1. Consider a two-period lived agent with preferences $u(c_1) + \kappa u(c_2)$ where $u(c) = \frac{c^{1-\rho}}{1-\rho}$, $\rho > 0$, $\rho \neq 1$, and $\kappa \in (0,1)$. Suppose his endowment in period 1 is $W$ and he has nothing in period 2; the gross return on saving is $R > 0$.

(a) Prove that $c_1^* = \lambda W$; $c_2^* = (1 - \lambda) RW$

where $\lambda \equiv \lambda (\kappa, R) \in (0,1)$.

(b) Prove that lifetime indirect utility is proportional to $u(W)$.

2. Consider a three-period lived agent (alive over periods $-1, 0, 1$ with selves $-1, 0, +1$). Naive Self 1 has preferences $u(c_{-1}) + \beta \delta u(c_0) + \beta \delta^2 u(c_{+1})$ where $\beta, \delta \in [0,1]$. Assume $u(c) = \frac{c^{1-\rho}}{1-\rho}$, $\rho > 0$, $\rho \neq 1$. The agent works in period $-1$ and has to retire in period $+1$. She can choose to work in period 0; if she does, she gets $\Delta$ but suffers utility cost of $e$.

(a) Suppose Self 0 inherits $W_0$ from Self -1. Prove that only if

$$u(\lambda (W_0 + \Delta)) - u(\lambda W_0) + \beta \delta u(R (1 - \lambda) (W_0 + \Delta)) - \beta \delta u(R W_0 (1 - \lambda)) \geq e$$

holds will Self 0 want to work. $\lambda$ and $R$ are defined in the problem above.

(b) Prove that Self $-1$ would want Self 0 to work if

$$u(\lambda (W_0 + \Delta)) - u(\lambda W_0) + \delta u(R (1 - \lambda) (W_0 + \Delta)) - \delta u(R (1 - \lambda) W_0) \geq e.$$ 

(c) Suppose (1) holds with equality at $W_0 = \overline{W}_0$. Prove that at $W_0 = \overline{W}_0$, while Self 0 is indifferent between working and not working, Self $-1$ would definitely want her to work.
3. Consider the two-period pure-exchange overlapping generations model where all members of a generation are identical and have the additively-separable utility function \( u(c_1) + v(c_2) \). The young and old period endowments are given by \( \omega_1 \) and \( \omega_2 \) respectively. In addition, the population grows at the rate \( n \). Suppose

\[
 u(c_1) + v(c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma}
\]

where \( \beta > 0 \) and \( \gamma \neq 1 \). Suppose \( \omega_1 > 0 \) and \( \omega_2 = 0 \).

(a) Carefully write down the maximization problem faced by a young agent.

(b) Compute the solution to the above problem.

(c) Under what conditions on \( \gamma \) is optimal savings a non-decreasing function of the interest rate?

(d) Draw the savings function, i.e., draw \( S^* \), the optimal savings as a function of \( R \), for different ranges of \( \gamma \).

(e) Is there a value for \( R \) (say \( \tilde{R} \)) for which optimal savings is equal to 0? Explain.

(f) When is this economy a “Samuelson case” economy?

4. We did the “case of early endowments” in class where the endowment of each agent was tilted towards their youth, or, in particular, they had endowments (1,0). This was the Samuelson case according to David Gale (JET, 1973). We showed that in such a case, autarky was not Pareto optimal. Now analyze the “case of late endowments”, i.e., (0,1). This is called the “Classical case”. Is autarky a Pareto optimal equilibrium for this economy? Discuss. Draw a diagram to explain.