(1: Getting Started in MATLAB):

Note that $y = \ln x$ and thus $x = \exp(y)$. Therefore, the Jacobian of the transformation from $x$ to $y$ is $\exp(y)$, which is always positive.

It follows that

$$p(y) = \theta^{-1} \exp\left[-\theta^{-1} \exp(y)\right] \exp(y)$$

or

$$p(y) = \theta^{-1} \exp\left[y - \theta^{-1} \exp(y)\right], \quad -\infty < y < \infty.$$

An accompanying MATLAB file is provided that plots this density, calculates the area under the curve, and compares the mean in two different ways.

(2) First, note that the mean-zero property follows trivially since

$$E(v_{it}) = E(a_i) + E(u_{it}) = 0$$

and

$$E(\pi_i) = E[1/T \sum_t v_{it}] = 1/T \sum_T E(v_{it}) = 0.$$

As for homoscedasticity, note that

$$\text{Var}(\epsilon_{it}) = \text{Var}(v_{it} - \lambda \pi_i) = \text{Var}(v_{it}) + \lambda^2 \text{Var}(\pi_i) - 2\lambda \text{Cov}(v_{it}, \pi_i).$$

Let’s individually consider each of these pieces comprising this variance.

$$\text{Var}(v_{it}) = \text{Var}(a_i + u_{it}) = \text{Var}(a_i) + \text{Var}(u_{it}) = \sigma^2_a + \sigma^2_u.$$  

$$\text{Var}(\pi_i) = \text{Var}(a_i + 1/T \sum_t u_{it}) = \sigma^2_a + \sigma^2_u/T.$$  

Finally,

$$\text{Cov}(v_{it}, \pi_i) = \text{Cov}(a_i + u_{it}, a_i + 1/T \sum_t u_{it})$$
\[
\begin{align*}
&= \text{Var}(a_i) + \text{Cov}(u_{it}, 1/T \sum_t u_{it}) \\
&= \sigma_a^2 + \sigma_u^2 / T.
\end{align*}
\]

Putting all of this together, we have
\[
\text{Var}(\epsilon_{it}) = (\sigma_a^2 + \sigma_u^2) + \lambda^2[\sigma_a^2 + \sigma_u^2 / T] - 2\lambda[\sigma_a^2 + \sigma_u^2 / T] \\
= (\sigma_a^2 + \sigma_u^2) + (\lambda^2 - 2\lambda)[\sigma_a^2 + \sigma_u^2 / T] \\
= (\sigma_a^2 + \sigma_u^2) + [(\lambda - 1)^2 - 1][\sigma_a^2 + \sigma_u^2 / T] \\
= (\sigma_a^2 + \sigma_u^2) - [T\sigma_a^2 / (\sigma_a^2 + T\sigma_u^2)][\sigma_a^2 + \sigma_u^2 / T] \\
= \sigma_a^2 + \sigma_u^2 - \sigma_a^2 \\
= \sigma_u^2.
\]

The third-to-last line follows from the definition of \( \lambda \). Thus, the errors are homoscedastic.

As for serial uncorrelation,
\[
\begin{align*}
\text{Cov}(\epsilon_{ik}, \epsilon_{ij}) &= \text{Cov}(v_{ik} - \lambda \pi_i, v_{ij} - \lambda \pi_i) \\
&= \text{Cov}(v_{ik}, v_{ij}) - 2\lambda \text{Cov}(v_{ik}, \pi_i) + \lambda^2 \text{Var}(\pi_i) \\
&= \sigma_a^2 + [(\lambda - 1)^2 - 1][\sigma_a^2 + \sigma_u^2 / T] \\
&= \sigma_a^2 - \sigma_a^2 \\
&= 0.
\end{align*}
\]

The second line expands the covariance and notes \( \text{Cov}(v_{ik}, \pi_i) = \text{Cov}(v_{ij}, \pi_i) \).

(3a) It is clear that \( c_i \) captures all time-invariant factors in the model and thus
\[
c_i = \alpha + z_i \gamma + h_i.
\]

(3b)
\[
\begin{align*}
\text{Var}(c_i) &= \text{Var}(\alpha + z_i \gamma + h_i) \\
&= \text{Var}(z_i \gamma + h_i) \\
&= \sigma_h^2 + \text{Var}(z_i \gamma) \\
&= \sigma_h^2 + E [(z_i - \mu_z) \gamma' (z_i - \mu_z)']
\end{align*}
\]
This second term is clearly non-negative, and equals zero in the rather uninteresting case where \( \gamma = 0 \). Thus, \( \text{Var}(c_i) > \text{Var}(h_i) \).

(3c) The fact that the variance of the unobserved fixed effect exceeds the variance of the unobserved random effect is intuitive, and is really all about conditioning. In the random effect approach, \( z_i \) has already been “taken out” of our regression equation, leaving less unexplained error. In the fixed effect approach, the observables \( z \) enter through \( c \). In other words, this is simply a result of a fact that the conditional variance is smaller than the unconditional variance, and suggests nothing deeper than that.