Part a.

First-period budget constraint is:

\[ C_1 = W_1 - S \ ; \ c_1 \geq 0 \quad (1) \]

Second-period budget constraint is:

\[ C_2 = W_2 + (1+r) \cdot S \ ; \ c_2 \geq 0 \quad (2) \]

Here, \( S \) denotes savings of Gemma in the first period.

Part b.

Use Equation 1 to obtain:

\[ S = W_1 - C_1 \]

Substitute this result into Equation 2:

\[ C_2 = W_2 + (1+r) (W_1 - C_1) \]

Thus,

\[ C_2 = W_2 + (1+r) W_1 - (1+r) C_1 \quad (3) \]
Equation 3 can be rewritten as:

\[ C_2 + (1+r)C_1 = \bar{W}_2 + (1+r)\bar{W}_1 \]

Divide both sides of the above equation by \((1+r)\), to obtain:

\[ \frac{C_2}{1+r} + C_1 = \frac{\bar{W}_2}{1+r} + \bar{W}_1 \]  \(4\)

Equation 4 is Gemma's intertemporal budget constraint. It shows that the present value of Gemma's life-time consumption equals to the present value of her life-time income/wealth.

Part c.
Recall that Gemma's intertemporal budget constraint can be written in form of Equation 3. That is,
\[ c_2 = \bar{w}_2 + (1+r)\bar{w}_1 - (1+r)c_1 \]  

This is an equation of a straight line in \((c_1, c_2)\) space, with the intercept of \([\bar{w}_2 + (1+r)\bar{w}_1]\) and the slope of \(-(1+r)\).

To draw a straight line, one needs to know two points. Let us choose the first point by choosing \(c_1 = 0\). Then, from Equation 3, we have:

\[ c_2 = \bar{w}_2 + (1+r)\bar{w}_1 \]

Similarly, set \(c_2 = 0\) and substitute it into Equation 3 to determine \(c_1\):

\[ 0 = \bar{w}_2 + (1+r)\bar{w}_1 - (1+r)c_1 \]

\[ \Rightarrow (1+r)c_1 = \bar{w}_2 + (1+r)\bar{w}_1 \]
\[ c_4 = \frac{1}{1+r} (\omega_2 + (1+r)\omega_4) \]

and \[ c_4 = \frac{-\omega_2}{1+r} + \omega_4 \]

Thus, we have obtained two points in \((c_4, c_2)\) space.

First point: \[ \begin{cases} c_1 = 0 \\ c_2 = \omega_2 + (1+r)\omega_4 \end{cases} \]

Second point: \[ \begin{cases} c_1 = \frac{-\omega_2}{1+r} + \omega_4 \\ c_2 = 0 \end{cases} \]

We can draw the graph now, by connecting these two points:

Set of Gemma's feasible consumption possibilities.
Consider the first choice of consumption:

\[
\begin{align*}
C_1 &= 0 \\
C_2 &= 0
\end{align*}
\]

If Gemma decides to consume nothing in the first period, she can put all of her first-period income (i.e., $W_1$) into her savings to obtain an additional income of $W_1(1+r)$ in the second period. Thus, she can consume a total of

\[-W_2 + W_1(1+r)\]
in the second period. However, we know from 5 that she actually chooses to consume nothing in the second period. Thus, she is throwing away all of her money. Clearly, this is not rational, and
Gemma will not choose the suggested consumption bundle. To see this analytically, plug zero values of consumption into Gemma’s utility function. Recall that:

$$U(c_1, c_2) = d \log c_1 + (1-d) \log c_2$$

Therefore, Gemma’s utility from zero consumption in both periods will be

$$U(0, 0) = d \log 0 + (1-d) \log 0$$

$$= \log 0 = -\infty$$

Thus, Gemma is “extremely unhappy” with the above choice of the consumption bundle (i.e., she obtains infinitely-large negative (!) utility).
Moreover, consider Gemmy's marginal utility with respect to her first period consumption:

$$MU_1 = \frac{\partial U(c_1, c_2)}{\partial c_1} = \frac{d}{c_1}$$

If $c_1 = 0$, then $MU_1 = +\infty$. This means that even a very small (i.e., marginal) increase in Gemma's first-period consumption will induce an infinitely large, positive change in her utility. Therefore, Gemma will never choose zero consumption in the first period. Similarly, one can show that Gemma will never choose zero consumption for the second period either.

Using the same approach, one can show that Gemma will never choose
to consume \( c_1 = 0 \) and \( c_2 = \omega_2 \)

---

Part f

The maximum amount Greemma can consume during her old age was determined in Part c. In order to maximize her second-period consumption, Greemma has to sacrifice all of her first-period consumption. Therefore, \( c_1 \) should equal to zero.

In Part d, we determined the corresponding maximum value of \( c_2 \):

\[
c_2 = \omega_2 + (1+r)\omega_1
\]

---

Part g

\[
\max_{c_1, c_2} \left\{ d \log c_1 + (1-d) \log c_2 \right\}
\]
subject to \[
\begin{align*}
& c_1 \geq 0, \\
& c_2 \geq 0 \\
& c_1 + \frac{c_2}{1+r} = \omega_1 + \frac{1}{1+r} \omega_2
\end{align*}
\]

We can use Gemma’s per-period budget constraints from Part a (i.e., Equations 4 and 2) to reformulate the optimization problem as follows:

\[
\text{Max}_{s} \left[ a \log (\omega_1 - s) + (1-d) \log (\omega_2 + (1+r)s) \right]
\]

subject to:
\[
\begin{align*}
& c_1 = \omega_1 - s \geq 0 \\
& c_2 = \omega_2 + (1+r)s \geq 0
\end{align*}
\]

This is a simplified form of the original optimization problem, since we have to choose only one
parameter here (i.e., $S$) instead of two parameters (i.e., $c_1$ and $c_2$) of the original model. Solving this simplified model, should provide us with the optimal value of the savings function $S^*$. Using this value, we will obtain the optimal values of Gemma's per-period consumption from her per-period budget constraints.

**Part h**

\[
\max_S \left[ d \log (W_1 - S) + (1-d) \log (W_2 + (1+r)S) \right]
\]

subject to

\[
\begin{cases}
  c_1 = W_1 - S \geq 0 \\
  c_2 = W_2 + (1+r)S \geq 0
\end{cases}
\]

F.O.C.: \[
\frac{dU}{ds} = -\frac{d}{W_1 - S^*} + \frac{(1-d)(1+r)}{W_2 + (1+r)S^*} = 0
\]
\[
\frac{\alpha}{\bar{W}_1 - S^*} = \frac{(1-\alpha)(1+r)}{\bar{W}_2 + (1+r)S^*}
\]

\[
\Rightarrow \quad \alpha \left[ \bar{W}_2 + (1+r)S^* \right] = (1-\alpha)(1+r)(\bar{W}_1 - S^*)
\]

\[
\Rightarrow \quad \alpha \bar{W}_2 + \alpha (1+r)S^* = (1-\alpha)(1+r)\bar{W}_1 - (1-\alpha)(1+r)S^*
\]

\[
\Rightarrow \quad \alpha (1+r)S^* + (1-\alpha)(1+r)S^* = (1-\alpha)(1+r)\bar{W}_1 - \alpha \bar{W}_2
\]

Taking the term \((1+r)S^*\) outside of the parenthesis, we obtain:

\[
(1+r)S^* \cdot (\alpha + (1-\alpha)) = (1-\alpha)(1+r)\bar{W}_1 - \alpha \bar{W}_2
\]

\[
\Rightarrow \quad (1+r)S^* = (1-\alpha)(1+r)\bar{W}_1 - \alpha \bar{W}_2
\]

\[
\Rightarrow \quad S^* = (1-\alpha)\bar{W}_1 - \frac{\alpha}{1+r} \bar{W}_2
\]
Using the above result together with the per-period budget constraints, we obtain the optimal values of Gemma's per-period consumption:

\[ c_1^* = \overline{w}_1 - s^* \]

\[ = \overline{w}_1 - \left[ (1-d)\overline{w}_1 - \frac{d}{1+r} \overline{w}_2 \right] \]

\[ = \overline{w}_1 - (1-d)\overline{w}_1 + \frac{d}{1+r} \overline{w}_2 \]

\[ \Rightarrow c_1^* = \frac{d}{1+r} \overline{w}_1 + \frac{d}{1+r} \overline{w}_2 \]

\( c_1^* > 0 \), since \( d, \overline{w}_1, r, \overline{w}_2 \) are all greater than zero.

Similarly,

\[ c_2^* = \overline{w}_2 + (1+r) s^* \]

\[ = \overline{w}_2 + (1+r) \left[ (1-d)\overline{w}_1 - \frac{d}{1+r} \overline{w}_2 \right] \]
\[ C^*_2 = \omega_2 + (1+r)(1-\delta)\omega_1 - 2\omega_2 \]

\[ = \omega_2(1-\delta) + (1+r)(1-\delta)\omega_1 \]

Thus,

\[ C^*_2 = (1-\delta)(1+r)\omega_1 + (1-\delta)\omega_2 \quad (8) \]

Again, it is easy to see that

\[ C^*_2 > 0 \]

Part i

Using Equations 7 and 8, we can compute the new consumption choice of Gemma in terms of her new wealth.

\[ C^*_{1, \text{new}} = \alpha \omega_{1, \text{new}} + \frac{\alpha}{1+r} \omega_{2, \text{new}} \]

Where, \( C^*_{1, \text{new}} \) denotes Gemma's new
The choice of her first-period consumption is:

\[ \begin{cases} \bar{W}_1, \text{new} = \bar{W}_1, & \text{and} \\ \bar{W}_2, \text{new} = 2 \bar{W}_2 \end{cases} \]

\[ \Rightarrow \quad c_1^*, \text{new} = \bar{X} \bar{W}_1 + \frac{\bar{X}}{1+r} \times (2 \bar{W}_2) \]

\[ = \bar{X} \bar{W}_1 + \frac{\bar{X}}{1+r} \bar{W}_2 + \frac{\bar{X}}{1+r} \bar{W}_2 \]

Therefore,

\[ c_1, \text{new} = c_1^* + \frac{\bar{X}}{1+r} \bar{W}_2 \]

\[ c_1^*, \text{new} > c_1 \]

Similarly, we calculate Gemma's new utility maximization choice of her second-period consumption, \( c_2^*, \text{new} \):

\[ c_2^*, \text{new} = (1-\bar{d})(1+r) \bar{W}_1, \text{new} + (1-\bar{d}) \bar{W}_2, \text{new} \]
\[ C^*_{z, new} = (1-d) (1+r) W_1 + (1-d) \cdot 2W_2 \]

Therefore,

\[ C^*_{z, new} = C^*_{z} + (1-d) W_2 \]

\[ C^*_{z, new} > C^*_{z} \]

Thus, the inheritance has increased both-period consumption of Gemma. Gemma does not consume all of the increase in her wealth during her young age. Instead, she uses part of it for her second-period consumption in accordance with her time preferences.
Graphically,

\[ W_2, \text{new} + (1+r)W_1, \text{new} \]

\[ W_2 + (1+r)W_1 \]

\[ \frac{W_2}{1+r} + W_1 \]

\[ \frac{W_2, \text{new}}{1+r} + W_1, \text{new} \]

\[ \text{new budget line} \]

\[ \text{old budget line} \]

---

Part j

We obtained expression for \( S^* \) in Part h (i.e., Equation 6):

\[ S^* = (1-\alpha)W_1 + \frac{\alpha}{1+r}W_2 \] (6)
Part k:

It follows from Equation 6, that

\[ S^* > 0 \] if and only if:

\[ (1-d) \bar{w}_1 - \frac{d}{1+r} \bar{w}_2 > 0 \]

Or, equivalently:

\[ (1-d) \bar{w}_1 > \frac{d}{1+r} \bar{w}_2 \]

Or,

\[ \bar{w}_1 > \frac{d}{1-d} \cdot \frac{\bar{w}_2}{1+r} \quad \text{(7)} \]

Thus, \( S^* > 0 \) if and only if:

\[ \bar{w}_1 > \frac{d}{1-d} \cdot \frac{\bar{w}_2}{1+r} \]

Example: \( d = \frac{1}{2} \); \( \bar{w}_1 = \bar{w}_2 = 1 \); \( r = 10\% \)
Then,

\[
\frac{d}{1-d} \cdot \frac{W_2}{1+r} = \frac{1/2}{1-1/2} \cdot \frac{1}{1+0.1}
\]

\[
= \frac{1/2}{1/2} \cdot \frac{1}{1.1}
= 0.9
\]

Thus

\[
W_1 = \bullet \quad 1 > 0.9 = \frac{d}{1-d} \cdot \frac{W_2}{1+r}
\]

and

\[
W_1 > \frac{d}{1-d} \cdot \frac{W_2}{1+r}
\]

Therefore,

\[
S^* > 0.
\]

**Question:** Can \( S^* \) be positive if \( W_2 > W_1 \)?

We need to find such \( W_2 \) that

a) \( W_2 > W_1 \), and

b) \( W_1 > \frac{d}{1-d} \cdot \frac{W_2}{1+r} \) (Equation 7)
In order to satisfy the above inequalities, we must have:

\[ w_2 > \frac{d}{1-d} \frac{w_2}{1+r} \]

\[ \iff 1 > \frac{d}{1-d} \cdot \frac{1}{1+r} \]

\[ \iff (1+r) > \frac{d}{1-d} \]

**Example:** \( r = 10\% ; \ d = 1/2 ; \)

\[ w_1 = 0.95 ; \ w_2 = 1 ; \]

\[ \implies (1+r) = 1.1 > 1 = \frac{d}{1-d} \]

Thus, Inequality 8 is valid.
\[ W_1 = 0.95 > 0.9 = \frac{d}{1-d} \cdot \frac{W_2}{1+r} \]

Thus, Inequality 7 is valid, and savings are positive:

\[ s^* > 0 \]

Finally,

\[ W_2 = 1 > 0.95 = W_1. \]

It is possible to have positive savings even if the second-period wealth/income is greater than that of the first period. Savings are determined through the combination of (i) time preferences, (ii) initial distribution of wealth, and (iii) market interest rate.
Recall that

\[ s^* = (1-\delta) \bar{w}_1 - \frac{\delta}{1+r} \bar{w}_2 \]

Let us denote \((1+r)\) by \(R\)

\[ R = (1+r) \]

\((R\) is gross interest rate\)

Then,

\[ s^* = (1-\delta) \bar{w}_1 - \frac{\delta \bar{w}_2}{R} \]

In order to find how \(s^*\) changes with \(R\), we need to find the following derivative.

\[ \frac{ds^*}{dR} = \frac{d\bar{w}_2}{R^2} > 0 \]
Thus, $S^*$ is an increasing function of $R$.

Notice that $R$ is an increasing function of $r$. The greater the net interest rate, the greater the gross interest rate. Therefore, $S^*$ is also an increasing function of $r$. A larger interest rate means that you will consume more tomorrow. For each unit saved today, you will have larger return tomorrow. An increase in savings caused by an increase in the interest rate is called a "substitution effect."