Testing for Cointegration

There are two basic approaches that are commonly used to test for cointegration.

- **Residual Based Tests**  
  \( H_0: \) no CI vs. \( H_A: \) CI  
  Use single-equation regression residuals  
  Engle-Granger; Phillips-Ouliaris

- **Likelihood-Ratio Based Tests**  
  \( H_0: \) CI of rank \( r \) vs. \( H_A: \) CI of rank \( r+1 \)  
  \( H_0: \) CI of rank \( r \) vs. \( H_A: \) CI of rank \( > r \)  
  Use restricted and unrestricted VECMs  
  Johansen
Let $y_t$ be an n-dimensional I(1) process.

Consider

$$H_0: y_t \text{ is not CI}$$
$$H_A: y_t \text{ is CI}$$

So, under $H_0$, $\alpha'y_t \sim I(1)$ for all nonzero $\alpha$ in $\mathbb{R}^n$.

The Engle-Granger (EG) Test

1. Regress $y_{1t}$ on $1, y_{2t}, \ldots, y_{nt}$ by OLS to get the residual series, $\hat{u}_t$.
2. Fit $\hat{u}_t$ to an ADF regression (no intercept or trend). That is, regress $\hat{u}_t$ on $\hat{u}_{t-1}, \Delta \hat{u}_{t-1}, \ldots, \Delta \hat{u}_{t-p+1}$
3. Compute the t-statistic for $H_0$: $\rho=1$.
4. Use the appropriate asymptotic null distribution for this test statistic (which is NOT the DF distribution).
The Phillips-Ouliaris (PO) Test

1. Regress $y_{1t}$ on $1, y_{2t}, \ldots, y_{nt}$ by OLS to get the residual series, $\hat{u}_t$.

2. Fit $\hat{u}_t$ to a DF regression (no intercept or trend). That is, regress $\hat{u}_t$ on $\hat{u}_{t-1}$.

3. Compute the t-statistic for $H_0: \rho=1$ and modify it as in the PP procedure.

4. Use the appropriate asymptotic null distribution for this test statistic (which is the same as the asymptotic distribution of the EG stat).
Notes

• The asymptotic distribution of the EG and PO test stat’s does not depend on the normalization chosen, i.e., which element of y is place on the l.h.s. of the regression. But, the test will have low power against alternatives in which $y_t$ is CI but the element of each CI vector corresponding to $y_1$ (or whichever variable is placed on the l.h.s.) is zero. (For example, if $y_{1t}, y_{2t}, y_{3t}$ are CI but only because $y_{2t}$ and $y_{3t}$ are CI, then a residual-based test with $y_{1t}$ on the l.h.s. will have very low power.)

• Aside from the issue above, the normalization selected may affect the test result in finite samples.

• Suppose $n > 2$ and we reject $H_0$. What is the CI rank of $y$?