Problemset 2

1. Find the Yule-Walker equations for the AR(2) process

\[ X_t = \frac{1}{3} X_{t-1} + \frac{2}{9} X_{t-2} + \varepsilon_t \]

and show that

\[ \rho_k = \frac{16}{21} \left( \frac{2}{3} \right)^{|k|} + \frac{5}{21} \left( -\frac{1}{3} \right)^{|k|} \]

2. This problem concerns invertibility of MA processes. We say that a MA process is invertible when the MA lag polynomial has all roots outside the unit circle. In this problem we demonstrate the theorem that every MA process can be written such that it is invertible. First consider the following two MA(1) processes:

\[ y_t = \varepsilon_t + 2\varepsilon_{t-1} \]
\[ z_t = \tilde{\varepsilon}_t + \frac{1}{2} \tilde{\varepsilon}_{t-1} \]

where \( \{\varepsilon_t\} \) and \( \{\tilde{\varepsilon}_t\} \) are white noise processes.

(a) Prove that \( \{z_t\} \) and \( \{y_t\} \) have the same covariance structure.

(b) Now consider the two MA(2) processes

\[ y_t = \frac{1}{a_2} (L - \alpha_1) (L - \alpha_2) \varepsilon_t \]

and

\[ z_t = \frac{1}{a_2} (L - \alpha_1) \left( L - \frac{1}{\alpha_2} \right) \tilde{\varepsilon}_t \]

Find the value of \( \tilde{\sigma}^2 \) for which these have the same covariance structure.

(c) Now consider a MA(q) process where one root lies inside the unit circle. Describe precisely how to obtain an invertible MA(q) process with the same covariance structure.
3. Consider the following ARMA(2,2) model
\[
x_t = 1.3x_{t-1} - 0.4x_{t-2} + \varepsilon_t - 1.2\varepsilon_{t-1} + 0.2\varepsilon_{t-2}
\]
\[
\varepsilon_t \sim iid (0,1)
\]

(a) Is \( x_t \) weakly stationary? If so compute the autocovariance function.
(b) Is this process invertible?
(c) Can this process be written as a MA(\( \infty \)) process? If so, provide the expression.

4. Consider the usual estimator of the \( k^{th} \) autocovariance.
\[
\hat{\gamma}_k = \frac{1}{T} \sum_{t=k+1}^{T} (X_t - \bar{X}) (X_{t-k} - \bar{X})
\]
Prove that
\[
\hat{\gamma}_0 + 2 \sum_{k=1}^{T-1} \hat{\gamma}_k = 0
\]

Hint: Recall that \( \frac{1}{T} \sum_{t=1}^{T} (X_t - \bar{X}) = 0 \)