Econometrics 672
Problem Set #3 SOLUTIONS

(1a) \[
\sum_{j=1}^{\infty} f(y_j | \gamma) = \gamma + \gamma (1 - \gamma) + \gamma (1 - \gamma)^2 + \cdots \\
= \gamma [1 + (1 - \gamma) + (1 - \gamma)^2 + \cdots] \\
= \gamma [1/\gamma] \\
= 1,
\]
as desired. The last line uses the formula for a convergent infinite geometric series.

(1b) Note
\[
L(\gamma) = \prod_{i=1}^{n} \gamma (1 - \gamma)^{y_i} = \gamma^n \prod_{i=1}^{n} (1 - \gamma)^{y_i}.
\]
Taking logs gives
\[
\ln L(\gamma) = n \ln \gamma + \sum_{i=1}^{n} y_i \ln (1 - \gamma) = n \ln \gamma + \ln (1 - \gamma) \sum_{i=1}^{n} y_i = n \ln \gamma + n \overline{y} \ln (1 - \gamma).
\]
This yields the first derivative of the log likelihood:
\[
\frac{\partial \ln L(\gamma)}{\partial \gamma} = n/\gamma - n \overline{y} / [1 - \gamma].
\]
Setting this expression equal to zero and solving produces
\[
\hat{\gamma}_{MLE} = (1 + \overline{y})^{-1}.
\]

(2a) The likelihood function is given as
\[
L(\alpha, \beta) = \prod_{i=1}^{n} \alpha \beta x_i^{\beta - 1} \exp \left( -\alpha x_i^{\beta} \right) \\
= \alpha^n \beta^n \exp \left( -\alpha \sum_{i=1}^{n} x_i^{\beta} \right) \prod_{i=1}^{n} x_i^{\beta - 1}.
\]
Thus, the log likelihood is
\[
\ln L(\alpha, \beta) = n \ln \alpha + n \ln \beta - \alpha \sum_{i=1}^{n} x_i^{\beta} + (\beta - 1) \sum_{i=1}^{n} \ln x_i.
\]
(2b) Differentiating with respect to $\alpha$ and setting the result equal to zero implies

$$\frac{n}{\hat{\alpha}} - \sum_{i=1}^{n} x_i^\beta = 0,$$

or equivalently,

$$\hat{\alpha} = \frac{n}{\left| \sum_{i=1}^{n} x_i^\beta \right|}.$$  

This provides a MLE estimator for $\alpha$ as a function of the data and $\hat{\beta}$. When differentiating with respect to $\beta$, we obtain

$$\frac{n}{\hat{\beta}} + \sum_{i=1}^{n} \ln x_i - \hat{\alpha} \sum_{i=1}^{n} x_i^\beta \ln x_i = 0.$$  

Once we have subbed in $\hat{\alpha} = \frac{n}{\left| \sum_{i=1}^{n} x_i^\beta \right|}$ into the expression above, we obtain an expression which implicitly defines the MLE estimator for $\beta$. To obtain this estimator, we would have to search for the value of $\beta$ that sets this equation equal to zero. Once this is obtained, the MLE estimator for $\alpha$ follows immediately.

(3a)

$$\int_{0}^{1} f(x|\theta) \, dx = \int_{0}^{1} \theta x^{\theta-1} \, dx = x^{\theta}\bigg|_{0}^{1} = 1 - 0 = 1,$$

so that the density function is proper. As for the mean,

$$E(x) = \int_{0}^{1} x \theta x^{\theta-1} \, dx = \int_{0}^{1} \theta x^{\theta} \, dx = \frac{\theta}{\theta + 1} x^{\theta+1}\bigg|_{0}^{1} = \frac{\theta}{\theta + 1}.$$  

(3b) Given a random sample of $n$ observations from the power distribution, we obtain the likelihood:

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \prod_{i=1}^{n} x_i^{\theta-1},$$

and the log likelihood

$$\ln L(\theta) = n \ln(\theta) + (\theta - 1) \sum_{i=1}^{n} \ln x_i = n \ln(\theta) + n(\theta - 1)\ln x.$$

Taking derivatives with respect to $\theta$ and setting these equal to zero produces

$$\frac{n}{\hat{\theta}_{MLE}} + n\ln x = 0,$$
which implies,
\[ \hat{\theta}_{MLE} = -\frac{1}{\ln x}. \]

To characterize the asymptotic distribution, we first need to obtain the second derivative of the log likelihood, which in this case is just a constant:
\[ \frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -n/\theta^2. \]

Thus, we have
\[ \hat{\theta}_{MLE} \xrightarrow{d} N(\theta, \theta^2/n). \]

(3c) Note that the score is simply
\[ n/\theta + \sum_i \ln x_i. \]

To calculate the expectation of the score, we must calculate \( E(\ln x) \). Using the given formula, we obtain
\[
E(\ln x) = \int_0^1 \theta \ln x x^{\theta-1} \, dx \\
= \theta \left[ \frac{x^\theta}{\theta} \ln x - \frac{x^{\theta}}{\theta^2} \right]_0^1 \\
= \theta \left[ 0 - \frac{1}{\theta^2} - 0 \right] \\
= -1/\theta
\]

In evaluating this expression at the given limits, we note that \( x^\theta \) dominates \( \ln x \) as \( x \to 0 \).

Thus, the expectation of the score is
\[ n/\theta + \sum_i E(\ln x_i) = n/\theta - n/\theta = 0. \]

(3d) In (3c), we showed that \( E(\ln x) = -1/\theta \). When taking the sample analog of this population expectation, we would expect
\[ \bar{\ln x} \approx -1/\theta, \]
or equivalently,
\[ \theta \approx -\frac{1}{\bar{\ln x}}. \]

This is, in fact, the MLE estimator for \( \theta \) derived in part (b).
First, note that, given this data, $\bar{\ln x} = -0.946$ and thus $\hat{\theta}_{MLE} = 1.06$. Further, using our estimate of the asymptotic variance $\theta^2/n$ is $1.06^2/10 = .112$.

In forming the Wald statistic (assuming the null is true), we obtain $(1.06 - 1.2)^2/.112 = .175$. This test statistic is to be compared to a $\chi^2_1$ critical value. We cannot reject this null hypothesis at conventional levels of significance.