Lecture 23 – The Dickey-Fuller Test

We have seen that

- the dynamic behavior of \( I(1) \) processes is quite different from the behavior of \( I(0) \) processes
- the way we go about defining and estimating the trend and cyclical components of a time series may depend on whether we assume the series is (trend) stationary or difference stationary.
- regressions with difference stationary variables need special care.

For these reasons we might be interested in testing the null hypothesis of a unit root against the stationary or trend-stationary alternative.
Consider the following AR(1) model for $y_t$:

$$y_t = \rho y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \text{iid } (0, \sigma^2)$$

$-1 < \rho \leq 1$

If $\rho < 1$, $y_t \sim I(0)$, mean 0 and var $\sigma^2/(1-\rho^2)$
If $\rho = 1$, $y_t \sim I(1)$, a random walk

- The OLS estimator of $\rho$ is consistent for all $\rho$; it is super-consistent when $\rho = 1$.
- The OLS t-statistic

$$\tau = \frac{\hat{\rho} - \rho}{\text{s.e.}(\hat{\rho})}$$

where $\text{s.e.}(\hat{\rho})$ is the OLS s.e. of $\rho$-hat, is asymptotically standard normal when $\rho < 1$; it has a non-standard asymptotic distribution when $\rho = 1$, being skewed to the left.
Therefore we cannot apply the standard t-test to test the unit root null. This test will reject the null too often; the actual test size will be greater than the nominal test size.

White (1956) showed that under $H_0: \rho=1$, $\tau$ does have a stable limiting distribution.

Dickey (1976) and Dickey and Fuller (1979) tabulated the percentiles of the this distribution. (How?) The percentiles of the Dickey-Fuller distribution are available in many time series textbooks (including the Enders and Hamilton texts).

See Table
Notes –

- the unit root test is a one-sided test. Reject $H_0$ if $\tau$ is “too negative”, i.e., if $\hat{\rho}$ is too much less than one.
- the median of the DF distribution is about -0.5.
- the 0.025 percentile of the standard normal is -1.96. The 0.025 percentile of the DF distribution is -2.23. The 0.05 percentile of the standard normal is -1.65. The 0.05 of the DF is -1.95.
- The asymptotic distribution seems to be appropriate even if $T$ is as small as 25.
- an equivalent test: Regress $\Delta y_t$ on $y_t$ and use $\tau = \frac{\hat{\rho}}{s.e.(\hat{\rho})}$ as the test statistic.
- Under $H_0$ the test-statistic $T(\hat{\rho} - 1)$ also converges in distribution. The percentiles of this distribution were also tabulated by Dickey and Fuller. Unit root tests based on this test statistic seem to be less powerful and, so, less widely used.
Suppose we modify the model and null hypothesis as follows:

\[ y_t = \alpha + \rho y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \text{iid } (0, \sigma^2), \quad -1 < \rho \leq 1 \]

\( H_0: \rho = 1 \text{ and } \alpha = 0 \)

\( H_A: \rho < 1 \)

The difference between this and the previous case? Under the I(0) alternative, \( y_t \) can have a non-zero mean. But, in neither case does \( y_t \) have a deterministic trend component. (Once we allow the \( \varepsilon \)'s to be serially correlated, this case could be appropriate for testing for a unit root in the unemployment or inflation rates.)
Under the null hypothesis, the t-statistic
$$\tau_\mu = \frac{\hat{\rho} - 1}{se(\hat{\rho})}$$
from the regression of $y_t$ on 1, $y_{t-1}$, or, equivalently, the t-statistic
$$\tau_\mu = \frac{\hat{\rho}}{se(\hat{\rho})}$$
from the regression of $\Delta y_t$ on 1, $y_{t-1}$ converges in distribution to a DF distribution, though the limiting distribution of $\tau_\mu$ is different from the limiting distribution of $\tau$.

See Table
Notes –

- the distribution of $\tau_\mu$ is more highly skewed than the distribution of $\tau$. (Using a standard normal distribution to test $H_0: \rho = 1$ would be even more misleading in this case.)

- Dickey and Fuller (1981) tabulated the percentiles for the asymptotic distribution of the F statistic associated with $H_0: \rho = 1$ and $\alpha = 0$. The F-test of the joint hypothesis might seem more natural to apply in this setting than the t-test of $H_0: \rho = 1$. In practice, however, the t-test is much more commonly used. Then, if $H_0: \rho = 1$ is not rejected it is assumed that $\rho = 1$ and $\alpha = 0$. 
The $\tau$ and $\tau_\mu$ tests are appropriate unit root tests for non-trending zero mean ($\tau$) or non-zero mean ($\tau_\mu$) series. Consider the following model

$$y_t = \alpha + \beta t + \rho y_{t-1} + \varepsilon_t,$$

$\varepsilon_t \sim iid (0, \sigma^2), \ -1 < \rho \leq 1$

and consider

$H_0$: $\rho = 1$ and $\beta = 0$, $y_t$ is a rw with drift

$H_A$: $\rho < 1$, $y_t$ is trend-stationary
Under the null hypothesis, the t-statistic

$$\tau_t = \frac{\hat{\rho} - 1}{se(\hat{\rho})}$$

from the regression of $y_t$ on 1, t, $y_{t-1}$, or, equivalently, the t-statistic

$$\tau_t = \frac{\hat{\rho}}{se(\hat{\rho})}$$

from the regression of $\Delta y_t$ on 1, t, $y_{t-1}$ converges in distribution to a DF distribution. The limiting distribution of $\tau_t$ is different from the limiting distributions of $\tau$ and $\tau_{\mu}$.

See Table
Notes –

• the distribution of $\tau_t$ is more highly skewed than the distribution of $\tau_\mu$.

• Dickey and Fuller (1981) tabulated the percentiles for the asymptotic distribution of the F statistic associated with $H_0: \rho=1$ and $\beta=0$. The F-test of the joint hypothesis might seem more natural to apply in this setting than the t-test of $H_0: \rho=1$. In practice, however, the t-test is much more commonly used. Then, if $H_0: \rho=1$ is not rejected it is assumed that $\rho =1$ and $\beta=0$. 
Allowing for serial correlation in the DF error terms –

I. Augmented Dickey-Fuller Test
II. Phillips-Perron Test