Problem Set # 1 (Linear Difference Equations)

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ECON 602, Spring 2009
Due 15 Jan

1. Identify the order of the following difference equations:

(a) $x_{t+2}^2 - 2x_{t+1} + x_t = 10$
(b) $x_{t+2} - e^{xt} = 2$
(c) $x_{t+i+1} + \sin x_{t-i} - 4 = 0; \ i = 1, 2, ..., m.$

2. Which of the following are autonomous systems and why:

(a) $x_{t+2} - 2x_{t+1} + e^{xt} = 6$
(b) $y_{t+2} - 2ty_t = 8$
(c) $x_{t+2} - e^{xt} = 2$

3. Consider the following difference equation

$$g = \left[ (1 - \frac{\theta}{z})S(R_{t-1}) - K(R_t) \right] - (R_{t-1} - c)[(1 - \theta)S(R_{t-2}) - K(R_{t-1})]$$

where $\theta, z, g,$ and $c$ are positive constants, and $R$ is the variable of interest. Identify the order of (1). How would you reduce the dimensionality of (1)? In other words, rewrite (1) as a system of two difference equations of first order.

4. Consider the quadratic map

$$y_{t+1} = \theta y_t(1 - y_t) ; 1 \leq \theta \leq 4. \quad (2)$$

(a) Identify the order of (2).
(b) Plot $y_{t+1}$ as a function of $y_t$. Compute the maximum value of $y_{t+1}$.
(c) How many steady states does (2) have? Compute the values of the steady state(s).

5. Suppose the following hold:

$$D_t = ap_t + b ; S_t = Ap_t + B \ \forall t = 0, 1, 2, ...$$

and

$$p_t - p_{t-1} = kE(p_{t-1}) ; E(p_{t-1}) = D_{t-1} - S_{t-1}$$

(a) Prove that

$$p_t = [1 + k(a - A)]p_{t-1} + k(b - B). \quad (3)$$

(b) Suppose $p = p_0$ when $t = 0$. Calculate the steady state. Write down the complete solution to (3).
(c) Show that \( p_t \rightarrow \left( \frac{b-B}{A-a} \right) \) as \( t \rightarrow \infty \) (without oscillations) if \( a < A \) and \( k < \frac{1}{A-a} \).

6. Newton’s Law of cooling states: the change in temperature of an object over one period is proportional to the difference between the temperature of the object and the temperature of the surrounding environment (room temperature). Let \( S \) be the room temperature and \( T_n \) denote the temperature of the object after \( n \) periods. Let \( k \) denote the proportionality constant and \( T_0 \) is the initial temperature of the object.

(a) Write down an equation that describes Newton’s Law of cooling.

(b) Explain in plain english the implication/meaning of Newton’s Law of cooling.

(c) Suppose a cup of tea, initially at a temperature of 180 F, is placed in a room which is held at a constant temperature of 80 F. Moreover, suppose that after one period (one minute is one period) the tea has cooled to 175 F. Using Newton’s Law of cooling, figure out the temperature of the tea after 20 minutes?

(d) What do you reasonably expect the final (asymptotic) temperature of the tea to be?