Lecture 19 - Decomposing a Time Series into its Trend and Cyclical Components

It is often assumed that many macroeconomic time series are subject to two sorts of forces: those that influence the long-run behavior of the series and those that influence the short-run behavior of the series. So, for example, growth theory focuses on the forces that influence long-run behavior whereas business cycle theory focuses on the forces that influence short-run behavior.

Therefore, it is useful to have econometric methods that can be used to decompose a time series into its trend and cyclical components.

[Other reasons – to transform a nonstationary series into a stationary series by removing the trend; alleviate a spurious regression problem.]

Several Methods –

- Polynomial trend removal
- Beveridge-Nelson type decompositions
- Hodrick-Prescott Filter
I. Polynomial Trend Removal

Assume that $y_t$ is a trend stationary process. That is:

$$y_t = \tau_t + c_t$$

$\tau_t$ is a deterministic function, typically a low-order polynomial, of $t$ called the trend (or secular or long-run or permanent) component of $y_t$ and $c_t$ is a stationary process, called the cyclical component (or short-run or transitory) component of $y_t$.

Assume that the trend is a polynomial in $t$, so that

$$y_t = \tau_t + c_t$$

$$= \beta_0 + \beta_1 t + \ldots + \beta_p t^p + c_t$$

This is a standard linear regression model with a serially correlated process, $c_t$. How to efficiently estimate the $\beta$’s? OLS. (Grenander and Rosenblatt, 1957, show that OLS is asymptotically equivalent to GLS when $x_t = [1 \ t \ldots t^p]$.)
Then it makes sense to set

\[ \hat{\tau}_t = \hat{\beta}_0 + \hat{\beta}_1 t + ... + \hat{\beta}_p t^p \]

and

\[ \hat{C}_t = y_t - \hat{\tau}_t \]

where \( \hat{\beta} \)-hat denotes the OLS estimator.

Notes:

• In practice \( p = 1 \) (linear trend) and, in the case of rapidly growing nominal variables, \( p = 2 \) (quadratic trend) tend to be the most common choices of \( p \).

• Suppose \( y_t \) and some or all of the \( x_t \)'s are increasing (on average) over time. A regression of \( y \) on \( x \) is likely to be statistically significant even if there is no meaningful or causal relationship between the two. That is, a regression of a trending depending variable on a trending regressor is likely to suggest a statistically significant relationship even when none is present. (What is such a regression called?)
In this case it makes some sense to add the explanatory variable $t$ as a separate regressor in the regression:

\[(*)\quad y_t = \beta_0 + \beta_1 x_t + \beta_2 t + \varepsilon_t.\]

An alternative: Formulate the model as

\[(**)\quad y_t^* = \beta_1 x_t^* + \varepsilon_t\]

where $y_t^*$ is the residual series from the regression of $y_t$ on a linear time trend and $x_t^*$ is the residual series obtained from the regression of $x_t$ on a linear trend.

Fact: The OLS estimate of $\beta_1$ obtained from regression (*) and the OLS estimator of $\beta_1$ obtained from regression (**) will be exactly the same. The residual series will be exactly the same, too. (Frisch-Waugh-Lovell Theorem; see, Greene or Davidson & MacKinnon)

[This result extends to regressions with higher order terms in $t$.]
II. Decompositions Based on Differences

An alternative to the trend stationary assumption to account for trend behavior in a time series is to assume that the series is **difference stationary**, i.e., $y_t$ is stationary in differenced form.
A. Difference Stationarity

A time series $y_t$ difference stationary of order $d$, $d$ a positive integer, if

- $\Delta^d y_t$ is stationary
- $\Delta^{d-1} y_t$ is not stationary
- The MA form of $\Delta^d y_t$ does not have a “unit root”

In practice, $d = 1$ and, for some rapidly growing nominal time series, $d = 2$ are the most commonly used values of $d$.

{Suppose $y_t$ is the trend stationary process

$$y_t = \beta_0 + \beta_1 t + \epsilon_t,$$

where $\epsilon_t$ is a stationary process. Is $y_t$ difference stationary of order one? (Why or why not?)}
• The number of differences required to make a time series stationary is also called the “order of integration of the series.”

• A stationary series is called an integrated of order zero, I(0), series. A difference stationary series with \( d = 1 \) is called an integrated of order one, I(1), series…

• An I(1) process is also called a unit root process because the characteristic polynomial of the AR representation of an I(1) process will have a root equal to 1.
B. The Random Walk Process

The simplest case of an I(1) process is the random walk:

\[ y_t = y_{t-1} + \varepsilon_t, \quad \varepsilon_t \text{ a zero-mean iid process} \]

Note that for the rw –

- \( \Delta y_t \) is an iid process: changes in \( y_t \) are serially uncorrelated, independent, identically distributed.

- \( \text{dy}_{t+s}/d\varepsilon_t = 1 \) for all \( s > 0 \): Innovations have completely permanent effects on the time series!

\[
\begin{align*}
y_{t+1} &= y_t + \varepsilon_{t+1} = \\
y_{t+2} &= y_{t+1} + \varepsilon_{t+2} = y_{t-1} + \varepsilon_t + \varepsilon_{t+1} + \varepsilon_{t+2} \\
&\quad \cdots \\
y_{t+s} &= y_{t-1} + (\varepsilon_t + \varepsilon_{t+1} + \varepsilon_{t+2} + \ldots + \varepsilon_{t+s})
\end{align*}
\]