |2a(X_n - a)| = (X_n - a)^2 + 2|a||(X_n - a)| \]

Now, consider the event where $|X_n - a| \leq \delta$ for the above choice of $\delta > 0$. Then the inequality above implies
\[ |X_n^2 - a^2| \leq \delta^2 + 2|a|\delta < \epsilon, \]
where the last line follows from our choice of $\delta$. Thus, the term

$$\Pr\{ |X_n^2 - a^2| > \epsilon \mid |X_n - a| \leq \delta \}$$

is zero since $|X_n^2 - a^2| < \epsilon$ whenever $|X_n - a| \leq \delta$.

(2) As discussed in class, it is sufficient to show that the bias and variance go to zero. Note that

$$E(X_n + Y_n - (X + Y)) = E(X_n - X) + E(Y_n - Y) = Bias(X_n) + Bias(Y_n),$$

and both of these must go to zero since we have both $X_n$ and $Y_n$ converging in mean-square to their respective limits. Thus, the bias is going to zero.

Now, consider

$$\text{Var}(X_n + Y_n) = \text{Var}(X_n) + \text{Var}(Y_n) + 2\text{Cov}(X_n, Y_n)$$

$$= \text{Var}(X_n) + \text{Var}(Y_n) + 2\text{Corr}(X_n, Y_n) \sqrt{\text{Var}(X_n)} \sqrt{\text{Var}(Y_n)}.$$ 

Since $X_n$ and $Y_n$ are converging in mean square, both $\text{Var}(X_n)$ and $\text{Var}(Y_n)$ are going to zero. This implies that $\sqrt{\text{Var}(X_n)}$ and $\sqrt{\text{Var}(Y_n)}$ are also going to zero. (The proof of this is not necessary here; it follows since the square root is a continuous function, and by a simple modification to the proof in question 1). Since $|\text{Corr}(X_n, Y_n)| < 1$, it is clear that $\text{Var}(X_n + Y_n) \to 0$. Thus, we have convergence in mean square to $X + Y$.

(3) Note (since $X > 0$)

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$\geq \int_{-\infty}^{r} xf(x)dx$$

$$\geq r \int_{r}^{\infty} f(x)dx$$

$$= r \Pr(x \geq r)$$

Rearranging gives

$$\Pr(x \geq r) \leq E(x)/r,$$

as desired.
(4) First, let us derive the sampling distribution of the sample maximum. We will do this via the
cdf technique:

\[ F_{\text{max}}(c) = \Pr(x_1 \leq c, \ldots, x_n \leq c) \]

\[ = [\Pr(X \leq c)]^n \]

\[ = [c/\theta]^n, \quad 0 \leq c \leq \theta \]

Thus, by differentiation,

\[ f_{\text{max}}(x) = nx^{n-1}\theta^{-n}, \quad 0 \leq x \leq \theta \]

For consistency, we need to evaluate:

\[ \Pr\{|\text{max} - \theta| > \epsilon\} \]

for \(0 < \epsilon \leq \theta\) (since the above probability for the case where \(\epsilon > \theta\) is obviously zero). Noting that \(\text{max} \leq \theta\), this reduces to

\[ \Pr(\text{max} - \theta \leq -\epsilon) = \Pr(\text{max} \leq \theta - \epsilon). \]

Using the previous density function for the sample maximum, we obtain

\[ \Pr\{|\text{max} - \theta| > \epsilon\} = \int_{0}^{\theta-\epsilon} nx^{n-1}\theta^{-n}dx = (1 - [\epsilon/\theta])^n. \]

For \(\epsilon, \theta > 0, \epsilon/\theta < 1\) this clearly approaches 0 as \(n \to \infty\) so that \(\text{max} \overset{p}{\to} \theta\).

(5) Note that

\[ \hat{\beta} = \frac{\sum_i x_i(x_i\beta + \epsilon_i)}{\sum_i x_i^2} = \beta + \frac{\sum_i x_i\epsilon_i}{\sum_i x_i^2} = \beta + \frac{n^{-1}\sum_i x_i\epsilon_i}{n^{-1}\sum_i x_i^2}. \]

We can think about the variables \(x_i\epsilon_i\) and \(x_i^2\) as random variables to which a law of large numbers could apply. Specifically, \(E(x_i\epsilon_i) = E(x_i)E(\epsilon_i) = 0\) \(\forall i\) so that \(n^{-1}\sum_i x_i\epsilon_i \overset{p}{\to} 0\).

Similarly, \(E(x_i^2) = \mu_2\), say, so that \(n^{-1}\sum_i x_i^2 \overset{p}{\to} \mu_2 \neq 0\). Thus, we can state that

\[ \text{plim} \left( \frac{n^{-1}\sum_i x_i\epsilon_i}{n^{-1}\sum_i x_i^2} \right) = \frac{\text{plim}[n^{-1}\sum_i x_i\epsilon_i]}{\text{plim}[n^{-1}\sum_i x_i^2]} = 0 \]

so that

\[ \hat{\beta} \overset{p}{\to} \beta. \]