Solution to Prob. Set on Consumption and Savings

1. It discourages saving because it actually encourages the individual to run down his/her savings before applying for a college loan. That way they get a bigger grant!

2. (a) Here $y_1 = 90$, $y_2 = 110$, $r = 0.1$. His initial real wealth is $a = 20$. So, the present value of his lifetime resources (income), PVLR or PVLI = $y_1 + (1/(1 + r))y_2 + a = 210$; (b) Use the intertemporal budget constraint from the lectures. That is, since PVLR = PVLC, we know that $c_1 + (1/(1 + r))c_2 = PVLR = 210$. Since $r = 0.1$, the equation asked for is $c_1 + (1/1.1)c_1 = 210$. Check that the horizontal intercept is 210 and the vertical intercept is 231 (if you plot current consumption on the X-axis, and future consumption on the Y-axis); (c-i) Smoothing consumption completely means setting $c_1 = c_2$. Therefore, using $c_1 + (1/1.1)c_2 = 210$, and setting $c_1 = c_2 = c$ (say), gives us $c + (c/1.1) = 210$, or $c = 110$. Therefore, in the current (first) period, he saves $y_1 - c = 90 - 110 = -$20. (c-ii) If current income rises by 11, check that his PVLR rises to 221. Since, $c + (c/1.10) = 221$ now, c will have gone up to 115.76. His saving would then be 101- 115.76 = $-14.76. (c-iii) Here $y_2$ increases by 11, so PVLR rises by $11/1.10 = 10$. So new PVLR = 220. Again, $c + (c/1.10) = 220$, and so $c = 115.24$, and saving would be $90$ minus that, or $-25.24$; (c-iv) An increase in wealth $a$ looks like an increase in current income and so the answer will be the same as in (d). However, savings will be 90 - 115.76 = $-25.76$.

3. Rory Gilmore knows she will live for two time periods. She cares about her young-age consumption ($c_1$) and her old-age consumption ($c_2$) in a manner described by the following utility function:

$$U(c_1, c_2) = c_1^\alpha c_2^{1-\alpha}$$

where $0 < \alpha < 1$. Her income when young is $w_1 > 0$ and her income when old is $w_1 > 0$. The interest rate in the economy is $r\%$ (i.e., if she puts in $1 in a savings account when young, she will get back $1 (1 + r) when she is old). Assume $r > 0$. 

1
(a) \[ c_1 + \frac{c_2}{1+r} = \frac{w_1}{2} + \frac{w_1}{1+r} \]

(b) No. Consider a choice of \( c_1 = \frac{w_1}{2} \) and \( c_2 = w_1 \), a bundle she can clearly afford. You can check that she gets higher utility from this bundle than from choosing \( c_1 = 0 \) and \( c_2 = 0 \). If she consumes \( c_1 = w_1 \), then her budget (use her intertemporal budget constraint) would allow her to consume

\[
c_2 = (1 + r) \left[ \frac{w_1}{2} + \frac{w_1}{1+r} \right] - (1 + r) w_1 = \left( \frac{1}{2} \right) w_1 (1 - r)
\]

this is equal to \( (1 + r)w_1 \) only if \( r < 0 \) which cannot be.

(c) If she consumes nothing when young, and saves everything, she can consume

\[
c_2 = (1 + r) \left[ \frac{w_1}{2} + \frac{w_1}{1+r} \right] = (1 + r) \frac{w_1}{2} + w_1 = w_1 \left[ \frac{(1 + r)}{2} + 1 \right]
\]

when old. This is the maximum she can consume when old.

(d) \[
\max_{(c_1, c_2)} U(c_1, c_2) = c_1^\alpha c_2^{1-\alpha}
\]

subject to the intertemporal budget constraint

\[
c_1 + \frac{c_2}{1+r} = \frac{w_1}{2} + \frac{w_1}{1+r}
\]

\((c_1, c_2) \geq 0\)

(e) Using the notation used in class, her first period income is \( y_1 \) and second period income is \( y_2 \) [in this particular problem, you are told that \( y_1 = \frac{w_1}{2} \), and \( y_2 = w_1 \)]. Now, one way to solve this is to use the condition for optimal consumption choices in your lecture notes, i.e., \( MRS = (1 + r) \).

Here, \( MRS \) is computed as follows:

\[
MRS = \frac{\partial U(c_1, c_2)}{\partial c_1} / \frac{\partial U(c_1, c_2)}{\partial c_2} = \frac{\alpha c_1^{\alpha-1} c_2^{1-\alpha}}{(1-\alpha) c_1^\alpha c_2^{-\alpha}} = (1 + r)
\]
Or,

\[ \frac{\alpha}{(1 - \alpha)} \frac{c^*_2}{c^*_1} = (1 + r) \]

or,

\[ c^*_2 = (1 + r) \frac{(1 - \alpha)}{\alpha} c^*_1 \]  

[1]

this is one equation in \( c^*_1 \) & \( c^*_2 \). The other equation in these variables is the intertemporal budget constraint

\[ c^*_1 + c^*_2 = y_1 + \frac{y_2}{1 + r} \]  

[2]

Use (1) in (2) to get

\[ c^*_1 + \frac{(1 + r)(1 - \alpha) c^*_1}{\alpha} = y_1 + \frac{y_2}{1 + r} \]

\[ \Rightarrow c^*_1 + \frac{(1 - \alpha) c^*_1}{\alpha} = y_1 + \frac{y_2}{1 + r} \]

\[ \Rightarrow c^*_1 = \alpha \left[ y_1 + \frac{y_2}{1 + r} \right] \]  

[3]

and now, using (1) and (3) again,

\[ c^*_2 = (1 + r) \frac{(1 - \alpha)}{\alpha} c^*_1 = (1 + r) \frac{(1 - \alpha)}{\alpha} \left[ y_1 + \frac{y_2}{1 + r} \right] = (1 - \alpha) (1 + r) \left[ y_1 + \frac{y_2}{1 + r} \right] \]

So, her optimal consumption choices are

\[ c^*_1 = \alpha \left[ y_1 + \frac{y_2}{1 + r} \right] \]

\[ c^*_2 = (1 - \alpha) (1 + r) \left[ y_1 + \frac{y_2}{1 + r} \right] \]

Now, substitute \( y_1 = \frac{w_1}{2} \), and \( y_2 = w_1 \) to get

\[ c^*_1 = \alpha w_1 \left[ \frac{1}{2} + \frac{1}{1 + r} \right] \]

\[ c^*_2 = (1 - \alpha) (1 + r) w_1 \left[ \frac{1}{2} + \frac{1}{1 + r} \right] \]  

[4]

(f) Suppose he leaves her an inheritance of \( x \). Then her old-age income will become \( y_2 + x \). This means she can now consume

\[ c^*_2 = (1 - \alpha) (1 + r) \left[ y_1 + \frac{(y_2 + x)}{1 + r} \right] \]

\[ = (1 - \alpha) (1 + r) \left[ \frac{w_1}{2} + \frac{(w_1 + x)}{1 + r} \right] \]
To make her consume exactly $2w_1$ when old, we would have to equate

$$2w_1 = (1 - \alpha) (1 + r) \left[ \frac{w_1}{2} + \frac{(w_1 + x)}{1 + r} \right]$$

and then solve for $x$:

$$x = \frac{1}{2} (1 - r + 3\alpha + \alpha r) \frac{w_1}{1 - \alpha}.$$  

(g)

$$c_1^* = \alpha \left[ y_1 + \frac{w_2}{1 + r} \right]$$  

$$c_2^* = (1 - \alpha) (1 + r) \left[ y_1 + \frac{w_2}{1 + r} \right]$$

now, under the conditions of the problem, $y_1 = 2w_1$, and $y_2 = w_1$. Then, her new choices will be

$$c_1^* = \alpha \left[ 2w_1 + \frac{w_1}{1 + r} \right] = \alpha w_1 \left[ 2 + \frac{1}{1 + r} \right]$$  

$$c_2^* = (1 - \alpha) (1 + r) w_1 \left[ 2 + \frac{1}{1 + r} \right]$$

Comparing your answers to (e) above, it is clear [from (4)] that her consumption in both periods of her life is higher when her young-age income goes up. Why? Because consumption depends on the PVLI [PVLI in part (g) is $2 + \frac{1}{1 + r}$ as opposed to $\frac{1}{2} + \frac{1}{1 + r}$ in part (e)] and the PVLI in part (g) is higher. She spreads the increase in her young-age income to finance higher consumption in both periods.