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Dynamic efficiency in the two-sector overlapping generations model

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Abstract

This paper examines dynamic efficiency in the context of a two-sector overlapping generations model. First, conditions for dynamic efficiency in a centrally planned economy are derived. Then, in a competitive environment, the implications of dynamic (in)efficiency for the steady state relative price and steady state welfare are demonstrated. For the special case of a log-linear world, the golden rule savings rate is identified along with restrictions on parameters that yield dynamically efficient steady states. The results are further demonstrated via a welfare analysis of a simple tax/subsidy scheme.

JEL classification: O41; E13; D9

Keywords: Dynamic efficiency; Two-sector; Overlapping generations

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1. Introduction

It has been known since Diamond (1965) that a competitive equilibrium in the one-sector overlapping generations (OG) model with productive capital may be dynamically inefficient,[1] and by now it is equally well-known that the welfare effects

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[1] In the one-sector neoclassical growth model, a steady state is said to be dynamically inefficient when a Pareto improvement is made possible via a reduction in capital accumulation. Galor and Ryder (1991)

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associated with a variety of economic events will, in the one-sector OG model, depend at least in part on these efficiency properties.[2] As of yet, however, there has been little discussion regarding the conditions for dynamic efficiency in the two-sector OG model though, clearly, such conditions are as essential to the latter construct as they are to the former. It can be anticipated, therefore, that steady state welfare analyses conducted within two-sector OG framework will have results that depend upon on whether or not initial steady states are dynamically inefficient, as well as on whether or not the event being analyzed has served to mitigate or exacerbate that (in)efficiency. In contrast to the one-sector model, however, these effects will be indirectly channeled through changes in the relative price of the two produced goods. This and other less conspicuous considerations imply that properties familiar from the one-sector context will require a modified representation in the two-sector framework. This paper pursues that representation and in doing so provides a necessary building block for a wide range of welfare analyses that could potentially be undertaken using the two-sector OG model.

More specifically, this paper formally demonstrates requirements for and implications of dynamic efficiency in a generally specified two-sector OG model.[3] First, the paper establishes the central condition for dynamic efficiency in a centrally planned economy. Then, the implications of dynamic (in)efficiency for steady state welfare in a competitive setting are also established. With the assumption that the investment good is capital-intensive, it is shown that steady state welfare is decreasing (increasing) in the relative price of capital when the initial steady state price exceeds (is exceeded by) that associated with the golden rule. Under the same assumption, it is shown that it is equivalent to state that steady state welfare is increasing (decreasing) in the steady state capital–labor ratio when the latter is initially below (above) the capital–labor ratio that defines the golden rule. An analogous result is also obtained with reference to the return on capital and its golden rule value. Put simply, with regard to either the steady state return on capital, relative price of capital or capital–labor ratio, welfare is improved by an increased proximity to the golden rule value, whether that be approached from above or below, and vice versa. Other, well-known implications of dynamic efficiency are also

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(*footnote continued*)
describe technological conditions that are sufficient for dynamic efficiency in the one-sector OG model with productive capital.

[2] For example, Diamond (1965), Tirole (1985) and Rhee (1991) examine the respective roles for public debt, asset bubbles and land. Buiter (1981) conducts a welfare analysis of international financial openness. Galor (1986), (1988), and (1994) analyze the respective welfare effects associated with labor migration, Hicks-neutral technical progress, and tariffs in a small country. Matsuyama (1991) explores the possibility of immiserizing growth as a result of technological progress.

[3] This paper does limit its focus to the case of a globally unique perfect foresight equilibrium as this is the most likely setting for subsequent applied research that chooses to make use of the two-sector OG model. Galor (1992) provides sufficient conditions for existence of such an equilibrium when, as below, one sector produces consumption goods and the other produces investment goods. It is interesting to note that Farmer and Wendner (2003) have more recently explored the existence and stability properties of a two-sector OG model that utilizes heterogeneous, rather than homogeneous, capital. That is, both sectors are assumed to produce investment goods and only one sector also produces consumption goods.

reaffirmed in this extended setting.[4] Perhaps the most useful derivations in the paper, however, are those associated with the special case of log-linear utility and production functions. In addition to explicit derivations of the golden rule price and capital–labor ratio, the savings rate that delivers the golden rule is identified along with restrictions on parameters that yield dynamically efficient steady states. These results enable subsequent applied research to include or rule-out possibilities for dynamic inefficiency via simple parameter restrictions.

The paper is organized as follows. Section 2 describes the essentials of a standard two-sector OG environment. Section 3 solves the problem of a central planner whose objective is to maximize the utility of each steady state generation subject only to the constraints on resources and technology and the additional requirement of identical consumption profiles for each generation. From the solution to this problem, the conditions under which the golden rule capital–labor ratio may be obtained are identified. Section 4 examines the same environment from a competitive standpoint and demonstrates the relationship between the efficiency properties of the steady state relative price, return on capital ownership and/or capital–labor ratio and the utility of economic agents living in the steady state. Section 5 examines the special case of a log-linear world. Section 6 considers an application of the results to a simple, government imposed tax/subsidy scheme. Section 7 concludes.

2. The environment

Consider a world which is populated in each period $t > 1$ by two overlapping generations of two-period lived individuals. Generation t is comprised of individuals born at date t who are regarded as 'young' at date t and 'old' at date $t+1$. The world population at date t is therefore the composite of the young members of generation t and the old members of generation $t-1$. At $t=1$, the initial period, the world is also populated by two overlapping generations, the young of generation 1 and the initial old, members of generation 0, who are assumed to live only during the single period $t=1$.

At any date, only the young are endowed with labor, each in possession of a single unit. Let L_t denote the number of young and the endowment of labor at date t and further assume that the exogenous rate of population growth is governed by $L_t = (1+n)L_{t-1}, n \geqslant 0$, at each date. The population at date t is therefore equal to $\bar{L}_t = L_t + L_{t-1} = L_t(2+n)/(1+n)$. The amount of capital available at date t is denoted by K_t and the initial capital stock, K_1, is owned by the old members of generation 0. The capital stock evolves according to $K_{t+1} = (1-\delta)K_t + I_t$, where $\delta \in [0,1]$ is the rate of depreciation applicable to existing capital at date t and I_t is period t gross investment in new capital. The capital–labor ratio, $k_t \equiv K_t/L_t$, summarizes the resources available to the world economy at date t.

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[4] It is readily acknowledged that the derivations below build upon the approach and methods utilized in the previous literature related to two-sector dynamic models, OG models and dynamic efficiency in one-sector models. Accordingly, some aspects of the presentation might be familiar to knowledgeable readers. For a brief account of the two-sector OG model, see Azariadis (1993).

2.1. Consumption

Let c_t^{y} and c_{t+1}^{o} denote consumption during youth and old age, respectively, for a member of generation t. Then, aggregate consumption in period t is $C_t = [c_t^{\mathrm{y}} L_t + c_t^{\mathrm{o}} L_{t-1}] = [c_t^{\mathrm{y}} + c_t^{\mathrm{o}}/(1+n)]L_t$; consumption per capita is thus $C_t/\bar{L}_t = [c_t^{\mathrm{y}} + c_t^{\mathrm{o}}/(1+n)]L_t/\bar{L}_t = [(1+n)/(2+n)]C_t/L_t$, a constant multiple of consumption per worker. Thus, there is an equivalence between the maximization of consumption per worker and consumption per capita.

Individuals are assumed to maximize the utility derived from consumption in both their youth and old age. Let $U(c_t^{\mathrm{y}}, c_{t+1}^{\mathrm{o}})$ denote lifetime utility for a member of generation t. It is assumed that U is twice continuously differentiable, strictly quasi-concave and increasing in c_t^{y} and c_{t+1}^{o}. Letting U_i denote the partial derivative of the utility function with respect to the ith argument, it is also assumed that $U_1(0, c_{t+1}^{\mathrm{o}}) = U_2(c_t^{\mathrm{y}}, 0) = \infty$, $U_1(\infty, c_{t+1}^{\mathrm{o}}) = U_2(c_t^{\mathrm{y}}, \infty) = 0$ and that old age consumption is a normal good.[5]

2.2. Production

As in Galor (1992), two goods – a consumption good and an investment good (physical capital) – are produced at each date using the available capital and labor resources and linearly homogeneous production functions. For sector i, $i = 1, 2$, let f_i denote the technology when expressed in intensive form, k_{it} the date t factor intensity ratio and l_{it} the fraction of the working population employed at date t. The f_i are assumed to be twice continuously differentiable and satisfy, for $i = 1, 2$: $f_i(0) = 0$, $f_i' > 0$, $f_i'' < 0$, $f_i'(0) = \infty$, $f_i'(\infty) = 0$. Consumption per worker at date t is thus given by $C_t/L_t = l_{1t} f_1(k_{1t})$, and date t gross investment per worker is given by $I_t/L_t = l_{2t} f_2(k_{2t})$. If expressed in per capita terms, both terms would need to be premultiplied by the constant $(1+n)/(2+n)$. Again, for both goods, there is an equivalence between maximizing production per capita at any date and maximizing production per worker at that date. The capital–labor ratio then evolves according to

$$(1+n)k_{t+1} = (1-\delta)k_t + l_{2t} f_2(k_{2t}). \tag{1}$$

Date t resource constraints are given by $l_{1t} + l_{2t} = 1$, and $l_{1t}k_{1t} + l_{2t}k_{2t} = k_t$. A more useful representation of this pair of constraints is given by

$$\begin{aligned} l_{1t} &= (k_t - k_{2t})/(k_{1t} - k_{2t}), \\ l_{2t} &= (k_{1t} - k_t)/(k_{1t} - k_{2t}). \end{aligned} \tag{2}$$

3. Centrally planned economy

A first step in identifying dynamically inefficient steady states involves the characterization of the golden rule allocation; that is, the steady state allocation

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[5] Old age consumption is a normal good if $U_1 U_{12} > U_2 U_{11}$ for $c_t^{\mathrm{y}}, c_{t+1}^{\mathrm{o}} > 0$.

chosen by a central planner to maximize the utility of each individual living in the steady state, subject to the condition that all generations have identical consumption profiles and utility levels, and while adhering to the economy's resource and technology constraints (see Diamond, 1965; Galor and Ryder, 1991).[6] Thus, the central planner must choose nonnegative values for c^{y}, c^{o}, k, l_1, l_2, k_1, and k_2 (where the removal of time subscripts, here and henceforth, indicates that a steady state is under consideration) to maximize $U(c^{\mathrm{y}}, c^{\mathrm{o}})$ subject to

$$c^{\mathrm{y}} + \frac{c^{\mathrm{o}}}{1+n} = l_1 f_1(k_1) \tag{3}$$

and the steady state versions of (1) and (2). The left-hand side of the above condition is steady state consumption per worker at a point of time in the planned economy whereas the right-hand side is the steady state production of the consumption good per worker at the same point in time. The latter trio of constraints can be substituted into the right-hand side of (3) and, as then becomes apparent, the planning problem can be solved in parts: the central planner maximizes the production of consumption goods per worker and separately decides upon a distribution of these goods across the two living generations.

Beginning with the first of these objectives, for a given $k>0$, the central planner must choose nonnegative l_i, k_i, $i = 1, 2$, to maximize $l_1 f_1(k_1)$ subject to $l_1 + l_2 = 1$, $l_1 k_1 + l_2 k_2 = k$, $l_2 f_2(k_2) = (n+\delta)k$. It will be useful to note that the Kuhn–Tucker conditions for the associated Lagrangian, $\mathscr{L}$, require that at the optimum,

$$\frac{f_1 - k_1 f_1'}{f_1'} = \frac{f_2 - k_2 f_2'}{f_2'}. \tag{4}$$

For a given $k>0$, let $x_1(k) \equiv l_1(k) f_1(k_1(k))$ denote the value of the objective function when evaluated at the optimal distribution of factors across sectors. The central planner must further identify the particular steady state capital–labor ratio that maximizes consumption per worker. By the envelope theorem,

$$\frac{\mathrm{d}x_1(k)}{\mathrm{d}k} = \frac{\partial \mathscr{L}}{\partial k} = \frac{f_1'}{f_2'}(f_2' - (n+\delta)).$$

Thus, steady state consumption per worker is increasing, decreasing or constant as f_2' is greater, less than, or equal to $(n+\delta)$. By the properties of the technology, steady state consumption per worker is maximized when

$$f_2'(k_2(k)) = n + \delta. \tag{5}$$

Though this result resembles the familiar condition for the golden rule in one-sector settings, it should be cautioned that this is a requirement of the factor intensity ratio which, unlike the one-sector model, is differentiated from the economy-wide capital–labor ratio. Since f_2' is a monotone function, there is a unique golden rule factor intensity in the investment good industry. Furthermore, it can be shown that

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[6] This definition implicitly assumes that the initial capital–labor ratio is not the steady state capital–labor ratio; else, the initial old must be explicitly excluded from the list of those whose utility is being maximized.

$k_2'(k) > 0$ at an interior solution to the maximization problem,[7] implying that there is only one value of the aggregate capital–labor ratio consistent with that factor intensity ratio. That value, denoted k^g, defines the golden rule capital–labor ratio in the two-sector setting.

Returning to the distribution problem, with k^g now chosen by the central planner, the set of feasible steady state allocations of consumption to the young and old generations is given by $c^y + c^o/(1+n) = x_1(k^g)$. The central planner determines the optimal distribution of produced consumption goods across the two living generations by choosing c^y and c^o to satisfy this constraint and also the first-order conditions associated with utility maximization,

$$U_1(c^y, c^o) = (1+n)U_2(c^y, c^o).$$

Now, for capital–labor ratios exceeding the golden rule value, it follows from the properties of the investment good technology that $f_2' < n + \delta$; in this case, the steady state is characterized by over-investment and decreased consumption per capita relative to the golden rule. Similarly, for capital–labor ratios falling below the golden rule, it follows that $f_2' > n + \delta$. Then, relative to the golden rule, the steady state is characterized by under-investment, and again by decreased consumption per capita. Only in the first of these two cases, however, would a reduction in capital accumulation by a central planner enable a Pareto improvement: both the concurrent and all subsequent generations would benefit from a resulting increase in steady state consumption per capita. Thus, for capital–labor ratios exceeding that associated with the golden rule, the steady state is dynamically inefficient.

The following proposition sums the above discussion.

Proposition 1. *The golden rule allocation is identified by the unique steady state capital–labor ratio, k^g, and the uniquely associated factor intensity ratio for the (capital-intensive) investment good industry, $k_2(k^g)$, that together satisfy $f_2'(k_2(k^g)) = n + \delta$. Steady states characterized by $k < k^g$, and therefore by $f_2'(k_2(k^g)) > n + \delta$, are dynamically efficient. Steady states characterized by $k > k^g$, and therefore by $f_2'(k_2(k^g)) < n + \delta$, are dynamically inefficient.*

Other properties traditionally associated with golden rule allocations also follow from steady state capital–labor ratios satisfying (5). This section concludes with a few preliminary implications, that will be made further note of in Section 4.4, where these properties are established. To begin, observe that at the golden rule capital–labor ratio, the factor allocations chosen by a central planner maximizing steady state consumption per capita satisfy

$$k^g = \frac{l_2 f_2}{n+\delta} = \frac{l_2 f_2}{f_2'}. \tag{6}$$

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[7] Specifically, from the constraints of the planning problem, it follows that $k_1 = k[1 - k_2(n+\delta)/f_2][1 - k(n+\delta)/f_2]^{-1}$. Substituting this expression into (4) yields an implicit solution for $k_2(k)$. Total differentiation of (3.2), after the substitution, then yields the stated result.

Solving (4) for f_2/f_2' and then substituting in (6) together with the capital resource constraint, it follows that at the golden rule factor allocations satisfy

$$\begin{aligned} \frac{f_1 - k_1 f_1'}{f_1} &= l_1, \\ \frac{k_1 f_1'}{f_1} &= l_2. \end{aligned} \tag{7}$$

That is, labor's share of output per worker in the consumption industry is equal to the fraction of labor employed in that industry and capital's share of the same industry is equal to the fraction of labor employed in the investment good industry. Similar implications regarding golden rule factor shares in the investment good industry follow from (6) and the capital resource constraint,

$$\begin{aligned} \frac{f_2 - k_2 f_2'}{f_2} &= \frac{l_1 k_1}{k^{\mathrm{g}}}, \\ \frac{k_2 f_2'}{f_2} &= \frac{l_2 k_2}{k^{\mathrm{g}}}, \end{aligned} \tag{8}$$

where the right-hand side of these equations now denote the fraction of capital employed in the consumption and investment good industries, respectively. As will be shown, these golden rule allocations have familiar implications for aggregate consumption and investment in the competitive setting.

4. Competitive equilibrium

In this section, the same economy is reconsidered supposing that instead of a central planner, allocations are achieved via competitive markets. The purpose is to establish the relationship between dynamic efficiency and steady state welfare in a competitive environment.

4.1. Profit maximization

With competitive markets for the factors of production, workers and capital owners will be paid their marginal value products by the industry in which they are employed. Furthermore, resources are intersectorally mobile. Letting w_t and r_t, respectively, denote the wage and rental payments at date t, it therefore follows that

$$\begin{aligned} w_t &= f_1 - k_{1t} f_1' = p_t [f_2 - k_{2t} f_2'], \\ r_t &= f_1' = p_t f_2', \end{aligned} \tag{9}$$

where p_t is the relative price of the investment good at date t and the consumption good serves as numeraire. It follows immediately that

$$p_t = \frac{f_1'}{f_2'} = \frac{f_1 - k_{1t}f_1'}{f_2 - k_{2t}f_2'}. \tag{10}$$

Let $\omega_t \equiv w_t/r_t$, so that for $i = 1, 2$

$$\omega_t = \frac{f_i - k_{it}f_i'}{f_i'}, \tag{11}$$

where the right-hand side is strictly increasing in k_{it}. Then, ω_t is an invertible function of k_{it}, so that the competitive factor intensity ratios can be expressed by $\tilde{k}_i(\omega_t)$, for $i = 1, 2$. It is assumed that $\tilde{k}_2(\omega_t) > \tilde{k}_1(\omega_t)$, for all $\omega_t > 0$, or, that the investment good industry is always the capital-intensive sector.[8] It is shown in the Appendix that ω_t is further determined by p_t. Substituting $k_i(\omega(p_t))$ into (9), yields $w_t = w(p_t)$ and $r_t = r(p_t)$. As is well-known, the assumption that the investment good is capital-intensive further implies that $\mathrm{d}w/\mathrm{d}p < 0$ and $(\mathrm{d}r/\mathrm{d}p)p/r > 1$.

From (2), it then follows that $l_{1t} = \tilde{l}_1(p_t, k_t) \equiv (k_t - \tilde{k}_2(\omega(p_t)))/(\tilde{k}_1(\omega(p_t)) - \tilde{k}_2(\omega(p_t)))$ and $l_{2t} = \tilde{l}_2(p_t, k_t) \equiv (\tilde{k}_1(\omega(p_t)) - k_t)/(\tilde{k}_1(\omega(p_t)) - \tilde{k}_2(\omega(p_t)))$. Per worker output in sector i, $i = 1, 2$, in the competitive setting is then given by $x_{it} = \tilde{x}_i(p_t, k_t) \equiv \tilde{l}_i(p_t, k_t) f_i(\tilde{k}_i(\omega(p_t)))$.

4.2. Utility maximization

While young, individuals sell their labor endowment inelastically to firms thereby generating wage income. This income is either spent on youthful consumption, or is saved to provide asset income to finance consumption when old. All savings are held in the form of physical capital; let k_{t+1}^{y} denote the amount of capital purchased by a member of generation t during period t. This capital is rented to firms during period $t+1$, earning a rental payment at that time, and remaining (undepreciated) capital is sold at the end of $t+1$ to a member of the subsequent generation. Thus, the maximization of lifetime utility subject to the pair of budget constraints imposed during youth and old age for a representative agent of generation t is then given by

$$\begin{aligned} \max_{c^{\mathrm{y}}, c^{\mathrm{o}} \geqslant 0} \quad & U(c_t^{\mathrm{y}}, c_{t+1}^{\mathrm{o}}) \\ \text{s.t.} \quad & c_t^{\mathrm{y}} + p_t k_{t+1}^{\mathrm{y}} = w_t \\ & c_{t+1}^{\mathrm{o}} = [r_{t+1} + (1-\delta)p_{t+1}]k_{t+1}^{\mathrm{y}}. \end{aligned}$$

The latter two constraints can be stated equivalently in the form of a lifetime budget constraint: $c_t^{\mathrm{y}} + c_{t+1}^{\mathrm{o}}/\rho_{t+1} = w_t$ where $\rho_{t+1} = [r_{t+1} + (1-\delta)p_{t+1}]/p_t$ denotes the gross return on capital ownership. Let the savings of generation t be denoted by $s_t = w_t - c_t^{\mathrm{y}} = p_t k_{t+1}^{\mathrm{y}}$ so that old age consumption is given by $c_{t+1}^{\mathrm{o}} = \rho_{t+1}s_t$. All prices are expressed in units of the consumption good.

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[8] This factor intensity assumption is made so as to ensure the existence of a globally unique equilibrium (see footnote 9).

The unique solution to the utility maximization problem can then be expressed in the form of a savings function, $s(w_{t,}\rho_{t+1})$, where the normality of old age consumption implies that $s_w(w_t, \rho_{t+1}) > 0$ for all $w_t, \rho_{t+1} > 0$. It is assumed that $s_\rho(w_t, \rho_{t+1}) > 0$ for all $w_t, \rho_{t+1} > 0$. This savings function implies an indirect utility function for a member of generation t denoted by $V(w_t, \rho_{t+1})$. Note also that date t savings per worker satisfies $L_t^{-1} \sum s(w_{t,}\rho_{t+1}) = s(w_{t,}\rho_{t+1})$.

4.3. Market-clearing

The market for physical capital at date t clears if

$$(1+n)k_{t+1} = \tilde{x}_2(p_t, k_t) + (1-\delta)k_t = \frac{s(w_t, \rho_{t+1})}{p_t}$$

or equivalently if

$$(1+n)k_{t+1} = \tilde{x}_2(p_t, k_t) + (1-\delta)k_t = \frac{s(w(p_t), [r(p_{t+1}) + (1-\delta)p_{t+1}]/p_t)}{p_t}.$$

An equilibrium is then a sequence $\{p_{t.}k_t\}$, given k_0, that satisfies this condition for all $t \geqslant 1$. It is assumed that a nontrivial equilibrium exists and is unique.[9]

A steady state associated with the unique equilibrium is a $p > 0$, $k > 0$ such that

$$(1+n)k = \tilde{x}_2(p, k) + (1-\delta)k = \frac{s(w(p), \rho(p))}{p}, \tag{12}$$

where $\rho(p) \equiv r(p)/p + 1 - \delta$. From the first equality, it will prove convenient to note that $(n+\delta)k = \tilde{x}_2(p, k)$.

4.4. Steady state welfare

The relationship between steady state welfare and dynamic efficiency in the competitive setting is now found by differentiation of the indirect utility function of an agent born to the steady state,

$$\frac{\mathrm{d}V(w, \rho)}{\mathrm{d}\rho} = V_w \left[\frac{\mathrm{d}w}{\mathrm{d}\rho} + \frac{V_\rho}{V_w} \right], \tag{13}$$

where V_w, $V_\rho > 0$ and the bracketed term signs the expression. The first term in this expression represents the slope of the factor price frontier, defined here as w/ρ, and is

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[9] Galor (1992) has demonstrated that equilibrium in the two-sector OG model may be characterized by nonexistence, uniqueness, or multiplicity of nontrivial steady state equilibria. To obtain a globally unique perfect foresight equilibrium, as has been assumed to exist here, the dynamic system must satisfy the following conditions: (a) the investment good is capital-intensive, (b) second period consumption is a normal good, (c) saving is an increasing function of the real return to capital, and (d) additional regularity conditions most hold. The first three conditions have been explicitly imposed above. The fourth condition is assumed to hold though, to avoid a cumbersome investment in additional notation, readers are referred to Galor (1992) for an explicit statement of the regularity conditions.

negative since (9) implies that

$$\frac{\mathrm{d}w}{\mathrm{d}\rho} = \frac{-k_1 k_1' f_1''}{k_2' f_2''} < 0. \tag{14}$$

That is, if profits are being maximized, an increase in the rental for capital services (and hence the return on capital) will encourage labor to be substituted for capital in both production processes, thereby driving downward the marginal productivity of labor and the wage. The second term is the negative of the slope of an iso-utility curve in (ρ, w) space. Thus,

$$\frac{\mathrm{d}V(w,\rho)}{\mathrm{d}\rho} \gtreqqless 0 \quad \text{as} \quad \frac{\mathrm{d}w}{\mathrm{d}\rho} \gtreqqless -\frac{V_\rho}{V_w}.$$

That is, from an initial steady state, a small increase in ρ will raise (lower, hold constant) utility as the slope of the factor price frontier is flatter than (steeper than, the same as) the slope of the iso-utility curve. So, for example, the initial steady state is located at a point of intersection between the factor price frontier and an iso-utility curve. An increase in ρ moves the factor price ratio from its initial steady state value to a point further along the factor price frontier, to the southeast. When the factor price frontier is flatter than the iso-utility curve, this point will lie above the initial iso-utility curve. It remains to show the conditions under which each possibility prevails.

First, it can be shown using Roy's identity that $V_\rho = s(w,\rho)V_w/\rho$, so that together with (12),

$$-\frac{V_\rho}{V_w} = -\frac{s(w,\rho)}{\rho} = -\left(\frac{1+n}{\rho}\right)pk. \tag{15}$$

Furthermore, from (10) and related derivations found in the Appendix, (14) can be further simplified,

$$\frac{\mathrm{d}w}{\mathrm{d}\rho} = -pk_1\frac{f_1' f_2}{f_1 f_2'} = -pk\left[\frac{(n+\delta)k_1}{(n+\delta)k + (k_1 - k)f_2'}\right] < 0, \tag{16}$$

where the second equality is obtained by solving the implied equality in (11) for f_1/f_1', and making substitutions from (2) and the steady state version of (1). Simple algebra reveals that the positive bracketed term is less than (greater than, equal to) one as f_2' is greater than (less than, equal to) $n+\delta$. Therefore, at the golden rule,

$$\frac{\mathrm{d}w}{\mathrm{d}\rho} = -pk. \tag{17}$$

In this case, movements along the factor price frontier in either direction are such that the income gain of one factor is equal and opposite in sign to the income loss of the other factor. In contrast, (16) implies that a move along the factor price frontier when the initial equilibrium is away from the golden rule results in uneven income changes for the two factors. Income lost by laborers, for example, will exceed income gained by capital owners when the initial equilibrium is dynamically inefficient, reflecting the shortfall between the proceeds from capital ownership and the cost of investment. And, when the initial capital–labor ratio is dynamically efficient – so that the proceeds on capital ownership exceed the cost of investment – the inequality is reversed; a loss to labor will be of smaller magnitude than the associated gain of capital owners.[10]

Using (16), it is possible to express (15) by

$$-\frac{V_\rho}{V_w} = \left(\frac{1+n}{\rho}\right)\left[\frac{(n+\delta)k + (k_1 - k)f_2'}{(n+\delta)k_1}\right]\frac{\mathrm{d}w}{\mathrm{d}\rho}.$$

Thus, the slope of the iso-utility curve is a multiple of the slope of the factor price frontier. The first bracketed term is clearly less than (greater than, equal to) one as ρ is greater than (less than, equal to) $1+n$. Since this is equivalent to the requirement that f_2' is greater than (less than, equal to) $n+\delta$, the multiple (the product of the two bracketed terms) is less than (greater than, equal to) one as ρ is greater than (less than, equal to) $1+n$. Therefore, it follows that $\mathrm{d}w/\mathrm{d}\rho \gtreqqless -V_\rho/V_w$ as $\rho \gtreqqless 1+n$ and furthermore that

$$\frac{\mathrm{d}V(w,\rho)}{\mathrm{d}\rho} \gtreqqless 0 \quad \text{as} \quad \rho \lesseqqgtr 1+n. \tag{18}$$

The equality in these expressions hold when competitive pricing yields a factor intensity ratio for the investment sector equal to that associated with the golden rule; that is, when $f_2'(k_2(\omega(p))) = n+\delta$. Thus, at the golden rule factor intensity, not only is steady state consumption per capita (and hence welfare for agents constrained to have identical consumption profiles) maximized in the centrally planned economy, but steady state utility of a representative economic agent is also maximized in the competitive setting. This occurs because, at the golden rule factor intensity ratio, the rate at which consumption is substituted over the lifetime of an individual in the competitive setting is identical to the rate at which it is substituted across steady state generations in the planned economy; that is, it can be shown that $\rho = 1+n$ implies that the slope of the stationary decentralized per capita consumption possibilities locus equals the slope of the central planner's stationary per capita consumption possibilities locus (see Appendix).

Of course, in an OG model, the competitive steady state need not occur at the golden rule. If, instead, the competitive factor intensity ratio in the investment good industry exceeded that associated with the golden rule, then at the initial steady state $\rho < 1+n$ and (18) implies that welfare is improved by increases in the return on capital ownership. And, if the competitive factor intensity ratio is less that of the golden rule, then initially $\rho > 1+n$ and increases in the return on capital worsen welfare. To go just a bit further, note that properties established in the Appendix, imply that $f_2'' k_2' \omega' > 0$. Thus, by monotonicity, there is a unique relative price, p^{g}, satisfying $f_2'(k_2(\omega(p))) = n+\delta$.[11] Furthermore, ρ is determined by p, and $\mathrm{d}\rho/\mathrm{d}p =$

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[10] It is interesting to note that multiplying (17) by $-\rho/w$ converts the expression to an equality between the elasticity, $-(\rho/w)\mathrm{d}w/\mathrm{d}\rho$, and capital's relative income share, rk/w. A similar exercise using (16), however, shows that the equality does not prevail away from the golden rule.

[11] See Section 5 for an explicit derivation of p^{g} as a function of k^{g} and the golden rule factor intensities.

$[\mathrm{d}r/\mathrm{d}p - r/p]/p > 0$ by the factor intensity assumption. Thus, (18) can be restated as

$$\frac{\mathrm{d}V(w,\rho)}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}p} = \frac{\mathrm{d}V(w,\rho)}{\mathrm{d}p} \gtreqless 0 \quad \text{as} \quad p \lesseqgtr p^{\mathrm{g}}, \tag{19}$$

that is, welfare increases or decreases with the relative price of the investment good as the initial steady state relative price is, respectively, less than or greater than p^{g}.

This discussion yields Proposition 2.

Proposition 2. *Steady state welfare is maximized in the competitive environment at the unique relative price for the (capital-intensive) investment good,* p^{g}, *that satisfies* $f_2'(k_2(\omega(p^{\mathrm{g}}))) = n + \delta$; *that is, at the price that yields the golden rule factor intensity ratio in the investment good industry. If the initial steady state is characterized by* $p > p^{\mathrm{g}}$, *then* $f_2'(k_2(\omega(p^{\mathrm{g}}))) > n + \delta$ *and steady state welfare is decreasing in* p. *If the initial steady state is characterized by* $p < p^{\mathrm{g}}$, *then* $f_2'(k_2(\omega(p^{\mathrm{g}}))) < n + \delta$ *and steady state welfare is increasing in* p.

Now, given p^{g}, and therefore a competitive factor intensity ratio for the investment good industry that is equal to that associated with the golden rule, (4) and (11) further imply that the factor intensity ratios for the consumption good industry are identical in the centrally planned and competitive economies. Furthermore, from the common requirements implied by (2) and the steady state version of (1), the associated aggregate capital–labor ratio is also common to the two economies, as are the labor allocations. The former half of this statement implies that, in the competitive economy, p^{g} obtains only when the steady state capital–labor ratio is k^{g}, as defined in Section 3.

Corollary 1. *If the steady state equilibrium of the competitive economy is characterized by* $p = p^{\mathrm{g}}$, *then* $k = k^{\mathrm{g}}$; *that is, the steady state capital–labor ratio of the competitive equilibrium is the golden rule capital–labor ratio of the centrally planned economy.*

The commonality of the centralized and competitive allocations when $k = k^{\mathrm{g}}$ has two important implications. First, and most importantly, it is possible to state a relationship between capital-accumulation and steady state welfare that depends upon the efficiency characteristics of the initial steady state capital–labor ratio. The steady state satisfies $(n+\delta)k = \tilde{x}_2(p,k)$ so that total differentiation implies $\mathrm{d}p/\mathrm{d}k = [\tilde{x}_2/k - \tilde{x}_{2k}]/\tilde{x}_{2p}$. It is well-established that $\tilde{x}_{2p} > 0$ and, under the assumption that the investment good is capital-intensive, that the Rybczynski theorem implies $\tilde{x}_{2k}(k/\tilde{x}_2) > 1$. Thus, under the assumptions stated, it follows that $\mathrm{d}p/\mathrm{d}k < 0$; that is, there is an inverse relationship between the steady state relative price of capital and the steady state capital–labor ratio. Moreover, this relationship is independent of the magnitude of the steady state capital–labor ratio relative to that of the golden rule. Together with the previously established relationship between welfare and the relative price of capital, this gives

$$\frac{\mathrm{d}V(w,\rho)}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}k} = \frac{\mathrm{d}V(w,\rho)}{\mathrm{d}k} \gtreqless 0 \quad \text{as} \quad k \lesseqgtr k^{\mathrm{g}} \tag{20}$$

for small changes in k. In other words, it has been verified that steady state welfare is improved by capital accumulation when the initial steady state is dynamically efficient, but is improved by the decumulation of capital when it is dynamically inefficient. Thus, an important relationship from one-sector analyses of dynamic efficiency indeed carries over to the two-sector environment. Moreover, when considering (18), (19) and (20), there is the following simple, summative statement that applies: steady state welfare is improved as the steady state return on capital, relative price of capital and capital–labor ratio approach their respective golden rule values, and is worsened as they become more distant. This statement holds regardless of whether the variables approach from above or below.

The following proposition formalizes the relationship between capital accumulation and welfare.

Proposition 3. *If the investment good is capital-intensive and the initial capital–labor ratio is below* (*above*) *that associated with the golden rule, then capital accumulation decreases p and* ρ *in the competitive economy and welfare is unambiguously improved* (*worsened*). *If, under the same factor intensity assumption, the initial capital–labor ratio is below* (*above*) *that associated with the golden rule then capital decumulation increases p and* ρ *in the competitive economy and unambiguously worsens* (*improves*) *welfare.*

The second implication of identical allocations in the centrally planned and competitive economies at the golden rule is that some frequently cited properties related to golden rule allocations can now be reaffirmed in the two-sector context. From (9) and (7), it follows that

$$w = f_1 - k_1 f_1' = l_1 f_1.$$

Also, at the gross return associated with the golden rule, it is immediate that

$$\rho k^{g} = (1+n)k^{g}.$$

This pair of results, familiar from the one-sector context (see Phelps, 1961), affirms that at the golden rule all wages are consumed and all of capital income is invested. Moreover, under the assumed factor intensity rankings, steady state capital–labor ratios below that of the golden rule yield a larger value for ρ and a smaller value for w, via an increased relative price for the investment good. Also then, it can be concluded that the sufficient condition for dynamic efficiency of Abel et al. (1989) is satisfied for all steady state capital–labor ratios below that of the golden rule; that is, capital income exceeds investment.

5. A log-linear economy

As is well-known, with log-linear utility functions, $U(c_t^{y}, c_{t+1}^{o}) = c_t^{y}(c_{t+1}^{o})^{\beta}$, the per worker savings function becomes

$$s_t = \sigma w_t = \sigma p_t[f_2 - k_{2t} f_2'],$$

where $\beta \in (0,1]$ is the discount factor[12]and $\sigma = \beta/(1+\beta)$ is the (constant) savings rate.[13] Therefore, the competitive steady state capital–labor ratio follows from (12),

$$k = \sigma[f_2 - k_2 f_2']/(1+n). \tag{21}$$

As for the central planner's choice, with general production technologies, (6) and (7) together imply that the golden rule capital–labor ratio satisfies

$$k^{\mathrm{g}} = \frac{k_1 f_1'}{f_1}\left(\frac{f_2}{n+\delta}\right). \tag{22}$$

Solving for f_1' and substituting into (10) together with (5) implies that the associated relative price in the competitive economy satisfies

$$p^{\mathrm{g}} = \frac{f_1}{f_2}\frac{k^{\mathrm{g}}}{k_1}. \tag{23}$$

For general production technologies, (22) and (21) imply that the steady state capital–labor ratio of the competitive equilibrium equals the golden rule capital–labor ratio only if $\sigma = \sigma^{\mathrm{g}}$, where

$$\sigma^{\mathrm{g}} = \frac{1+n}{n+\delta}\left(\frac{k_1 f_1'}{f_1}\right)\left(\frac{f_2}{f_2 - k_2 f_2'}\right). \tag{24}$$

Thus σ^{g} denotes the savings rate associated with the golden rule steady state.

Now, specifying the technologies so that $f_i(k_i) = k_i^{\alpha_i}, i = 1,2$, then (6), (7), and (8) together imply that the golden rule factor allocations are given by $l_1 = 1 - \alpha_1$, $l_2 = \alpha_1$, $k_1 = [(1-\alpha_2)/(1-\alpha_1)]k^{\mathrm{g}}$, and $k_2 = (\alpha_2/\alpha_1)k^{\mathrm{g}}$, where the factor intensity assumption requires that $\alpha_2 > \alpha_1$. Thus, for this specification of the technologies, (22) and (23), respectively, imply that

$$k^{\mathrm{g}} = \alpha_1\left[\frac{\alpha_2^{\alpha_2}}{n+\delta}\right]^{1/(1-\alpha_2)},$$

$$p^{\mathrm{g}} = \left(\frac{\alpha_1}{\alpha_2}\right)^{\alpha_2}\left(\frac{1-\alpha_1}{1-\alpha_2}\right)^{1-\alpha_1}(k^{\mathrm{g}})^{\alpha},$$

where $\alpha = \alpha_1 - \alpha_2 < 0$. Furthermore, (24) gives

$$\sigma^{\mathrm{g}} = \frac{1+n}{n+\delta}\left[\frac{\alpha_1}{1-\alpha_2}\right].$$

Clearly, it is possible to find feasible parameter values that yield a savings rate either greater or less than the golden rule value; consequently, parameters can be set so that

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[12] Whereas monotonic preferences require $\beta > 0$, the further restriction $\beta \leqslant 1$ is not needed for the results below. This restriction is imposed only so as to enable the usual interpretation of β as the rate at which future consumption is discounted relative to current consumption.

[13] For purpose of comparison with the application provided in Section 6, it is here noted that the steady state consumption profile associated with this savings function is given by $c^{\mathrm{y}} = (1-\sigma)w$ and $c^{\mathrm{o}} = \sigma w \rho$.

dynamic efficiency is either incorporated or ruled-out by assumption. Moreover, for this log-linear world, it is instructive to further highlight the role of the savings rate. The steady state capital–labor ratio can be expressed as a function of the savings rate via (21) and, more particularly, it can be shown to be increasing in the latter.[14] Moreover, under the factor intensity assumption, p is inversely related to k. The savings parameter, σ, is exogenous and will in general circumstances be either greater than or less than σ^{g}. If $\sigma > \sigma^{g}$, it follows that $k > k^{g}$ and $p < p^{g}$; that is, the steady state will be characterized by an overaccumulation of capital and is therefore dynamically inefficient. Therefore, given σ, an economic event that serves to reduce k from its initial steady state, will also increase p, bringing both closer to their golden rule values. Consequently, the capital decumulation introduces a positive influence on welfare as a result of this event (recall (20)). Of course, when $\sigma < \sigma^{g}$, analogous reasoning implies that the opposite welfare result will hold for the same event.

Focusing a moment longer on σ, the circumstances under which a competitive equilibrium will prove to be dynamically inefficient can be further expounded upon. Clearly, σ^{g} is positively related to the α_i and negatively related to δ. Thus, a steady state is more likely to be dynamically inefficient when the former is small and the latter is large in that a larger range of feasible savings rates imply capital overaccumulation. On the other hand, since $(1+n)/(n+\delta) \geqslant 1$ for $\delta \in [0,1]$, $n \geqslant 0$, it may be useful to note that $\alpha_1 + \alpha_2 > 1$ becomes a sufficient condition for the steady state of the competitive equilibrium to be dynamically efficient. That is, $\alpha_1 + \alpha_2 > 1$ implies that $\sigma^{g} > 1$ and thus $\sigma < \sigma^{g}$ for all feasible $\sigma \in (0,1]$. Intuitively, when technologies are together sufficiently capital-intensive, it becomes impossible to oversave relative to the golden rule. Otherwise, that is, when technologies are less capital-intensive, oversaving occurs when the saving rate is high relative to economywide capital-intensiveness.

6. An application

To demonstrate the role of dynamic efficiency conditions in a two-sector welfare analysis, a permanent, government-imposed tax/subsidy scheme is now examined. More specifically, an investment subsidy that is fully financed by a consumption tax is introduced into the log-linear environment of Section 5.[15] The effect of this scheme on capital accumulation is intentionally obvious; rather, the objective is to demonstrate that the policy's effect on welfare will additionally depend upon whether or not the initial steady state is dynamically efficient.

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[14] Though seemingly intuitive, this relationship is tedious to demonstrate. The complication arises from the fact that the steady state wage is also determined by the steady state capital–labor ratio; thus, functional forms which have been left implicit until now, must first be specified. The derivation is shown, however, in the appendix.

[15] For more involved applications of these dynamic efficiency conditions to two-sector welfare analyses see Cremers (2005), Sen (2005) and Cremers and Sen (2005).

Maintaining the previously defined notation, the policy-modified budget constraints are given by

$$(1+\tau)c_t^{\mathrm{y}} + (1-\gamma)p_t k_{t+1}^{\mathrm{y}} = w_t,$$

$$(1+\tau)c_{t+1}^{\mathrm{o}} = r_{t+1}k_{t+1}^{\mathrm{y}},$$

where $\tau > 0$ and $\gamma > 0$, respectively, denote the tax and the subsidy, and $\delta = 1$ has been assumed. Thus, individual consumption is taxed in both periods of life whereas purchases of investment goods are subsidized only when young. The lifetime budget constraint is then

$$(1+\tau)\left[c_t^{\mathrm{y}} + \frac{c_{t+1}^{\mathrm{o}}}{\rho_{t+1}}\right] = w_t + \gamma p_t k_{t+1}.$$

The revenue collected by the government at time t is given by

$$T_t \equiv \tau L_t c_t^{\mathrm{y}} + \tau L_{t-1} c_t^{\mathrm{o}} = \tau L_t [c_t^{\mathrm{y}} + c_t^{\mathrm{o}}/(1+n)]$$

and the revenue required for the subsidy is given by

$$S_t \equiv \sum \gamma p_t k_{t+1}^{\mathrm{y}} = \gamma p_t L_t k_{t+1}^{\mathrm{y}} = \gamma p_t k_{t+1} L_{t+1}$$

since, by definition, $L_{t+1}^{-1} \sum k_{t+1}^{\mathrm{y}} \equiv k_{t+1}$. Thus, a balanced budget requires $T_t = S_t$, which is satisfied when

$$\gamma = \tau\left[\frac{c_t^{\mathrm{y}} + c_t^{\mathrm{o}}/(1+n)}{(1+n)p_t k_{t+1}}\right]. \tag{25}$$

Using this equality to eliminate γ, the steady state lifetime budget constraint can be expressed

$$\left[c^{\mathrm{y}} + \frac{c^{\mathrm{o}}}{\rho}\right] + \tau\frac{c^{\mathrm{o}}}{\rho}\left[1 - \frac{\rho}{1+n}\right] = w.$$

It can be shown that steady state utility is maximized while satisfying the lifetime budget constraint when

$$c^{\mathrm{y}} = (1-\sigma)w,$$

$$c^{\mathrm{o}} = \frac{\sigma w \rho}{1 + \tau(1 - \rho/(1+n))}.$$

Thus, the direct effect of the policy on consumption is experienced only by the old. That portion of the wage not spent on youthful consumption, σw, must now be divided between tax payments on that consumption and purchases of investment goods. Thus, the tax works against the subsidy in determining the demand for investment goods. At given prices, old age consumption may either increase or decrease under the tax/subsidy scheme, depending only upon the relationship between ρ and $1+n$. This is despite the fact that, at constant prices, the quantity of investment goods demanded unambiguously increases under the scheme. To see this, let $\tilde{s}$ denote policy-modified individual savings,

$$\tilde{s} = \frac{w - (1+\tau)c^{y}}{1-\gamma}, \tag{26}$$

where the opposing effects of the consumption tax and the investment subsidy are made explicit. To net out these effects, recall that steady state market-clearing, in per worker terms, requires

$$\tilde{s} = (1+n)pk = p\tilde{x}_2(p,k). \tag{27}$$

Substituting (27) and (25), and the solution for c^{y} into (26) then yields an associated savings function:

$$\tilde{s}(w,\rho,\tau) \equiv \frac{\sigma w(1+\tau)}{1+\tau(1-\rho/(1+n))}.$$

That the net effect of the policy is to increase investment at given prices is made apparent by the fact that, for all $\tau > 0$, $\tilde{s}(w,\rho,\tau) > \sigma w$. That old age consumption may nevertheless decline is then accounted for by the fact that it is also taxed.

The utility maximizing solutions imply that the indirect utility function can be written

$$V(w,\rho,\tau) \equiv \sigma^{\beta}(1-\sigma)w^{1+\beta}\rho^{\beta}\left[1+\tau\left(1-\frac{\rho}{1+n}\right)\right]^{-\beta}.$$

The impact of the tax/subsidy scheme on welfare is found via differentiation of the indirect utility function:

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} = \frac{\beta V}{\sigma w}\left[\left(\frac{\mathrm{d}w}{\mathrm{d}\rho} + \frac{\tilde{s}}{\rho}\right)\frac{\mathrm{d}\rho}{\mathrm{d}\tau} + \frac{\tilde{s}}{1+\tau}\left(\frac{\rho}{1+n} - 1\right)\right], \tag{28}$$

where the direct effect of the policy on old age consumption is now reflected by the last term and, as was previously noted, is positive when $\rho > 1+n$. In addition to this direct effect, there are indirect effects on factor prices that are channeled through the policy-induced change in ρ. This portion of the welfare effect is clearly rooted in the more general analysis of Section 4 (see (13) and (15)). Therefore, it is through this indirect effect that the relationships characterized above – between welfare and changes in the return on capital, the relative price of the investment good and/or the capital–labor ratio – can be observed.

Before deriving the alternative representations of the indirect welfare effects, it is first noted that the general derivations in Section 4.4 imply that the coefficient for $\mathrm{d}\rho/\mathrm{d}\tau$ in (28) should also have a sign that depends upon whether or not ρ exceeds $1+n$. To be thorough, this fact is confirmed before employing technological specifications. Use (16) to obtain

$$\frac{\tilde{s}}{\rho} = \frac{(1+n)pk}{\rho} = -\frac{(1+n)}{\rho}\left[\frac{(1+n-\rho)k + \rho k_1}{(1+n)k_1}\right]\frac{\mathrm{d}w}{\mathrm{d}\rho}.$$

It can then be shown that

$$\frac{\mathrm{d}w}{\mathrm{d}\rho}+\frac{\tilde{s}(w,\rho,\tau)}{\rho}=pk\left(\frac{1+n}{\rho}-1\right)\left(1+\frac{\rho}{p}\frac{\mathrm{d}p}{\mathrm{d}\rho}\right),$$

so that the specified technologies together with (27) imply that (28) can be expressed by

$$\frac{\mathrm{d}V}{\mathrm{d}\tau}=\beta V\left(\frac{\rho}{1+n}-1\right)\left[-\frac{1}{\rho}\left(1+\frac{\rho}{p}\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)\frac{\mathrm{d}\rho}{\mathrm{d}\tau}+1\right]$$

when evaluated at the initial equilibrium with $\tau=0$. This factorization confirms that the sign of the indirect effect will depend in part upon whether the initial steady state is or is not dynamically efficient.[16] Moreover, under the factor intensity assumption, it is immediate that a policy-induced increase in ρ reduces welfare when the initial equilibrium is dynamically efficient and increases welfare when it is not. With the assumed technologies the welfare effect can be further simplified:

$$\frac{\mathrm{d}V}{\mathrm{d}\tau}=\beta V\left(\frac{\rho}{1+n}-1\right)\left[\left(\frac{1-\alpha_1}{\alpha_2-1}\right)\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}\tau}+1\right]. \tag{29}$$

A second representation of the same welfare effects highlights the role of the relative price of the investment good, rather than the return on capital ownership. Recalling that $\rho \gtreqless 1+n$ as $p \gtreqless p^{\mathrm{g}}$ and further recognizing that

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau}=\frac{\rho}{p}\left(\frac{\alpha_2-1}{\alpha}\right)\frac{\mathrm{d}p}{\mathrm{d}\tau}, \tag{30}$$

so that ρ and p are positively related under the factor intensity assumption, it is clear that results analogous to those just described also obtain with regard to a change in the relative price of the investment good. That is, together (29) and (30) demonstrate that a policy-induced increase in p worsens welfare when the initial equilibrium is dynamically efficient, and vice versa.

The final representation, which centers on the effect of the policy on capital accumulation and welfare, requires first that the relationship between equilibrium values of p and k be made explicit. To do so, totally differentiate (27) and evaluate at $\tau=0$ to get

$$\sigma w\left[\frac{\alpha_1}{\alpha}\frac{\mathrm{d}p}{p}+\frac{\rho}{1+n}\mathrm{d}\tau\right]=p\tilde{x}_2\left[\frac{\mathrm{d}\tilde{x}_2}{\tilde{x}_2}+\frac{\mathrm{d}p}{p}\right]. \tag{31}$$

The specified technologies then imply that

$$\frac{\mathrm{d}\tilde{x}_2}{\tilde{x}_2}=\frac{1}{\alpha}\left[\frac{\alpha_2k_1+(1-\alpha_2)k}{k_1-k}\right]\frac{\mathrm{d}p}{p}-\frac{k}{k_1-k}\frac{\mathrm{d}k}{k}.$$

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[16] In passing, it should also be noted that the assumption of specific functional forms allows the factored term, $\rho/(1+n)-1$, to have explicit alternative representations that highlight the relationship between the initial relative price of the investment good, the investment good factor intensity, or the steady state capital–labor ratio and savings rate, relative to their respective golden rule values. For brevity, these derivations are left to the interested reader.

Also, $\mathrm{d}\tilde{x}_2/\tilde{x}_2 = \mathrm{d}k/k$, so that

$$\frac{\mathrm{d}k}{k} = \frac{1}{\alpha}\left[\frac{\alpha_2 k_1 + (1-\alpha_2)k}{k_1}\right]\frac{\mathrm{d}p}{p}. \tag{32}$$

Then,

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = \frac{\mathrm{d}p}{\mathrm{d}k}\frac{\mathrm{d}k}{\mathrm{d}\tau} = \frac{p}{k}\left[\frac{\alpha k_1}{\alpha_2 k_1 + (1-\alpha_2)k}\right]\frac{\mathrm{d}k}{\mathrm{d}\tau}. \tag{33}$$

Since the factor intensity assumption implies that $\alpha < 0$, this expression exhibits the expected negative relationship between p and k. Also, as established above, $p \gtreqless p^{\mathrm{g}}$ as $k \lesseqgtr k^{\mathrm{g}}$. Together with (29) and (30), it therefore follows that policy-induced capital accumulation introduces a positive welfare effect when the initial capital–labor ratio is dynamically efficient but is otherwise negative, and vice versa for policy-induced capital decumulation.

Finally, to pin down the effects of the tax/subsidy policy on capital accumulation and welfare, using (32) in (31) gives

$$\frac{\mathrm{d}k}{\mathrm{d}\tau} = \frac{\rho k}{1+n}\left[1 + \frac{\alpha_2 k_1}{(1-\alpha_2)k}\right]$$

and hence accumulation unambiguously increases under the policy. From (29) and the string of relationships just described, it also follows that the bracketed term in (29) is strictly positive. Clearly though, both the indirect and the overall welfare effects may be either positive or negative. This is despite the fact that the tax/subsidy scheme always increases the steady state capital–labor ratio and always reduces both the relative price of the investment good and the return on capital ownership. If the initial steady state is dynamically efficient, then capital accumulation, a declining relative price for the investment good and a falling return on capital ownership all represent movements towards golden rule values and a positive indirect welfare effect is generated. If the initial steady state is dynamically inefficient, the same changes move the capital–labor ratio, the relative price and the return on capital ownership further from their golden rule values and hence results in a negative indirect welfare effect. This example thus illustrates that it is the effect of the policy on the proximity of equilibrium values for k, ρ and p to their respective golden rule values that matters in determining whether the indirect portion of the welfare effect is positive or negative.

As a final remark, given the specification of the technologies, the welfare implications of the tax/subsidy policy could have been obtained somewhat more directly. That is, using these technologies it can be shown

$$\frac{\mathrm{d}w}{\mathrm{d}\rho} + \frac{\tilde{s}}{\rho} = \left(\frac{\alpha_1}{\alpha_2 - 1}\right)\frac{w}{\rho} + \frac{\sigma w}{\rho},$$

so that the expression is positive or negative as $\sigma \gtrless \alpha_1/(1-\alpha_2) \equiv \sigma^{\mathrm{g}}$; that is, as the savings rate is less or greater than that associated with the golden rule allocation. From (28) and the policy-induced decrease in ρ just described, it follows that the indirect welfare effect associated with the tax/subsidy scheme is positive when $\sigma < \sigma^{\mathrm{g}}$, but negative when $\sigma > \sigma^{\mathrm{g}}$. To exclude the possibility of dynamic inefficiency and the subsequent negative welfare effects that arise from the tax subsidy scheme it is only necessary, therefore, to assume that $\sigma < \alpha_1/(1-\alpha_2)$.

7. Conclusion

This paper characterizes dynamic efficiency for a two-sector overlapping generations model with a globally unique steady state equilibrium. Both general and log-linear specifications are considered so as to facilitate further applied welfare analyses that choose to make use of the two-sector framework. To demonstrate a practical use of the conditions for dynamic efficiency, a government imposed tax/subsidy scheme is examined. The characterization provided may also be useful in re-examining old themes – including technological progress, the effects of national debt, money, land, and asset bubbles – in the two-sector context. In addition, and more importantly, this characterization will be useful when considering the welfare consequences of a wide range of issues that are inherently incompatible with the one-sector environment and often topically related to discussions of economic growth. In addition to disturbances that are thought to be sector specific, this would include any event that would reasonably be expected to influence a relative price for produced goods, important examples of which include the relative price of capital and, in open economies, a country's terms of trade and/or real exchange rate.

Acknowledgments

This paper is dedicated to the memory of my father, Clifford John Cremers, who passed away as it was being written and is sorely missed. The author is also grateful for the valuable suggestions of two anonymous referees.

Appendix

Proof. $\omega_t \equiv \omega(p_t)$

As in Oniki and Uzawa (1965), differentiation of (11) with respect to ω yields

$$\frac{\mathrm{d}k_{it}(\omega_t)}{\mathrm{d}\omega_t} = -\frac{\left(f_i'(k_{it}(\omega_t))\right)^2}{f_i(k_{it}(\omega_t))f_i''(k_{it}(\omega_t))} > 0.$$

Furthermore,

$$p_t = p(\omega_t) \equiv \frac{f_1'(k_{1t}(\omega_t))}{f_2'(k_{2t}(\omega_t))}$$

so that logarithmic differentiation, together with (11) and the first expression above, imply that

$$\frac{1}{p(\omega_t)}\frac{\mathrm{d}p(\omega_t)}{\mathrm{d}\omega_t} = \frac{1}{k_{2t}(\omega_t)+\omega_t} - \frac{1}{k_{1t}(\omega_t)+\omega_t}$$

which is positive or negative as $k_{1t}(\omega_t)$ is greater than or less than $k_{2t}(\omega_t)$. Thus, with the assumption that the investment good is capital-intensive, the relative price of the investment good is decreasing in the wage–rental ratio. Moreover, strict monotonicity also implies that $p(\omega_t)$ is invertible, so that $\omega_t \equiv \omega(p_t)$. □

Proof. If $\rho = 1+n$, the slope of the stationary decentralized per capita consumption possibilities locus equals $-(1+n)$.

Following Buiter (1981), the stationary decentralized per capita consumption possibilities locus is determined via the stationary budget constraints and market-clearing condition, which together imply that $c^{\mathrm{o}} = \rho pk(1+n)$ and $c^{\mathrm{y}} = w - pk(1+n)$. Total differentiation of each equation then implies that

$$\mathrm{d}c^{\mathrm{y}} = -k\left\{\left(\frac{\mathrm{d}r}{\mathrm{d}p} + 1 - \delta\right)\mathrm{d}p + p(1+n)\frac{\mathrm{d}k}{k}\right\},$$

$$\mathrm{d}c^{\mathrm{o}} = (1+n)k\left\{\left(\frac{\mathrm{d}r}{\mathrm{d}p} + 1 - \delta\right)\mathrm{d}p + p\left(\frac{r}{p} + (1-\delta)\right)\frac{\mathrm{d}k}{k}\right\},$$

where, in deriving the first equation, use has been made of the fact that $\mathrm{d}w/\mathrm{d}p + k\,\mathrm{d}r/\mathrm{d}p = x_2$, which follows from application of the envelope theorem to the equality, $w + rk = x_1 + px_2$. Thus, it follows immediately that if $\rho = r/p + 1 - \delta = 1 + n$, then $\mathrm{d}c^{\mathrm{o}}/\mathrm{d}c^{\mathrm{y}} = -(1+n)$. □

Proof. For the log-linear world, then the steady state capital–labor ratio of the competitive economy is increasing in the savings rate, σ.

First, with these specification, equality (9) implies that $k_i = \alpha_i/(1-\alpha_i)\omega,\ i = 1,2$. Also, (10) can be used to demonstrate that $\omega = Bp^{1/\alpha}$, where

$$B = [(\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2})/(\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1})]^{1/\alpha}$$

so that $k_i = \Gamma_i p^{1/\alpha}$, where $\Gamma_i = \alpha_i/(1-\alpha_i)B,\ i = 1,2$. Also, with this specification and (2), it can be shown that $l_2 = (\Gamma_1 - \Gamma_2)^{-1}[\Gamma_1 - kp^{-1/\alpha}] = (\alpha_1/\alpha)(1-\alpha_2)[1-(k/\Gamma_1)p^{-1/\alpha}]$. Also, $l_2 = \sigma(1-\alpha_2)[(n+\delta)/(1+n)]$ as follows from (12) and $k = l_2 f_2/(n+\delta)$. Together these expressions for l_2 imply that

$$p^{1/\alpha} = (k/\Gamma_1)[(\alpha_1(1+n) - \sigma\alpha(n+\delta))/(\alpha_1(1+n))]^{-1}.$$

Substituting this expression into (21) then implies that

$$k = \left\{\sigma\left(\frac{1-\alpha_2}{1+n}\right)\left(\frac{\Gamma_2}{\Gamma_1}\right)^{\alpha_2}\left[\frac{\alpha_1(1+n) - \sigma\alpha(n+\delta)}{\alpha_1(1+n)}\right]^{-\alpha_2}\right\}^{1/(1-\alpha_2)}.$$

Let

$$D \equiv \left[\left(\frac{1-\alpha_2}{1+n}\right)\left(\frac{\Gamma_2}{\Gamma_1}\right)^{\alpha_2}\right]^{1/(1-\alpha_2)} > 0,$$

and

$$G(\sigma) \equiv \left[\frac{\alpha_1(1+n) - \sigma\alpha(n+\delta)}{\alpha_1(1+n)}\right] > 0$$

so that $k = D\{\sigma G^{-\alpha_2}\}^{1/(1-\alpha_2)}$. Differentiating with respect to σ gives

$$\frac{\mathrm{d}k}{\mathrm{d}\sigma} = \frac{D}{1-\alpha_2}\left(\frac{\sigma}{G}\right)^{\alpha_2/(1-\alpha_2)}\left\{\frac{\sigma\alpha\alpha_2(n+\delta)}{\alpha_1(1+n)}G^{-1}+1\right\},$$

where the final bracketed term signs the expression. Simple algebra demonstrates that this term is always positive, and therefore $\mathrm{d}k/\mathrm{d}\sigma > 0$. □

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