

Technical Appendix: A Decomposition Strategy for Solving Empirical Models with Advanced International Trade Theories

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October 2010

Abstract

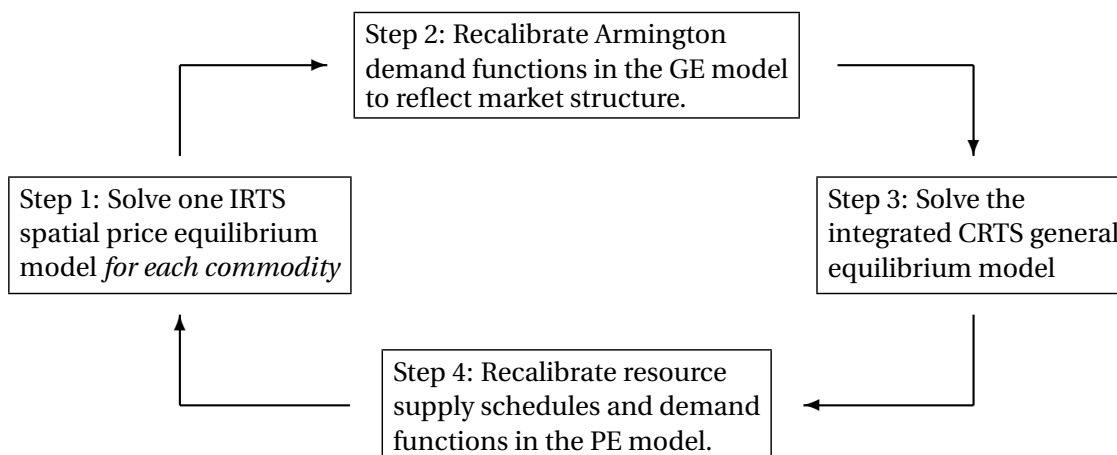
This working document outlines a solution method developed to solve large-scale multiregion general equilibrium models that include monopolistic competition among heterogeneous firms [consistent with the theory developed by Melitz (2003)]. The decomposition solution technique outlined here was used in the analysis for our paper, coauthored with Russell H. Hillberry, “Structural Estimation and Solution of International Trade Models with Heterogeneous Firms,” forthcoming in the *Journal of International Economics* [Balistreri et al. (forthcoming)], and this document serves as a technical appendix to that paper. The computer code is written in GAMS, and can be downloaded from the following web page:

<http://inside.mines.edu/~ebalistr/decomp/decomp.html>

1 Introduction

We represent the policy analysis model developed in Balistreri et al. (forthcoming) [which incorporates Melitz (2003) style monopolistic competition among manufacturing firms] on the basis of two related equilibrium problems. The first is a partial equilibrium (PE) model which captures the heterogeneous-firms industrial organization in manufacturing and the associated impact on productivity and prices. The PE model takes aggregate income levels and supply schedules as given. The second module is a *constant-returns* general equilibrium (GE) model of global trade in all products. The GE model takes industrial structure as given and determines relative prices, comparative advantage and terms of trade. We iterate between these two models in policy simulations, letting the first module determine industrial structure and the second module establish regional incomes and relative costs. Industrial structure (numbers of firms operating within and across borders) are passed from the first

Figure 1: A Decomposition Algorithm



module to the second whereas the structure of aggregate demand (income levels and supply prices) are passed back from the GE module to the PE module. Once the models are mutually consistent we have a solution to the multiregion general equilibrium with heterogeneous manufacturing firms. The four steps involved in the solution algorithm are depicted in Figure 1.

In most policy modeling exercises, applied economists prefer to work with integrated equilibrium models formulated as systems of equations in which prices and quantities are determined simultaneously. Indeed, the mixed complementarity format, in which we solve both the GE and PE modules, is particularly attractive as an integrated framework in which complementary slackness conditions, e.g. activity analysis, can be readily incorporated along with conventional neoclassical production functions. In the present application, however, dimensionality and out-of-equilibrium non-convexities argue strongly in favor of decomposition. When we solve the industrial organization model on a market by market basis, we avoid dealing with excessively high dimensionalities which otherwise arise when there are large numbers of both goods and markets. In addition, we find that decomposition leads to a significant improvement in robustness of the solution method.

The Melitz model incorporates two types of non-convexity. The first is the conventional interaction of prices, quantities, and incomes. Income effects are the source of most of the difficulties in proving convergence for complementarity algorithms [Mathiesen (1987)]. The second non-convexity is associated with the Dixit-Stiglitz aggregation and productivity effects. While it is possible to solve general equilibrium models including Dixit-Stiglitz effects [Markusen (2002)], it is well known that even small instances of the problem class can be extremely difficult. Our decomposition approach seems to avoid these computational difficulties by a *divide-and-conquer* strategy in which income effects are handled in one submodule and productivity effects in a second module.

In the following section we present the Melitz (2003) theory as it applies in Balistreri et al.

(forthcoming), and in Section 3 we develop the solution method.

2 Theory

Consumers have Cobb-Douglas utility over commodity bundles which are defined as constant-elasticity-of-substitution (CES) aggregates of differentiated products. Firms pay a fixed cost of entry. Entrants receive a random productivity draw. Firms with sufficiently low productivity draws exit, and the remaining firms produce with a technology exhibiting increasing returns to scale. Trade costs include *ad valorem* iceberg costs, revenue-generating tariffs, and a fixed cost of entering each market. Firms with higher levels of productivity will be able to profitably serve more markets. The model is simplified by isolating the characteristics and behavior of the average firm participating in each bilateral market. Melitz (2003) develops the critical links between the average and marginal firms, and shows how average firm characteristics relate to consumer utility.

2.1 Demand

Consumers in region $s \in R$ are assumed to have Cobb-Douglas preferences over composites from different sectors, A_{ks} , where the sector is indexed by k and α_k is the expenditure share;

$$U_s = \prod_k (A_{ks})^{\alpha_k}. \quad (1)$$

We drop the industry index at this point and isolate the Dixit-Stiglitz composite of manufactured goods consumed in region s ,

$$A_s = \left[\sum_r \int_{\omega_{rs} \in \Omega_r} q_s(\omega_{rs})^\rho d\omega_{rs} \right]^{\frac{1}{\rho}}, \quad (2)$$

where ω_{rs} indexes the differentiated products sourced from region $r \in R$ (and Ω_r is the set of goods produced in r). Substitution across the products is indicated by $\rho = 1 - 1/\sigma$, where σ is the constant elasticity of substitution. The dual price index, P_s , is given by

$$P_s = \left[\sum_r \int_{\omega_{rs} \in \Omega_r} p_s(\omega_{rs})^{1-\sigma} d\omega_{rs} \right]^{\frac{1}{1-\sigma}}. \quad (3)$$

Defining this in terms of the average variety's price, \tilde{p}_{rs} , we have

$$P_s = \left[\sum_r N_{rs} \tilde{p}_{rs}^{1-\sigma} \right]^{1/(1-\sigma)} \quad (4)$$

where N_{rs} is the number of varieties shipped from r to s . Melitz (2003) obtains this simplification by noting that \tilde{p}_{rs} is the price set by a small firm with the CES weighted average

productivity $\tilde{\varphi}_{rs}$.¹ Demand for the average variety to be shipped from r to s at a gross of trade and tax price of \tilde{p}_{rs} is

$$\tilde{q}_{rs} = \frac{\alpha E_s}{P_s} \left(\frac{P_s}{\tilde{p}_{rs}} \right)^\sigma \quad (5)$$

where E_s is the value of total expenditures in region s .²

2.2 Firm-level environment

We assume a single composite input price, c_r , associated with all fixed or marginal costs of manufacturing in region r . In application, we adopt an upstream Cobb-Douglas technology for generating the composite input. This is represented by a cost function of the form

$$c_r = (P_r^E)^{\beta_r} \prod_j (w_{jr})^{\beta_{jr}}, \quad (6)$$

where the w_{jr} are the prices of the factor inputs and P_r^E is the price of the composite intermediate input. Constant returns in the technology for forming the composite input indicates that the sum of the share parameters, the β , equals one.

Operating firms in a given market use the composite input to cover both fixed-operating and marginal costs, but firms also face an *entry* cost. The entry cost entitles the firm to a productivity draw. If the productivity draw is sufficiently high the firm will operate profitably. Let f_r^e indicate the entry cost (in composite-input units), and let M_r denote the number of entered firms in region r . Then each of the M_r firms incur the nominal entry payment $c_r f_r^e$, although this payment is spread across time (as there is a nonzero probability that the firm will survive beyond the current period).

Now consider the input technology for a firm from region r that finds it profitable to sell into market s . Let f_{rs} indicate the recurring fixed cost of operating on the r - s link, and let φ represent the firm-specific measure of productivity. A firm supplying q units to s uses

$$f_{rs} + \frac{q}{\varphi}$$

units of inputs. Higher productivity (higher φ) indicates lower marginal cost.

Once a firm incurs the *entry* cost, f_r^e , it is sunk and has no bearing on the firm's decision to operate in a given bilateral market. The profits earned by infra-marginal firms in the bilateral

¹The weighted average productivity is given by

$$\tilde{\varphi}_{rs} = \left[\int_0^\infty \varphi_{rs}^{\sigma-1} \mu_{rs}(\varphi_{rs}) d\varphi_{rs} \right]^{\frac{1}{\sigma-1}},$$

where $\mu_{rs}(\varphi_{rs})$ is the distribution of productivities of each of the N_{rs} firms.

² One problem we face in reconciling the empirical model with the established theory is the discrepancy between gross expenditures and value added, because of intermediate inputs. To simplify we assume that intermediate inputs are purchases of the aggregate consumption commodity. Gross expenditures, E_s , less the value of intermediate inputs (to all industries) equals regional income.

markets do, however, give firms the incentive to incur the entry cost in the first place. There is no restriction on the markets that can be served by a given member of M_r . If a firm's productivity is high enough such that it is profitable to operate in multiple markets it can replicate itself, maintaining the same marginal cost but incurring the fixed operating cost, f_{rs} , for each of the s markets it serves.

The small firms, facing constant-elasticity demand for their differentiated products, follow the usual optimal markup rule. Let τ_{rs} indicate the *iceberg* transport-cost factor, and let t_{rs} indicate the tariff. Focusing on the average firm (with productivity draw $\tilde{\varphi}_{rs}$) shipping from r to s , optimal (gross) pricing is given by

$$\tilde{p}_{rs} = \frac{c_r \tau_{rs} (1 + t_{rs})}{\rho \tilde{\varphi}_{rs}}. \quad (7)$$

2.3 Operation, Entry, and the Average Firm

We assume that each of the M_r firms choosing to incur the entry cost receive their firm-specific productivity draw φ from a Pareto distribution with probability density

$$g(\varphi) = \frac{a}{\varphi} \left(\frac{b}{\varphi} \right)^a; \quad (8)$$

and cumulative distribution

$$G(\varphi) = 1 - \left(\frac{b}{\varphi} \right)^a, \quad (9)$$

where a is the shape parameter and b is the minimum productivity.

Considering the fixed cost of operating, f_{rs} , on the r - s link there will be some level of productivity, φ_{rs}^* , at which operating profits are zero. All firms drawing a φ above φ_{rs}^* will serve the s market, and firms drawing a φ below φ_{rs}^* will not. A firm drawing φ_{rs}^* is the marginal firm from r supplying region s . This leads us to the fundamental condition which determines the number of operating firms in a given market, N_{rs} . Let $r(\varphi) = p(\varphi)q(\varphi)$ indicate the gross-of-tariff firm revenues as a function of the draw φ . Zero profits for the marginal firm requires

$$c_r f_{rs} = \frac{r(\varphi_{rs}^*)}{\sigma(1 + t_{rs})}. \quad (10)$$

We would like, however, to define this condition in terms of the average operating firm rather than the marginal firm.

Following Melitz (2003) we define $\tilde{\varphi}$ as the productivity of a firm pricing at \tilde{p} , such that our simplification in equation (4) is consistent. The probability that a firm will operate is $1 - G(\varphi^*)$, so we find the CES weighted average productivity,

$$\tilde{\varphi}_{rs} = \left[\frac{1}{1 - G(\varphi_{rs}^*)} \int_{\varphi_{rs}^*}^{\infty} \varphi^{\sigma-1} g(\varphi) d\varphi \right]^{\frac{1}{\sigma-1}}. \quad (11)$$

Using the Pareto distribution this becomes

$$\tilde{\varphi}_{rs} = \left[\frac{a}{a+1-\sigma} \right]^{\frac{1}{\sigma-1}} \varphi_{rs}^*. \quad (12)$$

Again, following Melitz (2003) optimal firm pricing and the input technology ($f_{rs} + q/\varphi$) we establish the relationship between the revenues of firms with different productivity draws:

$$\frac{r(\varphi_1)}{r(\varphi_2)} = \left(\frac{\varphi_1}{\varphi_2} \right)^{\sigma-1}. \quad (13)$$

Using (12) and (13) to simplify (10) we derive the zero cutoff profit condition in terms of average-firm revenues and the parameters:

$$c_r f_{rs} + \pi_{rs}^c = \frac{\tilde{p}_{rs} \tilde{q}_{rs}}{(1+t_{rs})} \frac{(a+1-\sigma)}{a\sigma}. \quad (14)$$

The variable π_{rs}^c is introduced to track any extra profits that are generated when each of the M_r firms operate in a market. We term these profits *capacity rents*. The value of π_{rs}^c must be zero in a steady-state, but if M_r is sticky a policy shock might lead to $N_{rs} = M_r$ indicating rents.³

Next we turn to the entry condition which determines the mass of firms, M_r . Firm entry requires a one-time payment of f_r^e , and entered firms face a probability δ in each future period of a *bad* shock, which forces exit. In a steady-state equilibrium δM_r firms are lost in a given period so total entry payments in that period must be $c_r \delta M_r f_r^e$. From an individual firm's perspective the *annualized* flow of entry payments is $c_r \delta f_r^e$.

Assuming risk neutrality and no discounting, firms enter to the point that expected operating profits equal the entry payment. A firm from r operating in market s can expect to earn the average profit in that market:

$$\tilde{\pi}_{rs} = \frac{\tilde{p}_{rs} \tilde{q}_{rs}}{\sigma(1+t_{rs})} - c_r f_{rs}. \quad (15)$$

Using the zero cutoff profit condition to substitute out the operating fixed cost this reduces to

$$\tilde{\pi}_{rs} = \frac{\tilde{p}_{rs} \tilde{q}_{rs}}{(1+t_{rs})} \frac{(\sigma-1)}{a\sigma}. \quad (16)$$

The probability that a firm in r will service the s market is simply given by the ratio of N_{rs}/M_r .⁴ Setting the firm-level entry-payment flow equal to the expected profits from each potential market gives us the free entry condition

$$c_r \delta f_r^e = \sum_s \frac{N_{rs}}{M_r} \frac{\tilde{p}_{rs} \tilde{q}_{rs}}{(1+t_{rs})} \frac{(\sigma-1)}{a\sigma} \quad (17)$$

³ The value of π_{rs}^c is determined by the variational-inequality presented in the next section, equation (20). We are only concerned with steady-state equilibria (where $\pi_{rs}^c = 0$) in this study, but we found that the computational model performed better with the extended condition, which avoids numeric moves where $N_{rs} > M_r$.

⁴ In Melitz (2003) the probability that a firm will operate, which equals the fraction of operating firms in equilibrium, is presented as $1 - G(\varphi^*)$.

Table 1: PE module; multiregion heterogeneous-firms partial-equilibrium

Equilibrium Condition	(Equation)	Associated Variable	Dimensions
Zero cutoff profits (ZCP)	(14)	N_{rs} : Number of operating firms	$R \times R$
Free entry (FE)	(17)	M_r : Mass of firms taking a draw	R
Dixit-Stiglitz preferences	(4)	P_r : Price index	R
Firm-level demand	(5)	\tilde{q}_{rs} : Average-firm quantity	$R \times R$
Firm-level pricing	(7)	\tilde{p}_{rs} : Average-firm price	$R \times R$
CES wtd. Average φ	(12)	$\tilde{\varphi}_{rs}$: Average-firm productivity	$R \times R$
Pareto dist. Marginal φ	(18)	φ_{rs}^* : Marginal-firm productivity	$R \times R$
Input-market clearance	(19)	c_r : Composite-input price	R
Capacity constraint	(20)	π_{rs}^c : Capacity rents	$R \times R$
Total Dimensions:			$3R + 6R^2$

which determines the mass of firms, M_r .

Finally we can recover the marginal productivity as a function of the fraction of operating firms, $N_{rs}/M_r = 1 - G(\varphi^*)$. Applying the Pareto distribution and inverting we have

$$\varphi_{rs}^* = \frac{b}{\left(\frac{N_{rs}}{M_r}\right)^{1/a}}. \quad (18)$$

In the following section we formalize a computational model based on the outlined heterogeneous-firms theory.

3 Solution Method

3.1 Partial Equilibrium Module

The exogenous links that make the PE module operational are the expenditure levels in each region, \bar{E}_r , (which establish demand for manufactured goods) and the prices, \bar{c}_r , and quantities, \bar{Y}_r , of the composite inputs to manufacturing. The model needs some flexibility to react to shocks, however, so we assume a constant-elasticity input-supply function centered (each iteration) on the quantity of inputs used by the sector in the general equilibrium (\bar{Y}_r). Input supply is thus $\bar{Y}_r (c_r/\bar{c}_r)^\eta$, where $\eta > 0$ is the elasticity. If the PE model is consistent with the general equilibrium $c_r = \bar{c}_r$, where c_r satisfies the equilibrium conditions in both modules.

Table 1 summarizes the nonlinear conditions in the PE module and establishes the complementarity between equations and associated variable. In addition to the conditions developed in the previous section we add the input-market clearance condition (which determines c_r)

$$\bar{Y}_r \left(\frac{c_r}{\bar{c}_r}\right)^\eta = \delta f_r^e M_r + \sum_s N_{rs} \left(f_{rs} + \frac{\tilde{q}_{rs} \tau_{rs}}{\tilde{\varphi}_{rs}}\right), \quad (19)$$

and the complementary-slack condition for determining capacity rents (π_{rs}^c)

$$M_r - N_{rs} \geq 0; \quad \pi_{rs}^c \geq 0; \quad \pi_{rs}^c (M_r - N_{rs}) = 0. \quad (20)$$

As noted above, in a steady-state equilibrium π_{rs}^c will equal zero, but the computational model benefits from an explicit constraint that prevents numeric moves where $N_{rs} > M_r$.⁵

3.2 General Equilibrium Module

The General Equilibrium Module (GE) is formulated as a standard constant-returns model of world trade in all products. Consumers have preferences over goods differentiated by region of origin (the Armington assumption). Consider the unit expenditure function associated with region- s purchases of goods of type k (we reintroduce the commodity index, $k \in K$, in the general equilibrium):

$$P_{ks} = \psi_{ks} \left[\sum_r \xi_{krs} [(1 + t_{krs}) c_{kr}]^{1-\sigma} \right]^{1/(1-\sigma)} \quad (21)$$

Notice that we define the Armington aggregation directly over the composite of the sector (k) inputs from region r , which trade at a price of c_{kr} . The total and relative productivity parameter ψ_{ks} and ξ_{krs} control the functional calibration. These are the instruments through which the GE module is affected by the PE solution.

Table 2 summarizes the full set of equilibrium conditions in the GE module. First we define the aggregate Cobb-Douglas expenditure function;

$$P_r^E = \prod_k (P_{kr})^{\alpha_k}. \quad (22)$$

The remaining transformation technologies are given by our characterization of the commodity-specific price index, (21), and the composite-input cost function, (6).

Each price (index) has an associated market. Let \bar{e}_{jr} be the exogenous endowment of factor j in region r . This will equal the quantity demanded;

$$\bar{e}_{jr} = \sum_k \frac{\beta_{jkr} c_{kr} Y_{kr}}{w_j}. \quad (23)$$

In turn the supply of the composite-input activity will equal demand (as derived from the Armington activity):

$$Y_{kr} = \sum_s \xi_{krs} \psi_{ks} A_{ks} \left(\frac{P_{ks}}{(1 + t_{krs}) c_{kr}} \right)^\sigma. \quad (24)$$

Supply of the Armington composite equals gross demand:

$$A_{kr} = \frac{\alpha_k E_r}{P_{kr}}. \quad (25)$$

⁵ This also indicates how the model might be extended into an intertemporal context where M_r cannot adjust instantaneously.

Table 2: GE module: multiregion constant-returns general equilibrium

Equilibrium Condition	(Equation)	Associated Variable	Dimensions
Optimality conditions:			
Expenditure function	(22)	E_r : Total Expenditures	R
Zero-profits Armington Activity	(21)	A_{kr} : Armington activity level	$K \times R$
Zero-profits Production	(6)	Y_{kr} : Sectoral Output	$K \times R$
Market-clearance conditions:			
Input markets	(23)	w_{jr} : Factor price by type	$J \times R$
Product markets	(24)	c_{kr} : Composite-input price	$K \times R$
Armington-Composite markets	(25)	P_{kr} : Price index by commodity	$K \times R$
Gross supply and demand	(26)	P_r^E : Aggregate price index	R
Income balance:			
Final demand	(27)	W_r : Hicks welfare index	R
Factor income + tariff revenue	(28)	I_r : Nominal income	R
Total Dimensions			$4R$ $+4(K \times R)$ $+(J \times R)$

Gross expenditures equal the value of final demand plus the value of intermediate use:

$$E_r = P_r^E W_r + \sum_k \beta_{kr}^E c_{kr} Y_{kr}. \quad (26)$$

The welfare index is calculated directly from the ratio of income to the price of the aggregate commodity:

$$W_r = \frac{I_r}{P_r^E}. \quad (27)$$

Income in a region equals the value of factor endowments plus tariff revenues:

$$I_s = \sum_j w_{js} \bar{e}_{js} + \sum_k \sum_r t_{krs} \xi_{krs} \psi_{ks} A_{ks} \left(\frac{P_{ks}}{(1 + t_{krs}) c_{kr}} \right)^\sigma. \quad (28)$$

3.3 Full Solution

The challenge to arriving at a fully consistent general equilibrium is to adjust the ψ_{ks} and ξ_{krs} (where $k = \text{Manufacturing}$) such that aggregate supply of the manufacturing composite and relative demands for inputs are consistent with the PE solution. Changes in the number of firms will indicate total and relative productivity changes in the composite inputs embodied in the trade flow. Once these productivity changes are incorporated the GE module can be solved to find a new set of gross expenditures, input prices, and input quantities to pass back to the PE module. At the global solution there are no additional adjustments in the ψ and ξ , and the common variables across the PE and GE modules have the same solution values.

In passing information from the PE module to the GE, we first establish total factor productivity by relating ψ_{ks} to the ratio of the price index and the total value of manufacturing expenditures:

$$\psi_{ks} = \frac{P_{ks}}{\alpha_{ks} E_s}. \quad (29)$$

An increase in the total number of varieties consumed in region s , $(\sum_r N_{rs})$, indicate a decrease in the computed PE price index. Through equation (29) the Dixit-Stiglitz effect is carried over to the GE Armington technology. Finding the relative productivity changes involves finding the set of ξ_{krs} that are consistent with the value of input demands in the PE module. All payments are to the factors so we must have:

$$c_{kr} \xi_{krs} \psi_{ks} A_{ks} \left(\frac{P_{ks}}{(1+t_{krs})c_{kr}} \right)^\sigma = N_{krs} \tilde{p}_{krs} \tilde{q}_{krs}. \quad (30)$$

Solving for ξ_{krs} , and noting that A_{ks} is the inverse of the new ψ , we have

$$\xi_{krs} = \frac{N_{krs} \tilde{p}_{krs} \tilde{q}_{krs}}{c_{kr} \left(\frac{P_{ks}}{(1+t_{krs})c_{kr}} \right)^\sigma} \quad (31)$$

The recalibration of the constant returns GE (ψ_{ks} and ξ_{krs}) based on the heterogeneous-firms PE solution, and the subsequent recalculation of the \bar{E}_r , \bar{c}_r , and \bar{Y}_r , has proven to be a robust solution method. The iterative procedure stops at the point that all variables common to the PE and the GE are consistent and there is no further recalibration indicated.

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