ABSTRACT. An issuer seeks to liquidate all or part of a portfolio of heterogeneous assets about which she has private information. In contrast with the single-asset case, greater optimism may lead her to sell more of a given asset. If the assets can be ranked by seniority then she will sell her senior assets first, as predicted by Myers’s pecking order hypothesis. If she can design new securities before and/or after learning her information then she has two equivalent, optimal strategies. She may design and sell a single standard debt security whose face value is decreasing in her information. Or she may pool and tranche her assets into many prioritized debt securities and sell those tranches whose seniority exceeds a threshold that is increasing in her information. In both cases, she retains more when information asymmetries are greater, as has been found in the case of no-documentation loans.
1. Introduction

This paper studies the general question of how an issuer should sell assets about which she has private information. Examples include a firm that raises capital by selling debt, equity, and/or hybrid securities, and a bank that sells claims to the repayments of loans that it has issued.

One can imagine various motivations for such a sale. A firm, for instance, may need cash to invest in a worthwhile project. A bank may seek funds in order to issue more loans or to comply with regulations. We capture these varied reasons by assuming, simply, that the issuer is less patient than investors. As this creates gains from trade, the issuer would sell her entire cash flow if information were symmetric. In the more realistic asymmetric information case, such an offer might be interpreted by investors as a negative signal. The issuer may thus choose to sell less, in order to signal optimism.

In most prior work on this problem, the issuer sells a single asset to investors. Examples include Leland and Pyle (1979), Myers and Majluf (1984), and DeMarzo and Duffie (1999). In the equilibria of these models, the issuer signals optimism by retaining a larger portion of her single asset. A setting with multiple assets is considered by DeMarzo (2005), who assumes the number of assets does not exceed the dimensionality of the issuer’s information. In this case, the quantity sold of each asset signals a different dimension of the issuer’s information, so the problem reduces to the single-asset case (DeMarzo 2005).

We depart from the above papers in assuming the issuer has more assets than there are dimensions in her information. Hence, when her information changes, she has a choice of which sale quantities to adjust. This is a complex combinatorial problem. In order to render it tractable, we assume the issuer’s information is one-dimensional and, moreover, that higher information entails a higher expected payout of each asset. These assumptions hold, e.g., if the payouts of the assets are monotone functions of some common underlying cash flow, and higher information raises the cash flow distribution in a first-order stochastic dominance (FOSD) sense. Monotone securities include equity and standard debt, as well as the prioritized tranches of most loan pools.¹ With the addition of a mild additional

¹ This assumption is violated, e.g., if a firm has private information about the variance of its future cash flow, as a higher variance moves the expected payouts of stocks and bonds in opposite directions. We leave this
assumption - the Hazard Rate Ordering property - we obtain strong predictions about optimal asset sales and security design that are consistent with observed practice and/or are borne out in the data.

In our base model, the issuer has a fixed set of assets to sell.\(^2\) We show by example that a more optimistic issuer may sell more of an asset at a higher price. This contrasts with the single-asset case. Intuitively, the issuer most efficiently signals a given increase in her information by retaining those assets whose expected payouts rise proportionally more as a result of her rosier expectations. But the identity of these “more informationally sensitive” assets may vary as the issuer becomes progressively more optimistic. Hence she may retain some shares of one asset to signal a given level of optimism, only to sell this asset in its entirety and retain shares of a different asset in order to signal yet greater optimism.

In the above story, two assets can alternate in their informational sensitivity as the issuer becomes more optimistic. In the remainder of the paper, we focus on settings in which the issuer’s assets do not alternate in this way. We define one asset to be more informationally sensitive than another if an increase in the issuer’s type always raises the expected payout of the first asset proportionally more than that of the second. We say, furthermore, that the assets in an issuer’s portfolio are informationally ordered if they can be ranked from most to least informationally sensitive.

In this setting, the issuer still signals greater optimism by retaining those assets whose expected payouts are more sensitive to the given increase. But with informationally ordered assets, if asset A is more sensitive than asset B for one change in the issuer’s information then it is more sensitive for all such changes. Hence, as the issuer’s optimism

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\(^2\) In order to obtain a unique equilibrium, we impose two mild restrictions. The first is the Intuitive Criterion of Cho and Kreps (1987): beliefs following an unexpected action must be concentrated only on types that could possibly hope to gain from the deviation. We also allow the issuer to underprice assets and ration their allocation. This helps eliminate implausible equilibria, even though no underpricing occurs in equilibrium. Under these assumptions there exists a unique equilibrium, which can be computed recursively using an algorithm that we provide.
grows, she retains her most informationally sensitive asset first, then her second most, and so on. In other words, she will always sell those assets whose informationally sensitivity lies below a threshold that falls as her optimism rises. In particular, she will not retain any portion of a given asset unless she also retains her more informationally sensitive assets in their entirety.

We next apply the above prediction to commonly observed securities such as equity, options, and prioritized debt. We assume, first, that the payouts of the issuer’s assets are increasing functions of a common cash flow. In this setting, the relative informational sensitivity of two assets depends on two factors: (a) how the issuer’s information affects the underlying cash flow and (b) how the cash flow affects the assets’ payouts.

Regarding (a), we assume that an increase in the issuer’s information lowers the hazard rate function of the underlying cash flow distribution. This Hazard Rate Ordering (HRO) property is stronger than the FOSD property discussed above, but weaker than the usual monotone likelihood ratio property.

Regarding (b), we use a generalized notion of seniority in which asset A is senior to asset B if the ratio of the A’s payout to B’s is a nonincreasing and nonconstant function of the underlying cash flow. This implies, as usual, that one bond is senior to another if the first bond’s claims take priority over the second’s in bankruptcy. But it also extends to other common assets that are not prioritized in terms of their claims on a cash flow. For instance, it implies that a stock is senior to a call option that is written on it.

Using this general notion of seniority, we show that if asset A is more senior than asset B, then it is less informationally sensitive. Hence, by the preceding results, an issuer with seniority-ranked assets will sell those securities whose seniority exceeds a threshold that is decreasing in her information. This result is consistent with Myers’s (1984) Pecking Order

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3 Our preceding results assume only that the expected payouts of the issuer’s assets are increasing in her information. For instance, these payouts may be increasing functions of the same or different cash flows, as long as the distribution of each such cash flow is increasing in the issuer’s information in a FOSD sense.  
Hypothesis, in which a firm minimizes the costs of asymmetric information by funding itself first with retained earnings, then with debt, and finally with equity.\textsuperscript{5,6}

The preceding results assume that the issuer is endowed with a fixed set of assets that she wishes to sell. We next permit security design: we let the issuer construct new securities that are secured by her initial assets. In prior work on security design, DeMarzo and Duffie (1999) studied the \textit{ex-ante}, single-security case: the issuer designs a single monotone security, learns her information, and chooses a proportion of her security to sell. We study the general case. We assume the issuer has a given set of initial assets that partition a common cash flow. She first designs any number of interim securities that are secured by her initial assets. She then learns her information and designs any number of \textit{ex-post} securities that are secured by her interim securities. This general setting includes DeMarzo and Duffie (1999) as a special case.

We show that each of the following two strategies of the issuer are optimal. They are also equivalent in the sense that they raise the same revenue at issuance and yield the same aggregate payout for any cash flow realization.

1. The issuer pools her initial assets, sees her information, and issues a single standard debt security whose face value is decreasing in her information.

2. The issuer pools her initial assets, tranches the result into a maximal set of prioritized debt securities, sees her information, and sells those tranches whose seniority exceeds a threshold that is decreasing in her information.\textsuperscript{7} Such behavior is commonly observed in practice.\textsuperscript{8}

\textsuperscript{5} Myers’s hypothesis finds empirical support in Fama and French (2002), Flannery and Rangan (2006), Opler \textit{et al} (1999), and Shyam-Sunder and Myers (1999). These papers also find some support for the alternative “static tradeoff” model in which a firm’s optimal capital structure is determined by balancing the tax and other benefits of debt against the costs of financial distress.

\textsuperscript{6} To clarify the role of distributional assumptions, FOSD (together with increasing security payout functions) suffices to show that an issuer will sell her less informationally sensitive assets first. Under the stronger HRO condition, these are her more senior assets.

\textsuperscript{7} In both strategies, the issuer first pools her initial assets. Intuitively, any function of these assets’ payouts can be rewritten as a function solely of the underlying cash flow, so pooling them does not limit the issuer’s subsequent actions in any way.

\textsuperscript{8} For empirical evidence, see Begley and Purnanandam (2016) and Franke and Krahnen (2004). With an uninformed issuer, retaining the equity tranche may preserve informational symmetries by discouraging information gathering by investors (Dang, Gorton, and Holmström 2015).
In the first strategy, the issuer writes a single standard debt security after seeing her information. DeMarzo and Duffie (1999) assume the issuer must design her security before seeing her information. While they also find debt is optimal, there is a key difference. In their model, the issuer signals higher information by retaining more of a debt security whose face value – while endogenous – does not vary with her information. In our case, a more optimistic issuer instead designs a debt security whose face value is lower.

Since the issuer in our setting is free to issue *ex ante* debt as in DeMarzo and Duffie (1999), issuing *ex post* debt must be better for her. A simple intuition is as follows. Selling fewer shares of a fixed debt security (as in DeMarzo and Duffie) lowers the payout to investors by the same proportion for any cash flow. In contrast, choosing a lower face value (as in our model) reduces the payout to investors only when the underlying cash flow exceeds the face value, which is more likely to occur if the issuer’s information is high. Thus, an optimistic issuer in our model retains more of the cash flow in states that, as a result of her greater optimism, have become more likely. This is a more efficient way to signal optimism than retaining more of the cash flow in every state. It is analogous to our finding that, with a fixed set of assets, the issuer most efficiently signals a given increase in her information by retaining those assets whose expected values have risen the most as a result of her higher expectations. In the security design setting, she instead redesigns her security so that she retains more of the cash flow in those states whose probabilities have risen the most as the result of her greater optimism.

The preceding results yield predictions that have been confirmed empirically. Begley and Purnanandam (2016) find that when the equity (retained) tranche of a pool of residential mortgage-backed securities (RMBS’s) makes up a larger proportion of the pool’s face value, the loans in the pool have lower subsequent delinquency rates conditional on observables and the securities that are sold fetch higher prices conditional on their credit ratings. This is implied by our model, in which the issuer retains more tranches in order to signal optimism.

Begley and Purnanandam (2016) also find that issuers retain larger proportions of the face value of RMBS pools that contain a higher proportion of no-documentation (“no-doc”) loans, controlling for other loan observables. Our model predicts this as well. To see why,
let use interpret the issuer’s information as a privately observed local macroeconomic shock that is independent of the proportion of no-doc loans in the issuer’s loan pool. Let us also assume, moreover, that no-doc loans are more fragile: their loan repayment chances – and thus their value to investors - are more sensitive to the local shock.9 Thus, a no-doc issuer gains more issuance revenue than an issuer of full-documentation loans if she convinces investors that the shock is high. In order to deter imitation by more pessimistic types, an optimistic no-doc issuer must therefore send a more costly signal: she must raise her retention rate more for a given increase in the shock. But by standard properties of signaling models, she sells her entire loan pool when she sees the lowest possible shock.10 The retention rate thus starts at zero in both cases and grows faster as the shock rises in the case of no-doc loans. The unconditional expected retention rate is therefore higher in the case of no-doc loan pools.

The preceding security design results assume that the issuer’s type and cash flow are discretely distributed. To facilitate the application of our results, we also study the limit as the issuer’s information and her realized cash flows become continuous.11 In this limit, the face value of the optimal ex-post debt security is given by a simple differential equation that has a unique solution. Moreover, this equation describes an equilibrium of the continuous model, and the issuer’s expected profits in the discrete model converge uniformly to her expected profits in this equilibrium.12 Hence, the continuous model is likely to give good predictions in applied settings.

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9 When the shock takes its lowest value, repayment probabilities may be lower for no-doc loans than for full-doc loans. This plays no role in the intuition since, for the lowest shock, the issuer’s behavior does not depend on her loans’ repayment chances: she simply sells her entire loan portfolio.

10 This holds since (a) the Intuitive Criterion selects a separating equilibrium in our model and (b) in any separating equilibrium of a signaling game, the lowest type cannot be punished with pessimistic beliefs and thus must take the action that maximizes her payoff under symmetric information. Hence, she must realize all the gains from trade with investors by selling her whole portfolio.

11 Previous applications of these continuum results include section 4.2 of DeMarzo (2005) and section 7.2 of Frankel and Jin (2015). DeMarzo (2005) cites an earlier version of our paper, DeMarzo (2003), for these results.

12 Manelli (1996, 1997) studies a general sequence of finite signaling games (which have finite type and message spaces) that converges to a continuous signaling game that, like ours, has compact type and message spaces. Manelli (1996) shows that any sequence of equilibria of the finite games has a convergent subsequence that converges to some equilibrium of the continuous game. Similarly, Manelli (1997) shows that if a sequence of equilibria of the finite games, each of which satisfies the Never a Weak Best Response criterion of Kohlberg and Mertens (1986), converges to some equilibrium of the continuous game, then this limiting equilibrium satisfies the same criterion. However, because Manelli’s finite games have finite
The rest of the paper is organized as follows. The base model with a fixed set of assets is studied in section 2. We turn to the important special case of informationally ordered assets in section 3. The general security design game is analyzed in section 4. We survey other related literature and summarize our results in section 5.

2. The Asset Sale Game

We begin with a base model (the “Asset Sale game”) in which a privately informed, risk-neutral issuer is endowed with a fixed portfolio of assets. The issuer has one-dimensional private information; an increase in this information weakly raises the expected return of each asset. The issuer may offer assets for sale to a set of uninformed, risk-neutral investors who behave competitively and have deep pockets. Selling all of her assets is efficient as the issuer is relatively impatient; for instance, she may have attractive alternative investments or face liquidity or capital requirements. However, because there is asymmetric information, she may retain some of her assets in order to signal that she is optimistic and thus to obtain higher prices.

Signaling games often have multiple equilibria. Moreover, the issuer’s strategy space in the Asset Sale game is unusually large as she sends a multidimensional signal: the quantity of each asset that she offers for sale. Nevertheless, we show that the game has a unique equilibrium that satisfies the Intuitive Criterion and that can be computed recursively – a result that underlies the rest of the paper.\textsuperscript{13} We also show, via two computed examples, that greater optimism may lead the issuer to sell more of a given asset, in contrast with single-asset models.

2.1. The Base Model

Our base model is as follows. The participants are a single issuer and a continuum of investors. All are risk-neutral and fully rational. The issuer is endowed with a portfolio of $n$ assets represented by the row vector $a \in \mathbb{R}_+^{1\times n}$, where $a_i > 0$ is the number of shares she

\textsuperscript{13} The uniqueness argument relies on our assumption that the issuer can set a price cap for each asset she sells; see below.
owns of asset $i$. Let $F_i$ denote the random future payoff of asset $i \in \{1, \ldots, n\}$. The issuer has private information about these payoffs, which is summarized by her type $t \in \{0, \ldots, T\}$. Conditional on the issuer’s type $t$, asset $i$ has an expected payoff of $f_i(t) = E[F_i | t]$.\footnote{More precisely, the payoff of asset $i$ is a function $F_i(\omega)$ of an exogenous, unknown random state $\omega$ that has some known unconditional distribution, as well as a family of conditional (on the issuer’s type) distributions. The conditional expectation $f_i(t)$ of asset $i$ is simply the expectation of $F_i(\omega)$ with respect to the conditional distribution of $\omega$ given $t$.} Let $F = (F_1, \ldots, F_n)' \in \mathbb{R}^{n \times 1}$ denote the column vector of random asset payoffs and let $f(t) = E[F | t] \in \mathbb{R}^{n \times 1}$ denote the column vector of expected payoffs conditional on $t$. We refer to $(a, f)$ as the issuer’s endowment.

We assume the issuer’s information can be ordered so that higher types have better news regarding the value of each asset:

**Assumption A (Monotone Expected Payoffs).** For $t > s$, $f(t) \geq f(s)$. In addition, $f(0) \geq 0$ and $af(0) > 0$.

This assumption states that a higher value of $t$ is (weakly) good news regarding the expected payoff of each asset, that an asset’s expected payoff is never negative (e.g., because of limited liability), and that the portfolio has a positive value even with the worst possible news.\footnote{Assumption A does not rule out zero (or even negative) realized payoffs for the securities, as long as the expected aggregate value of the portfolio remains positive.} If there is only one asset ($n = 1$), monotonicity is not restrictive since the types can be reordered. For $n > 1$, it implies that the ordering is common across the assets. Our leading example is a portfolio of securities backed by a common pool of assets, such as the debt and equity of a single firm or the mortgage-backed security tranches of a mortgage pool, with the issuer having private information about the future value of the underlying asset pool.\footnote{In the case of asset-backed securities, a sufficient condition for the expected payoffs $f(t)$ to be nondecreasing in the issuer’s type $t$ is that (a) the realized payoffs $F$ are nondecreasing in the random future value $Y$ of the underlying asset pool and (b) the distribution of $Y$ is nondecreasing in $t$ in a first-order stochastic dominance sense.}
On seeing her information $t$, the issuer chooses a quantity $q_i \in [0,a_i]$ of each asset $i$ to offer for sale. Let $q = (q_1, \ldots, q_n) \in \mathbb{R}^{1 \times n}$ denote the row vector of chosen quantities, where $0 \leq q \leq a$.

The issuer may also set a price cap $p_i \in [0,\infty]$ for each asset $i$. If the market clearing price for asset $i$ exceeds $p_i$, she charges $p_i$ and rations the asset.

Let $p = (p_1, \ldots, p_n) \in \mathbb{R}^{n \times 1}$ be the column vector of price caps.

We assume the issuer cannot borrow directly from investors. This prevents her from borrowing an arbitrarily large amount at the competitive riskless net interest rate of zero.

To justify this assumption, we make the standard assumption that a lender cannot force repayment of an unsecured loan. As for secured loans, we assume for the time being that investors are infinitesimal and dispersed and so cannot afford to pay the cost (assumed positive and arbitrarily small) of seizing the issuer’s collateral. We will permit the issuer to sell secured debt (which is equivalent to borrowing with collateral) in section 3.

The investors have a common, positive prior over the different possible realizations of the issuer’s type $t$. On seeing the issuer’s sale decision $(q, p)$, they form posterior beliefs $\mu(t \mid q, p)$. Investors are risk-neutral, behave competitively, and have deep pockets, so their demand for asset $i$ is perfectly elastic at the price $\sum_s f_s(t) \mu(t \mid q, p)$. Given the vector $p$ of price caps, the realized prices of the assets are thus given by the column vector

$$p(q, p) = p \land \sum_s f_s(t) \mu(t \mid q, p),$$

where $x \land y$ denotes the componentwise minimum of vectors $x$ and $y$.

The issuer is less patient than the investors: she discounts future cash flows at some rate $\delta \in (0,1)$, while investors’ discount factor is normalized to one. For instance, the issuer

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17 For vectors $x$ and $y$, the notation “$x \leq y$” means that for all $i$, $x_i \leq y_i$.

18 We could also allow the issuer to set reserve or minimum prices for the securities. However, since investors would refuse to buy overpriced securities, extending the strategy space in this way would play no role in equilibrium. (In other auction environments, a reserve price is useful to extract additional surplus from buyers. Our model differs in that investors are homogeneous and uninformed. Hence they earn no surplus even absent a reserve price.)

19 This restriction also justifies our assumption that the issuer cannot sell a quantity $q_i > a_i$ (“sell short”): there would be no way to force her to repay her future obligation of $(q_i - a_i)F_i$. 

may face capital requirements or need cash to invest in worthwhile projects. This difference in discount factors is the source of gains from trade in the model.

Suppose an issuer of type $t$ sells the quantities $q$ of her assets at prices $p = (p_1, \ldots, p_n)' \in \mathbb{R}_+^n$. She receives revenue $qp$ from the asset sale, plus the discounted expected payoff $\delta(a - q)f(t)$ of her retained assets. Her total payoff is thus

$$U(t, q, p) \equiv qp + \delta(a - q)f(t) = \delta af(t) + q(p - \delta f(t)).$$  \hspace{1cm} (2)

Our equilibrium concept is as follows.

**ASSET SALE EQUILIBRIUM.** A perfect Bayesian equilibrium for the Asset Sale game is an issuance strategy $(q(t), \bar{p}(t))$ for the issuer, a price response function $p(q, \bar{p})$, and a posterior belief function $\mu(t | q, \bar{p})$ for the investors, such that the following conditions hold.\(^{20}\)

1. **Payoff Maximization:** for any type $t$, the issuer’s sale decision $(q(t), \bar{p}(t))$ solves $\max_{q', \bar{p}'} U(t, q', p(q', \bar{p}'))$ subject to $0 \leq q' \leq a$.

2. **Competitive Pricing:** for any sale decision $(q, \bar{p})$, the price vector $p(q, \bar{p})$ satisfies equation (1).

3. **Rational Updating:** investors’ posterior beliefs $\mu(t | q, \bar{p})$ are given by Bayes’ rule whenever possible (i.e., on the equilibrium path).

Substituting the equilibrium price function $p(q(t), \bar{p}(t))$ for the price vector $p$ in (2) and omitting the fixed term $\delta af(t)$, we obtain the issuer’s equilibrium payoff function $u(t) \equiv q(t)[p(q(t), \bar{p}(t)) - \delta f(t)]$, which we refer to as the outcome of the equilibrium. It equals the additional profit that a type-$t$ issuer receives by virtue of her ability to sell a portion of her assets to investors.

\(^{20}\) The arguments of these functions are included for clarity. For instance, $q(t)$ should be interpreted as a function $q : \{0, \ldots, T\} \rightarrow \mathbb{R}_+^n$ that specifies the quantity vector chosen by each type of issuer.
2.2. Computation and Uniqueness

Like most signaling games, the above Asset Sale game has multiple equilibria. In order to obtain a unique prediction, we must use a signaling game refinement. The weakest such refinement is the Intuitive Criterion of Cho and Kreps (1987). In our context, this criterion states that if investors see an out-of-equilibrium sale decision \((q, \overline{p})\), their beliefs put weight only on those types who could possibly expect to gain from the deviation. More precisely, say a type-\(t\) issuer deviates to \((q, \overline{p})\). A sufficient condition for her to lose from this deviation is that her equilibrium payoff \(u(t)\) exceeds her maximum payoff \(q[\overline{p} - \delta f(t)]\) from the deviation – or, equivalently, that

\[
q\overline{p} < u(t) + \delta q f(t). \tag{3}
\]

Intuitively, if type \(t\) issues \(q'\) for the prices \(p'\) in equilibrium, her equilibrium payoff \(u(t)\) equals \(q'[p' - \delta f(s)]\). Substituting this expression for \(u(t)\) in (3), the right hand side of that equation can be rewritten as the sum of type \(t\)'s original revenue \(q'p'\) and her discounted cost \(\delta[q - q']f(t)\) of transferring \(q\) units of each asset to investors rather than \(q'\). This sum is the opportunity cost of deviating to \((q, \overline{p})\). Type \(t\) will not deviate to \((q, \overline{p})\) if this opportunity cost exceeds her maximum benefit \(q\overline{p}\) from doing so: if (3) holds.

The Intuitive Criterion states that on seeing the deviation \((q, \overline{p})\), investors must put zero probability on type \(t\) if she is not willing to choose \((q, \overline{p})\) but some other type might be: if condition (3) holds for \(t\) but fails for some other type \(s\). More precisely:

**The Intuitive Criterion.** A perfect Bayesian equilibrium of the asset sale game with posterior belief function \(\mu(\cdot \mid \cdot, \cdot)\) and outcome \(u(\cdot)\) is **intuitive** if, on seeing any quantity vector \(0 \leq q \leq a\) and price cap vector \(\overline{p}\), investors’ posterior probability \(\mu(t \mid q, \overline{p})\) is zero for any type \(t\) that satisfies (3) as long as there is some type \(s\) for which the inequality is reversed: \(q\overline{p} \geq u(s) + \delta q f(s)\).
An equilibrium that satisfies the Intuitive Criterion will be called \textit{intuitive}, as will investors’ beliefs in that equilibrium. If an outcome \( u(\cdot) \) is supported by an intuitive equilibrium, we will call the outcome intuitive as well.

Clearly, a simple way to ensure intuitive beliefs is for investors to respond to a deviation \((q, \bar{p})\) by putting all of their weight on a type \( t \) for which the opportunity cost \( u(t) + \delta q f(t) \) of the given deviation is at a minimum. Since no type has a lower opportunity cost of deviating to \((q, \bar{p})\) than this \( t \), there is no price vector \( p \leq \bar{p} \) that makes some type \( s \) willing to deviate to \((q, \bar{p})\) if it does not also make type \( t \) willing to choose this deviation: beliefs are intuitive. Since technically more than one type \( t \) may minimize the opportunity cost \( u(t) + \delta q f(t) \) of deviating, we must break ties in some arbitrary way. We will do so by putting all of investors’ weight on the minimum such type. Summarizing, a sufficient condition for the belief function \( \mu^* \) to be intuitive is that, following a deviation \((q, \bar{p})\), it satisfies

\[
\mu^*(\tau^*(q) \mid q, \bar{p}) = 1
\]  

where

\[
\tau^*(q) = \min\{\arg\min, [u^*(t) + \delta q f(t)]\}
\]  

is the lowest type \( t \) that minimizes \( u^*(t) + \delta q f(t) \).

We show below that any intuitive equilibrium of the Asset Sale game satisfies the following property.

\textbf{FAIR PRICING.} An equilibrium is \textit{fairly priced} if, for all \( i \) and \( t \), \( q(t) > 0 \) implies

\[
p_i(q(t), \bar{p}(t)) = f_i(t).
\]

That is, the price assigned to any asset \( i \) that is sold in equilibrium equals the conditional expected payout of this asset. This property is related, but not identical, to the notion of a separating equilibrium. Unlike separation, fair pricing implies that the issuer’s price caps never bind and that investors’ payoffs are identically zero, so social welfare is given by the issuer’s payoff function \( u(t) \). And unlike fair pricing, separation implies that investors can
infer the issuer's actual information $t$ in equilibrium, while fair pricing implies only that they can infer the conditional expected payout of any asset that the issuer chooses to sell – which is all they need to compute the asset’s fair-market value.

We now show that there exists a unique intuitive outcome $u^*(\cdot)$ of the Asset Sale game. This outcome is supported by a fairly priced equilibrium. For each type $t$, this fairly priced equilibrium maximizes the payoff of a type-$t$ issuer within the set of fairly priced equilibria and is efficient within this set. The outcome is given by the following maximization problem.

**Recursive Linear Program (RLP).** Let $A = \times_{i=1}^{n}[0,a_i]$ denote the set of feasible quantity vectors $q$. For type $t = 0$, let the quantity vector $q^*(0)$ equal the endowment vector $a$ and let $u^*(0) = (1 - \delta)af(0)$ denote the gains from trade from selling $a$ to investors. For any higher type $t > 0$, let

$$A_t = \{q \in A : \text{ for all } s < t, u^*(s) \geq q[f(t) - \delta f(s)]\}$$

(6)

denote the set of feasible quantity vectors $q$ for which the assigned payoff $u^*(s)$ of each lower type $s < t$ is at least as high as the payoff of this type $s$ from choosing $q$ and getting issuance revenue $qf(t)$. With this definition, let

$$u^*(t) = \max_{q \in A_t} (1 - \delta)qf(t)$$

be the highest payoff type $t$ can receive from a quantity vector $q$ in $A_t$ if the resulting issuance revenue is $qf(t)$. Finally, let

$$q^*(t) \in \arg\max_{q \in A_t} (1 - \delta)qf(t)$$

be any quantity vector in $A_t$ that yields this highest payoff.

The weak inequality in (6) states only that no type will be mimicked by a lower type. It does not explicitly rule out imitation by higher types. We ignore these constraints for now and verify later that they hold.

We construct a candidate equilibrium $e^* = (q^*(\cdot), p^*(\cdot), p^*(\cdot, \cdot), \mu^*(\cdot, \cdot))$ as follows. First, for each $t$, let type $t$’s quantity vector $q^*(t)$ be any solution to RLP and let type $t$’s
price cap vector be any solution to $\bar{p}^*(t) \geq f(t)$ (which will imply that the price caps do not bind if the equilibrium is fairly priced). Let posterior beliefs $\mu^*$ be given by Bayes’s Rule if $(q, \bar{p})$ is chosen by some type in equilibrium and by (4) and (5) if not. Finally, let the market price function $p^*(q, \bar{p})$ be result of substituting $\mu^*$ for $\mu$ in equation (1). Since beliefs $\mu^*$ are, by construction, intuitive, $e^*$ satisfies the Intuitive Criterion if it is an equilibrium.

The main result of this section is that $e^*$ is an equilibrium, is fairly priced, yields the outcome $u^*$, and is efficient (and optimal for the issuer) within the set of fairly priced equilibria. Moreover, any intuitive equilibrium of the Asset Sale game has the same outcome $u^*$ as $e^*$. Finally, in $e^*$, a higher quantity sold of any asset cannot raise the price of any asset, both on and off the equilibrium path.

**Proposition 1.** The Recursive Linear Program defined above has a unique solution $u^*$, which is strictly positive and nonincreasing in the issuer’s type $t$, and is the unique intuitive outcome. The profile $e^* = \left( q^* (\cdot), \bar{p}^* (\cdot), p^* (\cdot), \mu^* (\cdot | \cdot) \right)$ defined above is an intuitive equilibrium with outcome $u^*$, is fairly priced, and its price function is nonincreasing: for any $q, q'$ in $A$, if $q' \geq q$ then $p^* (q', \bar{p}) \leq p^* (q, \bar{p})$. Finally, for each type $t$, $u^* (t)$ is the highest attainable issuer’s payoff and social welfare in any fairly priced equilibrium.

**Proof:** See Appendix.

### 2.3. Asset Sale Equilibria

The Asset Sale game has the property that higher types may sell more of a given asset than a lower type. We show this in two examples. The first is a simple example with three types of issuer, whose solution can be easily computed by hand.

**Example 1.** Assume the issuer owns one share each of two assets: $a_1 = a_2 = 1$. The issuer has three possible types $t = 0, 1, 2$ and her discount factor is $\delta = 1/2$. The conditional expected payoffs of the assets are $f(0) = (1, 1), f(1) = (2, 3),$ and
\( f(2) = (5,4) \). That is, asset 2 (resp., 1) responds relatively more when the issuer’s type rises from 0 to 1 (resp., from 1 to 2). One can then verify that in the solution \( q^* \) to RLP, the issuer sells her entire portfolio when her type is 0, 2/3 shares of asset 1 and no shares of asset 2 when her type is 1, and no shares of asset 1 and 4/15 shares of asset 2 when her type is 2.

Intuitively, for each increase in her type, the issuer most efficiently distinguishes herself from the next lower type by retaining more of the asset whose valuation is raised proportionally more by the type increase. In particular, the increase in her type from 1 to 2 leads her to sell more of one asset, in contrast with the single-asset case where optimistic issuers always sell less. We next illustrate this principle in a more realistic example, with many issuer types:

**Example 2.** Suppose the issuer holds 1 share of each of 2 assets, with expected payoffs of each (conditional on \( t = 0,1,\ldots, 200 \)) given in Figure 1.

![Figure 1: Expected Payoffs of Assets for Example 2.](image)

Asset 1’s returns are less sensitive to information than asset 2’s returns for low \( t \), but are more sensitive for high \( t \).

While the overall sensitivity to information for each asset is similar (a 10% increase in value over the range of \( t \)), the right panel shows that asset 2’s return is more sensitive to increases in \( t \) for \( t < 50 \) and asset 1 is more sensitive when \( t > 50 \). Figure 2 depicts the equilibrium strategies for \( \delta = 0.9 \) (given by the solid line with circles for \( t = 0, 10, \ldots 200 \)). The lowest type sells all shares of both assets. Initially, she
signals type increases by selling fewer shares of asset 2, which is more sensitive than asset 1 is to these increases when the issuer’s type is low. Her approach changes when her type rises above 50: type increases then lead her to prize asset 1 relatively more than asset 2, so she signals these increases by selling fewer shares of asset 1 and more shares of 2. Finally, when her type is so high that asset 1 is retained in its entirety, the issuer has no choice but to sell less of asset 2 as a signal even though asset 2 is less informationally sensitive than asset 1 in this range.

Also depicted in Figure 2 is the equilibrium price function, as a contour plot showing the type investors infer given any quantity $q$; darker shades indicate lower types $t$, while white lines indicate the iso-price contours for $t = 0, 10, \ldots 200$. On each such contour, investors’ posterior beliefs and thus the prices of both assets are constant. As the contours are downward sloping, investors become more pessimistic as the issuer sells more shares of either asset. Moreover, for types $t$ between 50 and 120, beliefs drop discontinuously for a small increase in quantity. For example, starting from $q^*(60)$, a small increase in the quantity sold of either security will cause beliefs to drop discontinuously from 60 to below 20; similarly, a small increase from $q^*(90)$ would cause beliefs to drop to 0. This discontinuity occurs because in this example, the binding incentive constraint in RLP is non-local; for example, the type with the greatest incentive to mimic type 60 is type 18. The possibility that non-local constraints may bind distinguishes our setting from standard signaling models in which a single crossing property holds.
2.4. Splitting Securities

Thus far we have treated the issuer’s assets as exogenous. However, an issuer may be able to split the future payouts of her assets among several “subsecurities”. For example, investment banks holding mortgage pools often split these pools into distinct tranches prior to selling them to investors. Similarly, a corporation decides how to apportion the future cash flows of its assets among different securities, such as debt and equity. In this section we begin to consider the implications of the preceding analysis for optimal security design, and show that the issuer benefits by splitting existing assets into smaller sub-pieces prior to observing its type. Splitting assets gives the issuer additional flexibility in the issuance decision, allowing better types to signal their quality more efficiently. This result may
explain, for example, some of the gains associated with tranching mortgage pools into collateralized mortgage obligations (CMOs).\(^{22}\)

Below we provide a general result on the potential benefit of splitting securities:

**PROPOSITION 2.** Consider two endowments \((\hat{a}, \hat{F})\) and \((a, F)\) that both satisfy **ASSUMPTION A.** Suppose that for any non-negative portfolio \(q \leq a\), there exists a non-negative portfolio \(\hat{q} \leq \hat{a}\) such that \(\hat{q}\hat{F} = qF\). **ASSUMPTION A**Then for each type \(t\), \(\hat{u}^*(t) \geq u^*(t)\), so that the issuer’s payoff is weakly higher with \((\hat{a}, \hat{F})\).

**PROOF:** The proof is by induction on the type \(t\). The base case is trivial:

\[
\hat{u}^*(0) = (1 - \delta)\hat{a}\hat{f}(0) \geq (1 - \delta)\hat{q}(0)\hat{f}(0) = (1 - \delta)q^*(0)f(0) = (1 - \delta)af(0).
\]

Now let \(t > 0\) and suppose that for all \(s < t\), \(\hat{u}^*(s) \geq u^*(s)\). Consider the problem

**Error! Reference source not found.** for \(\hat{u}(t)\). Because \(\hat{q}(t)\hat{f} = q^*(t)f\),

\[
\hat{q}(t)\left[\hat{f}(t) - \delta \hat{f}(s)\right] = q^*(t)\left[f(t) - \delta f(s)\right] \leq u^*(s) \leq \hat{u}^*(s).
\]

Thus, \(\hat{q}\) lies in the set \(\hat{A}_t\) of feasible quantity vectors in RLP for type \(t\). Hence,

\[
\hat{u}^*(t) \geq (1 - \delta)\hat{q}\hat{f}(t) = (1 - \delta)q^*(t)f(t) = u^*(t) .
\]

As an application, suppose that before learning its information \(t\), the issuer has the opportunity to split all or some of the assets into tranches. For example, it might be possible to split asset \(F_1\) into securities \(F_{1a}\) and \(F_{1b}\) such that \(F_1 = F_{1a} + F_{1b}\). Because any portfolio involving \(F_1\) has the same conditional payoff as a portfolio with equal quantities of \(F_{1a}\) and \(F_{1b}\), **Error! Reference source not found.** immediately implies that this splitting cannot harm the issuer.

**EXAMPLE 3.** Consider the case of the issuer with two securities in Example 2. Suppose the issuer instead only had a single pooled asset with conditional payoffs \(f = f_1 + f_2\). By

\(^{22}\) It is not automatic that increased flexibility benefits the issuer, since in a strategic setting there can be gains to commitment. Indeed, in this setting the seller could gain by committing ex ante to quantities (e.g., committing to sell all assets before any information is learned). Our results here show that while committing to quantities may be helpful, committing to ratios (of the quantity of one asset to another) is not.
Error! Reference source not found., an issuer selling this pooled asset cannot do better than an issuer with the two separate securities. In particular, an issuer with a single asset is forced to issue $f_1$ and $f_2$ in equal proportions, whereas an issuer with the two securities finds it optimal to vary the proportions as shown in EXAMPLE 2. The lower two curves in Error! Reference source not found. illustrate the issuer’s payoffs in each case (the other two curves will be discussed below).

Error! Reference source not found. implies that absent transaction costs, there is no limit to the splitting that should occur. In Section 3 we consider the possibility of “unlimited” splitting – which generates the highest possible issuer payoff shown in Error! Reference source not found. – and show that it is equivalent to designing a single security after the issuer learns its private information.

In practice, even when securities are designed ex ante, there may be limits to the number of securities that can be issued. In that case there remains the question of how these securities should be designed. A further implication of Error! Reference source not found. is that optimal securities should not be “interior” in the space of feasible designs. To see why, consider two securities $(F_1, F_2)$ and without loss of generality suppose there is a quantity one of each. Suppose $F_1$ is interior, in the sense that $F_1 - \varepsilon F_2$ is a feasible design for sufficiently small $\varepsilon > 0$. Then the issuer could gain by using the following alternative feasible design:

$$(\hat{F}_1 = F_1 - \varepsilon F_2, \hat{F}_2 = [1 + \varepsilon]F_2).$$

Because any feasible portfolio of $(F_1, F_2)$ can be replicated as a portfolio of $(\hat{F}_1, \hat{F}_2)$, but not vice versa, by Error! Reference source not found. the issuer cannot be harmed (and generically will gain) by holding securities $(\hat{F}_1, \hat{F}_2)$ in place of $(F_1, F_2)$.

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23 DeMarzo and Duffie (1999) consider the case in which the issuer chooses a single security (tranch from a larger pool) to issue prior to learning the information $t$. The results here show that the issuer can do even better by splitting the cash flows further, creating multiple tranches to be sold.

24 For any $c_1, c_2 \in [0,1]$, the portfolio $c_1 F_1 + c_2 F_2$ is equivalent to the portfolio $c_1 \hat{F}_1 + \frac{c_2 + c_1 \varepsilon}{1 + \varepsilon} \hat{F}_2$, which is feasible as both assets weights lie in $[0,1]$. In contrast, the portfolio $c_1 \hat{F}_1 + c_2 \hat{F}_2$ is equivalent to $c_1 F_1 + \left[ c_2 (1 + \varepsilon) - c_1 \varepsilon \right] F_2$, which is infeasible if $c_2$ lies in the nonempty interval $\left[ 0, c_1 \varepsilon / (1 - \varepsilon) \right)$. 

19
In the space of monotone security designs, a firm’s debt and levered equity securities are commonly used “extreme” securities:25 the payoff to levered equity in low states cannot reduced without violating limited liability, and the payoff to debt cannot be reduced in high states without violating monotonicity. In the setting of EXAMPLE 2, consider pooling the original two securities as in Error! Reference source not found., but then tranching the pool into a debt and equity security. Error! Reference source not found. shows the issuer’s payoff from doing so when the debt has a face value of 172 given the pool’s payoff (conditional on the type \( t \)) is normal with a variance of 40.26 As shown in Error! Reference source not found., the issuer’s profits from pooling the original assets and tranching them into a senior debt security and a junior equity tranche are even higher than from selling them individually.

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25 By extreme we mean that the conditional payoffs cannot be replicated by a positive convex combination of other feasible securities.

26 It is now necessary to specify this conditional distribution (which we did not need in order to compute the prior equilibria) as the payoff of the debt and equity tranches are non-linear in the cash flows of the pool. The result that pooling and tranching into a debt-equity split is superior to individual sales does not depend on this distributional choice.
Debt and equity securities naturally differ in their priority and, consequently, in their informational sensitivity. In the following sections we give a formal definition of differential informational sensitivity, see how it can arise naturally as a result of priority versus subordination, and study the consequences for optimal asset sales and security design.

2.5. Informationally Ordered Assets

In this section we introduce an ordering of the issuer’s assets that is based on the sensitivity of their returns to the issuer’s information. For assets that can be so ordered, the equilibrium of the asset sale game has a simple form. The lowest type issuer sells all her assets; as her type rises, she retains her more informationally sensitive assets first as they are more efficient signals of greater optimism. Hence, the model predicts that firms will
tend to sell their informationally insensitive securities first in order to raise cash. This accounts for the common observation that a security is more liquid if its seller has less private information about its value.

2.5.1. Information Sensitivity

We say asset $i$ is more informationally sensitive than asset $j$ if its expected value changes by a larger percentage for a given change in the issuer’s type $t$. To ensure percentage changes are well-defined, we assume:

**Assumption B (Positive Expected Payoffs).** For all $i$, $f_i(0) > 0$.\(^{27}\)

This permits the following definition:

**Informational Sensitivity.** Asset $i$ is more informationally sensitive at $t$ than asset $j$ if, for all $s < t$, $f_i(t)/f_i(s) > f_j(t)/f_j(s)$.\(^{28}\) Asset $i$ is more informationally sensitive than asset $j$ if the above holds for all $t$. The assets display increasing information sensitivity (IIS) if for all $i > j$, asset $i$ is more informationally sensitive than asset $j$.\(^{29}\)

Intuitively, one would expect that the asymmetric information problem would be least severe for those assets that are least sensitive to the issuer’s private information. Hence, we might expect these assets to create the least price impact when sold, and thus be the most attractive to sell first. This intuition can be made precise as follows.

**Proposition 3.** Suppose asset $i$ is more informationally sensitive at $t$ than asset $j$.

Then for the equilibrium $q^*$, if $q_i^*(t) > 0$ then $q_j^*(t) = a$: a type $t$ issuer will not sell any of asset $i$ unless it sells asset $j$ in its entirety.

**Proof:** Omitting the constraint $0 \leq q \leq a$ in Error! Reference source not found. and terms that are independent of $q$, the Lagrangian is $qf(t) - \sum_{s < t} \lambda(s)q[f(t) - \delta f(s)]$. The

\(^{27}\) Under this requirement, a security can have zero payoffs in some states as long as, for any type $t$, there are states in which the security’s payoff is positive.

\(^{28}\) This condition is equivalent to $f_i(t)/f_j(t)$ increasing in $t$, or alternatively, $\frac{d}{dt} \ln f_i > \frac{d}{dt} \ln f_j$.

\(^{29}\) IIS is equivalent to the functions $f_i(t)$ being log-supermodular in $(i, t)$.
derivative with respect to \( q_i \) can then be written as \( \delta \sum_{s<t} [\lambda(s)f(s)] - kf(t) \), where \( k = \sum_{s<t} \lambda(s) \) - 1. Thus, \( q_i'(t) > 0 \) implies 
\[ \sum_{s<t} [\lambda(s)f(s)/f(t)] \geq k/\delta. \]
Since \( f(s)/f(t) < f(s)/f(t) \), it follows that \( \sum_{s<t} \lambda(s)f(s)/f(t) > k/\delta \), and thus \( q_i'(t) = \alpha_j. \) •

2.5.2. Hurdle Class Strategies

A further characterization of equilibrium is possible under IIS when all of the issuer’s assets can be ordered according to informational sensitivity. In this case, PROPOSITION 3 immediately implies that the issuer will choose to sell all of her less informationally sensitive assets and retain all of her more informationally sensitive ones, with the exact cutoff, or hurdle class, determined by her type. That is, she will choose a quantity vector in the set

\[ \hat{A} = \left\{ q \in A : \text{for some } \hat{i} \in \{0, \ldots, n\}, q_i \text{ equals } a_i \text{ for } i < \hat{i}, \right. \]
\[ \left. \text{zero for } i > \hat{i}, \text{ and some } \alpha \in [0, a_i] \text{ for } i = \hat{i} \right\} \]

of hurdle class strategies. In such a strategy, the issuer sells her entire endowment of all assets \( i < \hat{i} \) and no shares of any asset above \( \hat{i} \). Moreover, the hurdle class \( \hat{i} \) is nonincreasing in an issuer’s type:30

**PROPOSITION 4.** Suppose the issuer’s assets display Increasing Information Sensitivity. Then each type \( t \) chooses a quantity vector \( q^*(t) \) in \( \hat{A} \). Furthermore, for \( t > s \), \( q^*(t) \leq q^*(s) \) and hence \( t \) chooses a weakly lower hurdle class \( \hat{i} \) than \( s \) does.

**PROOF:** The fact that \( q^*(t) \) lies in \( \hat{A} \) follows from IIS and PROPOSITION 3. The existence of a hurdle class implies that \( q^* \) is ordered; that is, either \( q^*(t) \leq q^*(s) \) or \( q^*(s) \leq q^*(t) \). Also recall from PROPOSITION 1 that \( u^*(t) = (1-\delta)q^*(t)f(t) \leq u^*(s) = (1-\delta)q^*(s)f(s) \). Since \( f(t) \geq f(s) \) by ASSUMPTION A, it must be that \( q^*(t) \leq q^*(s) \) as claimed. •

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30 Intuitively, if a high type issuer had a higher hurdle class than a low type, she would receive higher securitization revenue. But the low type has a stronger incentive to raise her hurdle class than a high type, since her opportunity cost of parting with the additional securities is lower. So if the high type is willing to take her equilibrium action rather than that of the low type, the low type would strictly prefer the high type’s action to her own prescribed action.
Each vector \( q \) in \( \hat{A} \) can be summarized by a hurdle class \( \hat{i} \) and a proportion \( \alpha \) of asset \( \hat{i} \) to sell. Moreover, each such such pair \( (\hat{i}, \alpha) \) can be encoded as the real number \( \varphi(\hat{i}, \alpha) = \alpha + \sum_{i=0}^{\hat{i}-1} q_i \), from which \( \hat{i} \) and \( \alpha \) can each be easily recovered. Thus, under IIS the issuer essentially chooses a one-dimensional signal. The next result shows, moreover, that the local incentive compatibility constraint binds, whence this signal is determined by a simple difference equation.

**PROPOSITION 5.** If the assets display Increasing Information Sensitivity, then \( q^*(0) = a \), and for \( t > 0 \), \( q^*(t) \) is the unique element of \( \hat{A} \) such that

\[
q^*(t)[f(t) - \delta f(t-1)] = (1-\delta)q^*(t-1)f(t-1). \tag{7}
\]

**PROOF:** Appendix.

In the next section, we demonstrate an application in which IIS arises naturally.

**2.5.3. Application: Prioritized Securities**

We showed above that if assets can be ordered according to their informational sensitivity, then the asset sale game has a simple solution. We now show that in settings in which an issuer possesses securities that are backed by a common (or equivalent) asset pool, more senior securities will have lower informational sensitivity under a natural distributional assumption, the Hazard Rate Ordering.

Specifically, suppose the issuer possesses securities that are backed by pool of underlying assets with aggregate stochastic cash flow \( Y \in \mathbb{R}_+ \). The payoff of security \( i \in \{1, \ldots, n\} \) is given by \( F_i(Y) \) with \( F_i \) nondecreasing and nonnegative. For convenience, we write \( Y_t \) to represent the aggregate cash flow conditional on the issuer’s type \( t \), so that the conditional expected payoff of asset \( i \) is \( f_i(t) = E[F_i(Y_t) | Y_t] = E[E[F_i(Y_t) | Y_t]] \). We continue to assume **ASSUMPTION A** and **ASSUMPTION B**, which require that \( f_i(t) \) is positive and nondecreasing. While first order stochastic dominance is sufficient to assure this monotonicity, in order to obtain an even sharper characterization of the issuer’s behavior we make the following somewhat stronger assumption:
HAZARD RATE ORDERING (HRO). For all \( t, Y \) has a common support; and for all \( t > s \), \( \Pr(Y \geq y | t) / \Pr(Y \geq y) \) is decreasing in \( y \), for \( y \) in the support of \( Y \).\(^{31}\)

This property is weaker than the Monotone Likelihood Ratio Property (MLRP), which is commonly assumed in signaling environments.\(^{32}\)

In order to rank securities, we need to define a notion of seniority. Intuitively, a more senior security should receive a greater share of “early” cash flows than a more junior one. Roughly speaking, we say that security \( j \) is junior to security \( i \) if \( F_j(y) / F_i(y) \) is increasing in \( y \). The precise definition, which permits the denominator to be zero or the ratio to be locally constant, is as follows.

**SENIORITY AND PRIORITIZED SECURITIES.** Security \( j \) is junior to security \( i \) (and \( i \) is senior to \( j \)) if \( F_j(Y) = h(Y)F_i(Y) \) for some nonnegative, nondecreasing function \( h \) that takes more than one value with positive probability on that portion of the support of \( Y \) where \( F_i \) is positive.\(^{33}\) The set of securities \( \{F_i\} \) is prioritized if any asset \( i \) is senior to any asset \( j > i \).

Roughly speaking, a more senior security is paid a relatively greater share of early cash flows while a more junior security receives more of the later cash flows. If \( F_j \) is junior to \( F_i \), then its payoff can only cross that of \( F_i \) from below; the same is true of any positive scalar multiples of \( F_j \) and \( F_i \). We could also state the condition in terms of the sensitivity of final returns: the return of security \( j \) is more sensitive to \( Y \) than is that of security \( i \).

This definition generalizes standard notions of seniority, in which no payment is made to junior debtholders until senior debtholders have been repaid in full. This is a special case of the above definition: if senior and junior debt corresponds to securities \( i \) and \( j \) in the

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\(^{31}\) At the upper boundary of \( Y \)'s support, \( \Pr(Y \geq y | t) \) may be zero; we only require the ratio be decreasing up to this point. This definition is equivalent to \( g(y | t) /[1 - G(y | t)] \) decreasing in \( t \) where \( g \) (resp., \( G \)) is the conditional density (distribution) of \( Y \), but it applies also if \( Y \) has atoms or is discrete.

\(^{32}\) The MLRP states that the ratio of the conditional densities \( g(y | s) / g(y | t) \) increases with \( y \) for \( t > s \). This property implies HRO, which in turn implies FOSD (see, e.g., Ross (1983)).

\(^{33}\) That is, there is a constant \( c > 0 \) such that \( \Pr(F_i(Y) > 0 \text{ and } h(Y) < c) \) and \( \Pr(F_i(Y) > 0 \text{ and } h(Y) > c) \) are both positive.
definition, the function $h(Y)$ is zero while asset $i$ is being repaid, grows while asset $j$ is being repaid, and becomes constant once asset $j$ has been repaid in full. However, the definition is more general as repayments to both types of securities may be increasing over a range of cash flows as long as the junior security rises proportionally faster.

Some examples appear in Figure 4. The securities are ranked in order, with $F_1$ the most senior and $F_5$ the most junior, with the only exception that $F_4$ is noncomparable with $F_2$ or $F_3$. Thus $(F_1,F_4,F_5)$ and $(F_1,F_2,F_3,F_5)$ are prioritized sets.

![Figure 4: Examples of Prioritized Securities.](image)

Securities are ranked with $F_1$ most senior and $F_5$ most junior, with the exception that $F_4$ is non-comparable with $F_2$ and $F_3$.

We can now state the following important result, relating a security’s seniority to the conditions under which it will be liquidated: under HRO, if one security is junior to another, then it will be more informationally sensitive, and so the former security will be sold only after the latter has been completely liquidated.

**PROPOSITION 6.** Suppose HRO holds and the securities are prioritized. Then IIS holds, and the issuer will not sell any portion of a given security unless it also sells its more senior securities in their entirety.

**PROOF:** See Appendix.  

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As an immediate application of this result, if we consider a standard, strictly prioritized capital structure consisting of senior debt, junior debt, and equity claims, then PROPOSITION 6 implies that the quality of the issuer’s information will be perfectly correlated with seniority of the securities it issues: while all types will sell senior debt claims, the best issuers will refrain from selling junior debt, and only the worst types will sell equity securities.

The results of PROPOSITION 6 are applicable even if the underlying assets differ across securities, as long as the asset payoffs have equivalent distributions. For example, issuers of mortgage-backed securities may hold tranches of different mortgage pools, but as long as the pools have similar conditional distributions with respect to the issuer’s information (for example, with respect to prepayment risk), we should expect the issuer first to sell the more senior tranches of its various mortgage pools.\footnote{Even if the conditional distributions are not completely identical, the result will hold approximately in the sense that securities with very different seniority are still likely to be ranked by their information sensitivity even if the underlying assets differ.}

3. Security Design

We now apply the tools we developed in prior sections to the problem of security design. In particular, we generalize the single ex-ante security case studied by DeMarzo and Duffie (1999) by permitting an issuer to design multiple monotone securities before and/or after she receives her information. We first show that any intuitive equilibrium of this game is equivalent one in which the issuer pools her initial assets, sees her information, and then designs and sells a single security.

To further sharpen our predictions, we impose the Hazard Rate Ordering property studied above. We also assume *monotonicity*: the final payments to both the issuer and the investors are nonnegative and nondecreasing in the issuer’s cash flow. This is a common assumption in the security design literature;\footnote{See, for example, DeMarzo (2005), DeMarzo and Duffie (1999), Frankel and Jin (2015), Hart and Moore (1995), Matthews (2001), and Nachman and Noe (1994).} it can be justified as follows. Suppose the issuer has free disposal over her cash flow $Y$ and can also contribute her own cash to inflate it. Hence, if the payment to the issuer were decreasing in $Y$, the issuer would freely dispose of some cash in order to raise her payoff. And if, alternatively, the payoff to the security

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\footnote{See, for example, DeMarzo (2005), DeMarzo and Duffie (1999), Frankel and Jin (2015), Hart and Moore (1995), Matthews (2001), and Nachman and Noe (1994).}
holders were falling in \( Y \), the issuer would contribute her own cash to inflate \( Y \), pay the security holders less, and thus raise her own payoff. Finally, nonnegativity is motivated by the common feature of limited liability in securitization deals.

Under these two assumptions, we show that the issuer’s optimal ex-post security is standard debt with a face value that is decreasing in her information. We establish this by proving a formal equivalence between ex-post security design and a strategy in which the issuer pools her initial assets, designs a maximal set of prioritized debt securities or “tranches” that are secured by the pool, sees her information, and then chooses how much of each tranche to sell. And by our preceding findings on informationally ordered securities, under the Hazard Rate Ordering the issuer will sell those tranches whose seniority lies above a threshold that is increasing in her information. But this strategy is equivalent to selling a single standard debt security whose face value is decreasing in the issuer’s information. Thus, this simple ex-post strategy is also optimal.

In both strategies, the issuer pools her initial assets. In DeMarzo (2005), in contrast, it is sometimes better for the issuer not to do this. Why? In our setting, the initial assets partition a common underlying cash flow. Hence, each pre-existing asset can be exactly replicated by a new asset that is secured by the pool of initial assets. As pooling does not restrict the issuer, it cannot harm her. In DeMarzo (2005), in contrast, the initial assets are backed by distinct and imperfectly correlated cash flows. This gives rise to two new effects. First, pooling prevents the issuer from signaling that one cash flow is high while another is low, which is bad. Second, by diversifying away the residual risk in the cash flows, pooling also allows the issuer to issue debt with a lower default risk and thus to reap more of the gains from trade with investors. Whether pooling is optimal depends on the tradeoff between these two forces, as shown in Theorem 5 in DeMarzo (2005).

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36 The two strategies are equivalent because any monotone security can be mimicked by a suitable portfolio of tranches and vice-versa. Equivalence means that the portfolios sold by the issuer in the two strategies raise the same revenue from investors and, for any realization of the underlying cash flow, entail the same final aggregate payout to investors.

37 We emphasize, however, that DeMarzo’s Theorem 5 relies explicitly on an earlier version of the results of this section, which appeared in the working paper DeMarzo (2003) and have not been published elsewhere.
3.1. The General Security Design Game

In the Asset Sale game, we restricted the issuer to choosing how many shares to sell of each of a given set of assets. This limits her; for instance, she cannot borrow money using some of her assets as collateral. In this section, this constraint is lifted.

In particular, we assume as before that the issuer is endowed with a set of initial assets. We add a simplifying assumption: these assets are secured by, and partition, a common exogenous stochastic cash flow $Y$.\(^{38}\) Before learning her type, the issuer designs a set of interim securities whose payouts are functions of the payouts of the initial assets. After she sees her type, she can then design a set of ex-post securities whose payouts are functions of the payouts of the interim securities.\(^{39}\) We show that any intuitive equilibrium of this game is equivalent to one in which the issuer pools her initial assets and issues a single ex-post security whose payout is a function of the pool’s final value $Y$.

Formally, the game we study is as follows. Let $\#(V)$ denote the dimension (number of components) of a vector $V$.

**THE GENERAL SECURITY DESIGN (GSD) GAME.** The issuer owns a cash flow $Y$ whose support is $Y$. She is also endowed with a finite set $F(Y) = \left(F_i(Y)\right)_{i=1}^{n}$ of initial assets whose payout functions $F_i(Y)$ partition this cash flow:

$$
\sum_{i=1}^{n} F_i(Y) = Y. \quad ^{40}
$$

The issuer first designs a finite set $I = \left(I^k\right)_{k=1}^{K}$ of interim securities; the payout of each such security $k$ is some function $I^k(F(Y))$ of her initial asset payouts which we write, for brevity, as $I^k(Y)$.\(^{41}\) The interim assets satisfy limited liability:

$$
\sum_{k=1}^{K} I^k(Y) \in [0,Y].
$$

The issuer then sees her type $t$ and

---

\(^{38}\) This means that the sum of initial asset payouts equals the realization $Y$ of the cash flow. This assumption is equivalent to assuming that, in addition to her initial assets, the issuer can also use any residual cash flow as collateral for her interim securities. Hence this residual cash flow functions as an initial asset.

\(^{39}\) Requiring the issuer to issue interim assets is not restrictive: any of them can be a pass-through security whose payout is identically equal to the payout of some initial asset.

\(^{40}\) That is, we define the new function $I'(Y) = I(F(Y))$ and then rename $I'$ to $I$. 

\(^{41}\) The issuer then sees her type $t$ and
offers investors a finite vector \( P = \left( P^j \right)_{j=1}^L \) of some number \( L \geq 1 \) of *ex-post* securities, where the payout of *ex-post* security \( j \) is a function \( P^j_i(I(Y)) \) of her interim assets’ payouts. These payouts satisfy limited liability: 
\[ \sum_{j=1}^L P^j_i(I(Y)) \in \left[ 0, \sum_{k=1}^K I^k(Y) \right] \]. At the same time, the issuer specifies a global revenue cap \( \rho \in \mathbb{R}_+ \): she commits that if her sale of *ex-post* securities raises revenue exceeding \( \rho \), she will keep \( \rho \) and forego the rest.\(^{42}\) Investors then assign a price \( p^j \) to each *ex-post* security \( j \). Let
\[ W^j_p(Y) = \sum_{j=1}^{\#(P)} P^j_i(I(Y)) \] (8)
denote the aggregate payout to investors that results from the action \((I, P, \rho)\).\(^{43}\) By construction, it satisfies limited liability: \( W^j_p(Y) \in [0, Y] \). The payoff of a type-\( t \) issuer from the action \((I, P, \rho)\) is the sum of her fixed discounted expected cash flow \( \delta E[Y | t] \) and her securitization profits \( \left( \rho \wedge \sum_{j=1}^L p^j \right) - \delta E[W^j_p(Y) | t] \).

The GSD game permits the issuer to employ many securitization schemes observed in practice. For instance, she may design interim assets consisting of two prioritized securities: a senior tranche \( I^1(Y) = \min \{D, Y\} \) with face value \( D \) and a mezzanine tranche \( I^2(Y) = \min \{D', Y - I^1(Y)\} \) with face value \( D' \). After discovering her type \( t \), she may then sell a quantity \( z^i \) of each tranche \( i = 1, 2 \). Formally, her two *ex post* securities are then given by \( P^i_i(I(Y)) = z^i I^i(Y) \) for \( i = 1, 2 \). This is a two-security generalization of DeMarzo and Duffie (1999). Alternatively, the issuer may sell senior and mezzanine debt *ex post* with type-contingent face values \( D_i \) and \( D_i' \) respectively. Formally, she designs a single *ex-ante* pass-through security \( I^1(y) = y \), while her two *ex-post* securities are then given by \( P^1_i(I(y)) = \min \{D_i, y\} \) and \( P^2_i(I(y)) = \min \{D_i', y - P^1_i(I(y))\} \). This is a two-

\(^{42}\) For instance, she might simply discard the residual or refund it to investors.

\(^{43}\) The notation \( \#(V) \) denotes the dimension (number of components) of a vector \( V \).
security generalization of the one-security case used by DeMarzo (2005) and Frankel and Jin (2015).

We define equilibrium in the GSD game as follows.\(^{44}\) A strategy for the issuer is a triplet \((I, P,(\bullet), \rho, (\bullet))\) which instructs her first to issue interim asset vector \(I\) and then, on seeing her type \(t\) and having issued interim asset vector \(I'\) (which equals \(I\) unless she deviated), to select the revenue cap \(\rho, (I')\) and the ex-post security vector \(P, (I') = (P^j, (I'))\) where the number \(L\) of ex-post securities can depend on \(t\) and \(I'\). Let \(g(t)\) denote the prior probability that the issuer’s type is \(t\). Let

\[
U(I, P, \rho | t, p) = [\rho \sum_{j=1}^{n(p)} p^j (I, P, \rho)] - \delta E[W^j_p (Y) | t]
\]

(9)
denote the issuer’s expected payoff from the action \((I, P, \rho)\) conditional on her type \(t\) and the investors’ pricing function \(p\). The issuer’s ex-ante expected payoff from choosing the strategy \((I, P, (\bullet), \rho, (\bullet))\) when investors use the pricing function \(p\) is then given by

\[
U(I, P, (\bullet), \rho, (\bullet) | p) = \sum_t g(t) U(I, P, (I), \rho, (I) | t, p).
\]

We will refer to the action \((P, \rho)\) that the issuer chooses after learning her type as her paction (an abbreviation of “ex-post action”).

**EQUILIBRIUM: GSD GAME.** An equilibrium of the General Security Design game consists of a strategy \((\bar{T}, P, (\bullet), \rho, (\bullet))\) for the issuer,\(^{45}\) together with a price function \(p\) and a belief function \(\mu\), such that the following conditions hold.

1. Payoff Maximization: (a) the prescribed interim asset vector \(\bar{T}\) is contained in \(\arg \max, U(I, P, (\bullet), \rho, (\bullet) | p)\) and (b) for each interim asset vector \(I\), the paction \((P, (I), \rho, (I))\) chosen by each type \(t\) lies in \(\arg \max_{(p, \rho)} U(I, P, \rho | t, p)\).

\(^{44}\) As before, we restrict to pure-strategy equilibria for simplicity.

\(^{45}\) This strategy instructs the issuer to choose the interim asset vector \(\hat{I}\) and then, following any interim asset vector \(I\), to issue the paction \((P, (I), \rho, (I))\) when her type is \(t\).
2. Rational Updating: let $T(P, \rho | I)$ denote the set of types who choose the paction $(P, \rho)$ conditional on having chosen any given interim asset vector $I$. Upon seeing the action $(I, P, \rho)$, the investors’ posterior probability $\mu(t | I, P, \rho)$ equals the probability that the issuer’s type is $t$ conditional on her type being in $T(P, \rho | I)$. That is, it equals $g(t) \left[ \sum_{s \in T(P, \rho | I)} g(s) \right]^{-1}$ if $t \in T(P, \rho | I)$ and zero otherwise.\(^{46}\)

3. Competitive Pricing: for any action $(I, P, \rho)$ of the issuer, the price of each ex-post security equals its expected payout given investors’ posterior beliefs:

$$p(I, P, \rho) = \sum_t E[P(I(Y)) | t] \mu(t | I, P, \rho). \quad (10)$$

The outcome of the game – the expected payoff of type $t$ from sticking to her strategy - is $u(t) = u(t | \bar{T})$ where $u(t | I) = U(I, P_t(I), \rho_t(I) | t, \rho)$ is the issuer’s expected payoff from issuing some interim asset vector $I$, learning that her type is $t$, and finally choosing the paction that is prescribed by her strategy given $I$ and $t$.

The Intuitive Criterion is defined in the GSD game as follows. Consider an equilibrium and any interim asset vector $I$. If type $t$ sticks to her continuation strategy, she will get $u(t | I)$. Now suppose that instead, she chooses some other paction $(P, \rho)$. A sufficient condition for her to lose from this deviation is that her equilibrium payoff $u(t | I)$ exceeds her maximum payoff $\rho - \delta E[W^I_p(Y) | t]$ from the deviation – or, equivalently, that

$$\rho < u(t | I) + \delta E[W^I_p(Y) | t]. \quad (11)$$

The right hand side of (11) is the minimum revenue that type $t$ requires to be willing to deviate to $(P, \rho)$. The Intuitive Criterion for the GSD game states that on seeing the

\(^{46}\) In particular, if an unexpected interim asset vector $I$ is chosen, investors believe that it was simply a mistake that conveys no information about the issuer’s type. This is in the spirit of the “no signaling what you don’t know” condition of perfect Bayesian equilibrium (Fudenberg and Tirole 1991, p. 332): the issuer’s interim asset vector $I$ cannot signal her type $t$ as she chooses $I$ before learning $t$. 

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deviation \((P, \rho)\) following \(I\), investors must put zero probability on type \(t\) if she is never willing to choose \((P, \rho)\) following \(I\) but some other type might be: if condition (11) holds for \(t\) but fails for some other type \(s\). More precisely:

**The Intuitive Criterion (GSD Game).** An equilibrium of the GSD game with posterior belief function \(\mu\) and outcome \(u\) is *intuitive* if, on seeing any action \((I, P, \rho)\), investors’ posterior probability \(\mu(t | I, P, \rho)\) is zero for any type \(t\) that satisfies (11) as long as there is some type \(s\) for which the inequality is reversed: for which \(\rho \geq u(s | I) + \delta E[W^I_{\rho}(Y) | s]\).

GSD games have complex strategy spaces. The next result shows that, without loss of generality, we can restrict the issuer to the simpler set of strategies in which she designs a single pass-through interim security and, on seeing her type, a single ex-post security. In the following, let an action \((I, P, \rho)\) be *expected* in a given equilibrium if it is selected with positive probability in that equilibrium.

**Proposition 7.** Let \(E = (\hat{I}, P(\bullet), \rho(\bullet), \rho, \mu)\) be any equilibrium of the GSD game with outcome \(u(t)\). Then the following profile \(\hat{E} = (\hat{I}, \hat{P}(\bullet), \hat{\rho}(\bullet), \hat{\rho}, \hat{\mu})\) is also an equilibrium of the GSD game, with the same outcome \(u(t)\). And if \(E\) is intuitive, then so is \(\hat{E}\).

1. Securitization. In \(\hat{E}\), the issuer’s interim asset vector \(\hat{I} = (\hat{I}^1)\) consists of a single security whose payout equals her cash flow: \(\hat{I}^1(Y) = Y\). She then chooses a pact action as follows for any given \(t\) and interim asset vector \(I\). First, if she deviated in the prior stage (so that \(I \neq \hat{I}\)), she chooses the pact action \((P_t(I), \rho_t(I))\) that she would choose in \(E\) after choosing \(I\). Else she issues a pact action that consists of her equilibrium revenue cap \(\rho_t(\hat{I})\) in \(E\) together with an ex-post asset vector consisting of a single security whose payout
equals the aggregate payout \( \sum_{j=1}^{n(i)} P_j^i(\hat{i})(\hat{i}(Y)) \) of her equilibrium ex-post security vector \( P_i(\hat{i}) \) in \( E \).

2. Beliefs. Let \( \tilde{T}(P, \rho \mid I) \) denote the set of types whose strategies in \( \tilde{E} \) instruct them to choose the action \((P, \rho)\) conditional on having chosen some interim asset vector \( I \). Upon seeing an action \((\tilde{I}, P, \rho)\) that is expected in \( \tilde{E} \), the investors’ posterior probability \( \mu(t \mid I, P, \rho) \) equals the probability that the issuer’s type is \( t \) conditional on her type being in \( \tilde{T}(P, \rho \mid I) \). That is, it equals \( g(t) \left[ \sum_{s \in \tilde{T}(P, \rho \mid I)} g(s) \right]^{-1} \) if \( t \in \tilde{T}(P, \rho \mid I) \) and zero otherwise. Upon seeing an action \((I, P, \rho)\) that is unexpected in \( \tilde{E} \), \( \tilde{\mu}(t \mid I, P, \rho) \) equals \( \mu(t \mid I, P, \rho) \): its value in \( E \).

3. Pricing. Prices are given by equation (10) with \( \tilde{\mu} \) substituted for \( \mu \): for any action \((I, P, \rho)\), the resulting price vector is

\[
\tilde{p}(I, P, \rho) = \sum_t \mathbb{E}[P(I(Y)) \mid t] \tilde{\mu}(t \mid I, P, \rho).
\]

**Proof:** See Appendix. ♦

By Proposition 7, we can restrict the issuer to doing nothing *ex ante* and, after seeing her information, designing a single *ex-post* security \( S : \gamma \to [0, \gamma] \). We will refer to this restriction of the GSD Game as the *Ex-Post Security Design (EPSD) Game*.

While the EPSD game is simpler than the GSD game, the issuer’s security is still an infinite-dimensional signal of her type. The infinite-dimensional signaling problem has not been solved in general.\(^{47}\) However, we can solve it in an important special case, using results from prior sections. As in section 4, we will assume the issuer’s information

\(^{47}\) Researchers have studied the two-dimensional signaling problem; see, e.g., Quinzii and Rochet (1985). We are not aware of solutions in three or more dimensions.
satisfies the Hazard Rate Ordering property. Moreover, we will restrict the issuer to the set of monotone securities, which is defined as

\[ M = \{ S: \mathbb{Y} \rightarrow [0, \bar{y}]: S(y) \text{ and } y - S(y) \text{ are nonnegative and nondecreasing in } y \in \mathbb{Y} \}. \]

Finally, we will assume the cash flow is discrete:

**Assumption C (Discrete Cash Flow).** \( Y \in \mathbb{Y} = \{ y_0, \ldots, y_n \} \).

Henceforth we normalize \( y_0 \) to zero and write \( y_n \) as \( \bar{y} \). Discreteness will be relaxed in the following section.

Given her information \( t \), if the issuer sells security \( S \in M \) for the price \( p \), her total payoff is \( p + \delta E[Y - S(Y) \mid t] = \delta E[Y \mid t] + p - \delta E[S(Y) \mid t] \). As before, we ignore the exogenous discounted expected cash flow \( \delta E[Y \mid t] \) and simply define the issuer’s payoff to be her surplus from the sale, \( p - \delta E[S(Y) \mid t] \).

Let \( S_t \) denote the security designed by an issuer of type \( t \). For convenience, we now define intuitive, fairly priced equilibria in the context of EPSD games. Each definition is the natural restriction of the analogous definition for a GSD game.

**EPSD Equilibrium.** A perfect Bayesian equilibrium of the EPSD game is a security design \( S_t \in M \) and price cap \( \bar{\rho}_t \in \mathbb{R} \), for each type \( t \), as well as a price function \( \hat{p}(S, \bar{\rho}) \) and belief function \( \hat{\mu}(t \mid S, \bar{\rho}) \), with the following properties:

4. **Payoff Maximization:** for all \( t \), the issuer’s choice \((S_t, \bar{\rho}_t)\) solves

\[
\max_{S, \bar{\rho}} \left\{ \hat{p}(S, \bar{\rho}) - \delta E[S(Y) \mid t] \right\} \text{ subject to } S \in M.
\]

---

48 One can also define a corresponding notion of monotonicity in the GSD game. For any given vector \( A \) of ex-ante securities, let \( X_A \) denote the set \( \{A(y) : y \in \mathbb{Y}\} \) of possible ex-ante security payout vectors. Let an issuer’s securities in the GSD game be **monotone** if (a) each ex-ante security \( A(y) \) is nondecreasing in the cash flow \( y \) and (b) for each type \( t \), the payout \( P^j_{t} (x) \) of each ex-post security \( j = 1, \ldots, L_t \) is nonnegative and nondecreasing in the vector \( x \in X_A \) of ex-ante security payouts, as is the residual \( y = \sum_{j=1}^{L_t} P^j_{t} (x) \). It should be clear that if the issuer’s securities are monotone in the GSD game, then her corresponding ex-post security is monotone in the EPSD game.
5. Competitive Pricing: for any monotone security $S \in M$ and price cap $\bar{\rho}$, the price function $\hat{p}(S, \bar{\rho})$ equals $\min \left\{ \bar{\rho}, \sum_t E[S(Y) \mid t] \mu(t \mid S, \bar{\rho}) \right\}$.

6. Rational Updating: the investors’ belief function $\hat{\mu}(t \mid S, \bar{\rho})$ follows Bayes’s rule when applicable.

**FAIR PRICING (EPSD GAME).** An equilibrium $\left( (S_t, \bar{\rho}_t)^T, \hat{p}, \hat{\mu} \right)$ of the GSD game is *fairly priced* if, for each type $t$, the price of *ex-post* security $S_t$ equals its expected value conditional on the issuer’s type: $\hat{p}(S_t, \bar{\rho}_t) = E\left[ S_t(Y) \mid t \right]$.

The outcome of the EPSD game is the function $\hat{u}(t) = \hat{p}(S_t, \bar{\rho}_t) - \delta E\left[ S_t(Y) \mid t \right]$ giving the securitization payoff of each type $t$.

**THE INTUITIVE CRITERION (EPSD GAME).** A perfect Bayesian equilibrium $\left( (S_t, \bar{\rho}_t)^T, \hat{p}(\cdot, \cdot), \hat{\mu}(\cdot, \cdot, \cdot) \right)$ of the EPSD game, with outcome $\hat{u}(\cdot)$, is *intuitive* if, for any security $S \in M$ and revenue cap $\bar{\rho} \in \mathbb{R}_+$ for which $\bar{\rho} \geq \hat{u}(t) + \delta E\left[ S(Y) \mid t \right]$ for some type $t$,

$$\bar{\rho} < \hat{u}(s) + \delta E\left[ S(Y) \mid s \right] \text{ implies } \hat{\mu}(s \mid S, \bar{\rho}) = 0.$$ 

We solve the EPSD game as follows. Given ASSUMPTION C, we can define the following set of elementary monotone securities:

$$F_i^*(Y) = (y_i - y_{i-1}) \times 1[Y \geq y_i] \text{ for } i = 1, \ldots, n.$$ 

That is, elementary security $i$ pays off if and only if the cash flow $Y$ is at least $y_i$, in which case it pays the incremental cash flow $y_i - y_{i-1}$. Further, let $f^*(t) = E\left[ F^*(Y) \mid t \right]$ be the vector of expected payoffs of these $n$ elementary securities, conditional on the issuer’s type $t$. Letting $a^*$ be a row vector consisting of $n$ ones, the endowment $(a^*, f^*)$ is then the maximal splitting of the cash flows $Y$ into a set of monotone securities.

In particular, *any* monotone security $S$ can be replicated by an allowable portfolio $q^S$ from the endowment $(a^*, f^*)$ and vice versa. Specifically, let $q^S$ denote the row $n$-vector whose
ith component is \( q^\delta_i = \left[ S(y_i) - S(y_{i-1}) \right] / (y_i - y_{i-1}) \). Clearly, \( q^\delta F^*(Y) = S(Y) \) and, since \( S \) is monotone, \( 0 \leq q^\delta \leq a^\star \). Similarly, any portfolio \( q \) of elementary securities corresponds to the single security defined by \( S^q(y) = qF^*(y) \). Clearly, \( S^q(0) = qF^*(0) = 0 \), \( S \) is nondecreasing, and, since \( a^\star F^*(y) \) has slope of at most one, so does \( S^q \). Therefore, \( S^q \in M \): it is a monotone security whose payout is identical to the portfolio \( q \).

This isomorphism allows us to use our prior results to analyze the security design equilibrium. First we extend our results from Section 2:

**PROPOSITION 8.** Suppose the issuer is restricted to monotone securities \( S \in M \) in the EPSD game and the issuer’s information satisfies FOSD: for any cutoff \( y \) and types \( t > s \), \( \Pr(Y \leq y \mid s) \geq \Pr(Y \leq y \mid t) \). Then in any intuitive equilibrium of the EPSD Game, the issuer’s optimal security \( S^r_t(Y) \) equals \( q^\star(t)F^*(Y) \) where \( q^\star(t) \) solves Error! Reference source not found.RLP.

**PROOF:** See Appendix.

The intuition for **PROPOSITION 8** is that the portfolio \( F^* \) is a maximally fine, monotone tranching of the cash flow \( Y \). Hence, the ex post design of a single monotone security derived from a random cash flow \( Y \) is equivalent to the issuer’s first tranching the cash flow as finely as possible and then, on seeing its type \( t \), choosing optimal quantities of each tranche to issue.

### 3.2. The Optimality of Debt

Because the securities \( F^* \) are prioritized, we can also use the results of Section 2.5.3 to develop a much stronger characterization of the optimal ex post security design. Indeed, if we strengthen our informational assumption to HRO, then the securities \( F^* \) will display increasing information sensitivity, and thus \( q^\star(t) \) will have the hurdle class property from **PROPOSITION 4**. But if \( q \) has hurdle class \( c \), then the corresponding security design is

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49 A similar construction appears in Hart and Moore (1995), although they use it for a different purpose.
That is, $S^q(Y)$ is equivalent to a standard debt contract with face value $D = y_{c-1} + q_c^*(t)(y_c - y_{c-1})$.

This observation leads to the main result of this section. Let $v(D, t) = E[Y \wedge D | t]$ denote the expected payout of a standard debt contract with face value $D$ conditional on the issuer’s type $t$. For $t \in \{0, \ldots, T\}$, and starting with $D_0^* = y_n$, define the sequence of face values $D_0^* > D_1^* > \ldots > D_T^* > 0$ according to\(^\text{50}\)

\[
v(D_{t+1}, t+1) - \delta v(D_{t+1}, t) = (1 - \delta) v(D_t, t).
\]

Equation (12) is the analogue, in the case of simple debt, to condition (7), which was used to characterize the equilibrium under IIS: each type $t + 1$ chooses the debt contract that makes the next lower type $t$ just willing not to mimic. We then have the following characterization for the security design game:

**Proposition 9.** Suppose the issuer’s information satisfies HRO and the issuer is restricted to monotone securities. Then in the unique intuitive equilibrium of the Ex Post Security Design game, the optimal security design is standard debt, with type $t$ issuing debt with face value $D_t^*$; that is, $S_t^*(Y) = \min\{Y, D_t^*\}$. The equilibrium face value $D_t^*$ is positive and strictly decreasing in the issuer’s type $t$.

The equilibrium price for debt with face value $D$ is $v\left(D, \max\left\{t : D_t^* \geq D\right\}\right)$.

**Proof:** See Appendix.

**Proposition 9** establishes that under HRO, standard debt is the unique optimal ex post security design, in which the level of the debt (or equivalently, its seniority with respect to the underlying cash flows) signals the issuer’s information. In contrast, DeMarzo and

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\(^{50}\) The left side of (12) is strictly increasing in $D$ for $D < y_n$. Because $v(D, t+1) > v(D, t)$ and $v(0, t) = 0$, a unique solution to (12) exists, and $D_{t+1}^* \in (0, D_t^*)$. 
Duffie (1999) show that if the issuer must design a single security to sell before learning its type \( t \), standard debt is also the optimal security design, but with a crucial difference: In their model the debt level is fixed ex ante and the proportion of it that the issuer retains serves as the signal of quality. Intuitively, retaining more shares lowers the payout to investors by the same proportion given any realization of the underlying state \( Y \). Here, lowering the face value also lowers the payout by a constant proportion, but only in states in which the security is not in default. These states are more likely to occur when the issuer has high quality assets. Thus, a lower face value is more efficient than retention as a way for an issuer to signal high quality and, by Error! Reference source not found. and Error! Reference source not found., the Intuitive Criterion selects the efficient equilibrium.

Nachman and Noe (1994) also show that debt is the optimal ex post security design when an issuer must raise a fixed amount of cash in order to fund a worthwhile project.\(^{51}\) The all-or-nothing nature of the financing problem leads to a pooling equilibrium in which each type of issuer sells the same standard debt security. Not knowing the issuer’s type, investors assign a uniform price to this security: the equilibrium is not fairly priced. In our model, rather than having a fixed funding target, the issuer seeks to maximize its securitization profits. This yields a fairly priced equilibrium in which the issuer signals high quality by choosing a lower face value.

### 3.3. Continuous Security Design

The preceding results assume discrete distributions of the issuer’s type \( t \) and the final asset value \( Y \). In many applications it is more convenient to consider a continuous type space and assets with continuous payoffs. We first develop such an extension heuristically, and then show formally that the discrete models converge to the continuous one.

An advantage of the continuous model is that the optimal ex post security design is given by a simple and tractable differential equation. It is also easily used in other applied settings such as DeMarzo (2005) and Frankel and Jin (2015). Finally, we exploit its

\(^{51}\) Their assumptions include Conditional Stochastic Dominance, which is equivalent to our Hazard Rate Ordering property (Nachman and Noe 1994, n. 13, p. 19), and the D1 refinement of Banks and Sobel (1987), which is stronger than the Intuitive Criterion.
simplicity to derive two empirical applications that have been confirmed by Begley and Purnanandam (2016).

### 3.4. Characterizing the Optimal Security

As before, let $v(D, t) = E[Y \land D | t]$ denote the payoff of simple debt with face value $D$. Letting $\Delta$ be the increment between types $t$, the incentive compatibility condition (12) becomes

$$v(D^*_t + \Delta, t) - \delta v(D^*_t, t) = (1 - \delta) v(D^*_t, t).$$  \hspace{1cm} (13)

For a continuous type space with full support $[0, T]$, we consider the limit of this incentive constraint as $\Delta$ becomes small. Taking the derivative with respect to $\Delta$ at $\Delta = 0$ yields the following differential equation and initial value problem for the face value, which we denote $D^*_t$ to distinguish it from the discrete solution:

**CONTINUOUS INITIAL VALUE PROBLEM (CP).**

$$\frac{d}{d\alpha} D^*_t = \frac{-v_2(D^*_t, t)}{(1 - \delta) v_1(D^*_t, t)} = \frac{-\delta E[\min(D^*_t, Y) | t]}{(1 - \delta) Pr(Y > D^*_t | t)}$$  \hspace{1cm} (14)

with $D^*_t : [0, T] \rightarrow \mathbb{R}$, together with the initial value

$$D^*_0 = \bar{Y},$$  \hspace{1cm} (15)

where $\bar{Y} = \inf \{y : Pr(Y \geq y) = 0\}$ is the essential supremum of $Y$.

The boundary condition (15) holds as the lowest type $t = 0$ is unconstrained in Error! Reference source not found. and thus sells $Y$ in its entirety. Equivalently, we may interpret this contract as an outright sale of the asset.

We now show that if there exists a unique solution to the above differential equation, it is a separating equilibrium of the continuous model:

**PROPOSITION 10.** Assume HRO and the existence of a unique, continuous solution $D^*$ to CP. Then there is a separating equilibrium of the continuous model of the
security design game in which, given its type $t$, the issuer announces a security whose payout function is $S^t(Y) = \min(D^\infty_t, Y)$.

**PROOF:** See online appendix (DeMarzo, Frankel, and Jin 2015).

We illustrate this result with a numerical example.

**EXAMPLE 4.** Suppose the cash flows $Y$ are lognormally distributed conditional on the issuer’s information. Let the issuer’s information be the drift of $Y$ (the mean of $\ln(Y)$) so that $Y = 100e^{(t-\frac{\sigma^2}{2})+\sigma Z}$ where $Z$ is standard normal. Suppose $t \in [0, .25]$ and $\delta = 0.95$. We derive the optimal face value of the debt as a function of $t$ for different volatilities $\sigma$ in Figure 5.

![Figure 5: Optimal Face Value of Debt as a Function of Issuer Type and Asset Volatility](image)

Note that the debt choice is decreasing in the issuer’s type $t$. On the other hand, the debt choice is not monotone in the (publicly known) volatility $\sigma$. But because the quality of the debt depends upon both face value and volatility, a more meaningful comparison is to compute the amount of cash raised by the issuer, $E[\min(D^\infty_t, Y) | t]$, for these same three

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cases. Recall that the issuer’s payoff equals the gains from trade $(1 - \delta) E[\min(D^\tau, Y) \mid t]$: the difference in discount rates, $1 - \delta$, multiplied by the expected portion of the final cash flow $Y$ that is transferred from the issuer to investors.

![Figure 6: Total Cash Raised as a Function of Issuer Type and Asset Volatility](image)

As shown in Figure 6, the amount of cash raised (and thus the issuer’s payoff) is decreasing in the volatility of the cash flows: it is easier to borrow against less risky assets. Thus, the signaling problem can induce an implied risk aversion for the issuer even though all agents in the model are risk neutral.52

### 3.5. Existence, Convergence, and Comparative Statics

PROPOSITION 10 assumes the existence of a unique solution to CP. In this section we provide sufficient conditions for this property to hold. Under these conditions, we show

52 On the other hand, the convexity of the curves in Figure 6 implies that the issuer can benefit from additional private information. [PETER: This seems to suggest that a mean-preserving spread in $t$ raises the issuer’s expected payoff. But from the figure we can conclude only that it raises her expected securitization revenue. We would need a graph that shows her total payoff as a function of $t$.] The impact of both private information and idiosyncratic risk on the choice of assets to pool is explored in DeMarzo (2005).
also that the solution to the discrete problem (12) converges to the continuous solution in the continuous limit. Hence, the unique solution to CP is close to the unique intuitive equilibrium in the discrete case. This result is important since there may be equilibria other than CP in the continuous case. All results in this section are proved in our online appendix (DeMarzo, Frankel, and Jin 2015).

In addition, a researcher may wish to embed the security design problem in a continuous model in which the joint distribution of types and shocks depends on the actions chosen by the issuer and other players in a prior “pregame” period (e.g., Frankel and Jin 2015). We show that as the gaps between types and shocks shrink to zero, the gap between profits in the discrete and continuous model also shrinks to zero, uniformly in the joint distribution of types and shocks (and thus in the pregame action profile). This result may be used in applications to show that the issuer's optimal action in the discrete case converges to that of the continuous model.53

In order to state our result, we first specify a continuous model as well as a sequence of discrete models \( i = 1, 2, \ldots \) that converge to the continuous one. We first describe the continuous model, which we denote “model \( \infty \)”. Let the issuer’s type \( \sim G \) have full support \([0,1]\) and let the final asset value \( Y \) have full support \([0,\bar{y}]\) with conditional distribution \( H(y|t) \) given the issuer’s type \( t \). We assume \( H \) is continuously differentiable, has no atoms, and satisfies the Hazard Rate Ordering property. We also assume \( H \) satisfies the following technical continuity property.

**LIPSCHITZ-H (L-H).** The conditional distribution function \( H \) is Lipschitz continuous and also has some minimum sensitivity to its arguments: there are constants \( k_0, k_1 \in (0,\infty) \) such that for all \( t \) in \([0,1]\) and \( y \) in \([0,\bar{y}]\), the derivative \( H_1(y|t) \) is in \((k_0,k_1)\) and \(-H_2(y|t)\) lies in \([k_0,y(\bar{y}-y),k_1]\).54,55 Furthermore, both

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53 The convergence of the issuer’s optimal action will generally require additional assumptions, such as the strict quasiconcavity of the issuer’s expected total profits as a function of the issuer’s pregame action.
54 A higher type \( t \) is good news about the shock and thus lowers \( H(y|t) \). Accordingly, we state the bounds in terms of \(-H_2(y|t)\) which is nonnegative.
55 As \( H(0,t) \) and \( H(\bar{y},t) \) are identically zero and one, respectively, we cannot require that \( H_2(y|t) \) be sensitive to the type \( t \) for all cash flows \( y \). However, we can require that as \( y \) moves away from zero or \( \bar{y} \),
partial derivatives of $H$ are Lipschitz continuous in the type $t$: there is a constant $k_2$ in $(0,\infty)$ such that for all $y$ in $[0,\bar{y}]$ and $t',t''$ in $[0,1]$, both $|H_1(y|t')-H_1(y|t'')|$ and $|H_2(y|t')-H_2(y|t'')|$ are less than $k_2|t'-t''|$.

We now define a sequence of discrete models $i=1,2,\ldots$ that converge to model $\infty$. Let $(N_i)_{i=1}^\infty$ and $(N'_i)_{i=1}^\infty$ be any two increasing sequences of positive integers. In model $i$, let the gaps between adjacent types $t$ and shocks $Y$ be $\Delta_i = 1/N_i$ and $\Delta'_i = \bar{y}/N'_i$, respectively. That is, $t$ lies in $S_i = \{0,\Delta_i,\ldots,1-\Delta_i,1\}$ and $Y$ lies in $S'_i = \{0,\Delta'_i,\ldots,\bar{y}-\Delta'_i,\bar{y}\}$. By construction, both gaps $\Delta_i$ and $\Delta'_i$ converge to zero as $i$ goes to infinity. Let the conditional distribution of $Y$ in model $i$ be the restriction of the continuous distribution function $H$ to types $t$ in $S_i$ and payoffs $y$ in $S'_i$. Similarly, the distribution of $t$ in model $i$ is the restriction of $G$ to types $t$ in $S_i$.\(^{56}\)

Let $E^i$ and $E^\infty$ denote the expectations operators in models $i$ and $\infty$, respectively. Let $v^i(D,t) = E^i[\min(D,Y)|t]$ denote the expected payout of simple debt with face value $D$ in model $i$ given a type $t \in S_i$. Let $v^\infty(D,t) = E^\infty[\min(D,Y)|t]$ denote the expected payout of the same security in model $\infty$ given a type $t \in [0,1]$.

Fix equilibria in models $i$ and $\infty$ in which the issuer’s security is simple debt. Let $D^i_t$ and $D'^\infty_t$ be the equilibrium face values of these securities for a given type $t$. The equilibrium price of this security in model $i$, denoted $p^i(t)$, is simply the security’s expected payout $v^i(D^i_t,t)$. And the issuer’s expected issuance profit, denoted $u^i(t)$, is simply the gains from trade $(1-\delta)v^i(D^i_t,t)$ as competition drives investors’ payoffs to zero. Similarly, in

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\(^{56}\) That is, in model $i$, the probability that the type does not exceed some $t$ in $S_i$ is $G(t)$, while the probability, conditional on a type $t$, that the shock $Y$ does not exceed some $y$ in $S'_i$ is $H(y|t)$.

\(^{57}\) This sensitivity rises at least linearly in $y$. This is ensured by the factor $y(\tau-y)$ in the lower bound on $-H_2(y|t)$. 

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the continuous model the price \( p^\infty(t) \) of the security equals the expected payout \( v^\infty(D^\infty_t, t) \) and the issuer’s profit \( u^\infty(t) \) equals the expected gains from trade \( (1-\delta)v^\infty(D^\infty_t, t) \).

We now state our first result, which concerns existence and uniqueness.

**Proposition 11.** Assume Hazard Rate Ordering and Lipschitz-\( H \).

1. There exists a unique function \( D^\infty \) that satisfies CP for \( v = v^\infty \). This function is decreasing and differentiable in the type \( t \), and takes values in \((0, \bar{y}]\). The associated price and profit functions, \( p^\infty \) and \( u^\infty \), are decreasing and continuous in the type \( t \).

2. For each discrete model \( i = 1, 2, \ldots \), there exists a unique, decreasing function \( D^i \) with \( D^i_0 = \bar{y} \) and satisfying (13) with \( v = v^i \) for all \( t \in S_i \) and \( \Delta = \Delta_i \).

An intuition is as follows. The Picard-Lindelöf theorem is the usual tool for proving the existence and uniqueness of the solution to a differential equation. Unfortunately, we cannot apply this theorem directly because the differential equation (14) in CP is not Lipschitz continuous in \( D^\infty \): it approaches negative infinity as \( D^\infty \) approaches \( \bar{y} \).

We sidestep this difficulty in the following way. We define upper and lower bounds on \( D^\infty \) using a modification of equation (14) that is Lipschitz continuous with constant \( k \). We then show that as \( k \) grows, these upper and lower bounds approach the same limit, which satisfies (14) and thus must be its unique solution \( D^\infty \).

Having established the existence of a unique solution, we now show that the face value functions in the discrete model converge to the face value of the continuous model, uniformly in the issuer’s type \( t \). Since the discrete equilibria uniquely satisfy the intuitive criterion, this result gives a reason to focus on the continuous equilibrium that we have identified.
**Proposition 12.** Assume the Hazard Rate Ordering and Lipschitz-$H$. For any $\varepsilon > 0$ there is an $i^{*}$ such that for all models $i > i^{*}$ and all types $t$ in $[0,1]$, \[ |D_{i}^{t} - D_{i}^{*}| < \varepsilon. \] 57  

The idea of the proof is as follows. For any model $i = 1, 2, \ldots$ and constant $k > 0$, we first show that any solution $D_{i}$ must lie between fixed upper and lower bounds $\overline{D}_{k}^{i}$ and $\underline{D}_{k}^{i}$, where these bounds are Lipschitz continuous with constant $k$. Moreover, these bounds converge to the aforementioned upper and lower bounds on $D^{\infty}$ as $i$ grows. By the prior intuition, these bounds on $D^{\infty}$ converge in turn to $D^{\infty}$ as $k$ grows. Thus, by taking $i$ and $k$ to infinity simultaneously, $\overline{D}_{k}^{i}$ and $\underline{D}_{k}^{i}$ - and thus $D_{i}$ which lies between them – must converge to the unique solution $D^{\infty}$ of model $\infty$.

Begley and Purnanandam (2016) find that issuers retain larger proportions of the face value of RMBS pools that contain a higher proportion of no-documentation loans. This is predicted by our model, using the following results.

**Proposition 13.** Assume the conditional distributions $\hat{H}(y|t)$ and $\hat{\tilde{H}}(y|t)$ satisfy Hazard Rate Ordering and Lipschitz-$H$. Suppose also that $v_{2}(D,t)/v_{1}(D,t)$ (which is positive) is never higher for $H = \hat{H}$ than for $H = \hat{\tilde{H}}$. Then the solutions $\hat{D}^{\infty}$ and $\tilde{D}^{\infty}$ to CP that correspond to $\hat{H}$ and $\hat{\tilde{H}}$, respectively, satisfy $\tilde{D}_{i}^{\infty} \leq \hat{D}_{i}^{\infty}$ for any type $t$.

**Proof:** See Appendix.

**Proposition 14.** Assume that, for any $t \in [0,1]$ and $y \in [0, \bar{y}]$, $\hat{H}(y|1) \geq \hat{\tilde{H}}(y|1)$ and $\hat{H}_{2}(y|t) \leq \hat{\tilde{H}}_{2}(y|t)$ 58 Then $v_{2}(D,t)/v_{1}(D,t)$ is never higher for $H = \hat{H}$ than for $H = \hat{\tilde{H}}$.

**Proof:** Integration by parts yields

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57 Technically, the function $D_{i}$ is defined only for types $t$ in the discrete set $S_{i}$. We extend it to all types $t$ in $[0,1]$ by evaluating it at the highest type in $S_{i}$ that does not exceed $t$.

58 As both derivatives are negative by Lipschitz-$H$, this assumption means that the effect is not smaller in absolute value for the no-documentation loans.
\[
v(D,t) = \int_{y=0}^{T} \min \{y, D \} dH(y | t) = D - \int_{y=0}^{D} H(y | t) dy,
\]
whence \(v_2(D,t)\) equals \(-\int_{y=0}^{D} H_2(y | t) dy\) which is larger for \(H = \tilde{H}\) than for \(H = \hat{H}\).

And \(v_1(y,t)\) equals \(1 - H(y | t)\) which can be written as \(1 - H(y | 1) + \int_{y=1}^{1} H_2(y | s) ds\), which is smaller for \(H = \tilde{H}\) than for \(H = \hat{H}\). Thus, \(v_2(D,t) / v_1(D,t)\) (which is positive) is larger for \(\tilde{H}\) than for \(\hat{H}\). ♦

To apply these results to the findings of Begley and Purnanandam (2016), suppose a bank plans to lend a given \(Y\) dollars. Prior to lending, it flips a coin to decide the proportion of no-documentation loans to issue. After lending, it sees private information \(t\) in \([0,1]\) concerning a local macroeconomic shock that affects its loans’ repayment probabilities. Assume, plausibly, that the distribution of the bank’s information \(t\) does not depend on the outcome of the prior coin flip.

Let \(\tilde{H}(y | t)\) and \(\hat{H}(y | t)\) denote the conditional distributions of the bank’s aggregate loan repayments \(Y\) conditional on the bank’s information \(t\), in the low- and high-documentation cases respectively. The main effect of requiring documentation is to prevent loan applicants from exaggerating their financial strength. Thus, even if the bank receives the best possible information, \(t = 1\), about the macro shock, the probability that repayments fall below any fixed threshold \(y\) will be higher when there are more no-documentation loans in the pool. This is the first inequality assumed in Proposition 14.

Moreover, as no-documentation borrowers tend to be more financially fragile conditional on observables, the performance of their loans will be more sensitive to local macroeconomic conditions. Thus, the probability that repayments will fall below any given threshold \(y\) will be more sensitive to the bank’s information \(t\) about the local macro shock if the pool contains more no-documentation loans. This is captured by the second inequality assumed in Proposition 14.

Now combining the two preceding propositions, the model predicts that if a bank includes more no-documentation loans in its pool, the face value \(D_t\) of its security will be lower for
any given information \( t \) and hence the retained face value \( \bar{y} - D_t \) will be higher. Finally, as \( t \) is independent of the documentation decision, the unconditional expectation \( E(\bar{y} - D_t) \) of retained face value will also be higher for the low-documentation loan pool. Our thus model predicts that banks will retain higher proportions of the face value of loan pools that have more no-documentation loans, as found empirically by Begley and Purnanandam (2016).

We turn now to some useful technical results. The first gives conditions under which the convergence of the discrete model to the continuous model is uniform in various parameters. This property can be useful in applications in which the security design game is preceded by some interaction in which the issuer also chooses an optimal action, as in Frankel and Jin (2015). In such settings, the result can help establish that the issuer’s optimal choices in the discrete model are well-approximated by her optimal choice in the continuous model.

Henceforth, we assume the distribution of types \( G \) is continuous with a strictly positive density \( g \) that satisfies the following technical condition:

**Lipschitz-G (L-G).** There are constants \( k_3, k_4 \in (0, \infty) \) such that for all types \( t \) and \( t' \) in \([0,1]\), \( g(t) \leq k_3 \) and \( |g(t) - g(t')| \leq k_4 |t - t'| \).

In some applications (e.g., Frankel and Jin (2015)), the cash flow \( Y \) depends both on a shock and on the actions of the issuer and others. We capture this in a simple way as follows. Let \( Y \) be the product of an exogenous random variable \( z \in [0,1] \) and a *cash flow parameter* \( \bar{y} > 0 \) that may depend on the outcome of a prior stage of play and lies in some bounded interval \([0,y]\). Henceforth, let \( H \) denote the fixed distribution of the *relative cash flow* \( z \), conditional on the type \( t \). We also reinterpret Lipschitz-\( H \) as referring to this redefined function \( H \), with \( y \) and \( \bar{y} \) in that assumption replaced by \( z \) and 1, respectively.

We now show that the key functions of the model converge uniformly in the distributions \( G \) and \( H \), the type \( t \in [0,1] \), and the cash flow parameter \( \bar{y} \).\(^{59} \) These key functions are the

\(^{59}\) Uniformity in \( t \) does not apply to the expected profit functions \( E_u \) and \( E_{\Pi} \), as they do not depend on \( t \).
face value function $D^i$, price function $p^i$, and the conditional profit function $u^i$. We also show uniform convergence of the issuer’s unconditional expected issuance profits $E u^i = E^i \left[ u^i(t) \right]$ to its continuous analogue, $E u^\infty = E^\infty \left[ u^\infty(t) \right]$. Finally, let $\Pi^i(t)$ denote the issuer's conditional total profits in model $i$: the sum of issuance profits $u^i(t)$ and the conditional expected portfolio return $E^i \left[ Y | t \right]$. Let $E \Pi^i = E^i \left[ \Pi^i(t) \right]$ denote unconditional expected total profits.\(^{60}\) We show that these converge uniformly to their continuous counterparts, $\Pi^\infty(t) = u^\infty(t) + E^\infty \left[ Y | t \right]$ and $E \Pi^\infty = E^\infty \left[ \Pi^\infty(t) \right]$.

**Proposition 15.** Fix constants $k_0, k_1, k_2, k_3, k_4$, and $y$, all in $(0, \infty)$. Let $H$ be the set of conditional distribution functions $H(z | t)$ that satisfy Hazard Rate Ordering and Lipschitz-$H$ with constants $k_0, k_1, k_2$. Let $G$ be the set of distribution functions $G$ that satisfy Lipschitz-$G$ with constants $k_3$ and $k_4$. For all $\varepsilon > 0$ there is an $i^*$ such that for all models $i > i^*$, $G$ in $G$, $H$ in $H$, $y \in [0, y]$, and $t$ in $[0,1]$, $|\omega(t)-\omega^\infty(t)|$ is less than $\varepsilon$ for $\omega$ equal to $D, p, u, and \Pi$; and $|E\omega^i-E\omega^\infty|$ is less than $\varepsilon$ for $\omega$ equal to $u$ and $\Pi$.\(^{61}\)

Finally, we show a homogeneity property that can simplify the analysis of models in which security design is embedded (e.g., Frankel and Jin (2015)).

**Corollary 16.** The face value functions $D^i$ and $D^\infty$, the price functions $p^i$ and $p^\infty$, the profit functions $u^i$, $\Pi^i$, $u^\infty$, and $\Pi^\infty$, and the issuer’s expected profits $E u^i$, $E \Pi^i$, $E u^\infty$, and $E \Pi^\infty$, defined above, are all homogeneous of degree one in the cash flow parameter $y$.

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\(^{60}\) This last quantity is especially important in applications: if there is a pregame period, the issuer will act so as to maximize the sum of $E \Pi^i$ (perhaps multiplied by a discount factor) and any pregame payoff.

\(^{61}\) As in Proposition 14, we extend these functions to all types $t$ in $[0,1]$ by evaluating them at the highest type in $S_i$ that does not exceed $t$. 

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4. Final Remarks

4.1. Other Related Literature

Our work is related to He (2009), who considers the portfolio liquidation problem in a two asset generalization of Leland and Pyle (1979) with a risk-average issuer and risk-neutral investors. In that context, if the two assets’ returns are positively correlated conditional on the issuer’s information, then selling more of one asset lowers investors’ demand curve for the other. Intuitively, retaining more of the second asset is costlier, and thus a stronger quality signal, for a risk-averse issuer who keeps more of the first asset since asset returns are correlated. In our model, which instead assumes risk-neutrality, such cross-signaling also occurs but is due the correlation in the issuer’s information across assets. Also, the CARA-Normal framework in He (2009) precludes the consideration of securities whose payouts depend nonlinearly on underlying asset returns, which is a primary focus of our analysis.

We focus throughout on a market setting in which the issuer can signal quality through quantity choices. Williams (2015) instead adapts the competitive search framework of Guerrieri, Shimer, and Wright (2010) to demonstrate that it is also possible to achieve a similar equilibrium outcome in which issuers signal quality by choosing the liquidity of the market in which they choose to trade, and shows that debt is the optimal monotone security in that context as well. An interesting extension might be to consider equilibrium liquidity choices when the issuer can sell multiple securities as in our paper. Another possibility, in a dynamic context, might be to allow the issuer to signal by delaying its trades as in Varas (2014).

4.2. Summary of Findings

In this paper we consider study the problem of an informed issuer who wishes to sell securities to raise cash. Using the Intuitive Criterion, we identify the unique equilibrium when the issuer has a fixed set of securities, and her information has a monotone impact on the expected payout of each one. Moreover, “splitting” a security into smaller tranches before becoming informed cannot lower the issuer’s payoff.
We show, further, that when securities can be ordered according to their informational sensitivity, a “pecking order” result applies: the least informationally sensitive securities are sold first. When information satisfies the Hazard Rate Ordering property and securities are prioritized by seniority (which we define in a new and flexible way), information sensitivity coincides with seniority: the issuer will issue all securities whose seniority exceeds a given threshold. More optimistic issuers choose a higher threshold and thus sell fewer securities.

We also consider the optimal design of a set of securities that are secured by a common cash flow. We show, first, that the issuer optimally waits until receiving information about this cash flow, and then issues a single security. Under the additional assumption that her information satisfies the Hazard Rate Ordering property, this security is simple debt with a face value that is decreasing in her information. An equivalent strategy is to maximally tranche the cash flow \textit{ex ante} and then, \textit{ex post}, sell securities whose seniority exceeds a threshold that is increasing in the issuer’s information. In both cases, the issuer retains a larger portion of her cash flow when she is more optimistic, which Begley and Purnanandam (2016) confirm empirically.

The preceding security design results assume discrete information and shocks. By taking limits, we show that the optimal face value of the issuer’s \textit{ex post} debt security in the continuous case is given by a simple differential equation. Moreover, the issuer's expected profits in the discrete model converge uniformly to her profits in the continuous model. Our solution in the continuous case implies that issuers with worse asymmetric information problems will retain larger tranches of their cash flows, as Begley and Purnanandam (2016) also find.

5. References


6. **Appendix**

**Proof of Proposition 1.** We first show by induction that each $u^*(t)$ is well defined (exists and is unique) and positive. First, $u^*(0)$ is well-defined and positive by **Assumption A.** Now suppose $u^*(s)$ is well-defined and positive for $s < t$. Then for type $t$, $A_t$ is nonempty as it includes the zero vector; it is compact as it is defined using weak inequalities. Hence, $u^*(t)$ and $q^*(t)$ exist by the Extreme Value Theorem. Also, since there exists $\varepsilon > 0$ sufficiently small so that $\varepsilon a$ lies in $A_t$, so $u^*(t) \geq (1-\delta)\varepsilon af(t) \geq (1-\delta)\varepsilon af(0) > 0$: it is positive as well.

Why is $u^*$ nonincreasing? By construction, the payoff $u^*(s)$ that RLP assigns to any type $s$ is not less than the payoff $q^*(t)(f(t) - \delta f(s))$ that this type would get by imitating any type $t > s$ in any fairly priced equilibrium. This imitation payoff, in turn, cannot be less
than the payoff \( u^*(t) = (1 - \delta)q^*(t)f(t) \) assigned to type \( t \) since the sales revenue \( q^*(t)f(t) \) is the same and, by ASSUMPTION A, the opportunity cost of selling the assets is no higher for \( s \) than for \( t \): \( \delta q^*(t)f(s) \leq \delta q^*(t)f(t) \). Intuitively, while type \( t \) sells for a higher price, in order to separate from lower types it must raise less cash, reducing its attainable surplus.

We next show by induction that for any type \( t \), \( u^*(t) \) is an upper bound on issuer’s payoff and on social welfare in a fairly priced equilibrium. Investors must break even in any fairly priced equilibrium. This implies that the issuer’s payoff equals social welfare. It also implies our base case: the payoff of an issuer of type 0 is bounded above by the potential gains from trade \( u^*(0) \) for this type. As for types \( t > 0 \), if type \( t \)’s equilibrium quantity vector \( q \) does not lie in \( A_t \), then there is a type \( s < t \) which, by choosing \( q \), gets a payoff \( q[f(t) - \delta f(s)] \) (by fair pricing for type \( t \)), which exceeds the upper bound \( u^*(s) \) on her equilibrium payoff: type \( s \) will deviate. Hence type \( t \) must choose a quantity vector in \( A_t \), whence \( u^*(t) \) is an upper bound on \( t \)’s payoff as well by fair pricing.\(^62\)

From the prior result, any fairly priced equilibrium with outcome \( u^* \) must be optimal for the issuer and is efficient within the set of fairly priced equilibria. We now verify that the profile \( (q^*(\cdot), p^*(\cdot, \cdot), u^*(\cdot, \cdot), \mu^*(\cdot, \cdot, \cdot)) \) defined in the text is a fairly priced equilibrium with outcome \( u \).\(^63\) This will imply that it is an intuitive equilibrium since, as shown in the text, beliefs \( \mu^* \) are intuitive.

By construction, the beliefs function \( \mu^* (\cdot, \cdot, \cdot) \) is given by Bayes’s rule on the equilibrium path, whence Rational Updating holds in the definition of an Asset Sale Equilibrium. Since the price function \( p^*(\cdot, \cdot) \) is the result of substituting \( \mu^* \) for \( \mu \) in (1), Competitive Pricing holds. To verify that the profile is an equilibrium, it thus remains to verify Payoff Maximization. We do so below.

\(^{62}\) The definition of \( u^* \) does not rule out pooling. If \( q^*(t)f(s) \) equals \( q^*(t)f(t) \), then type \( s \) could pool with type \( t \) in equilibrium: \( q^*(s) \) might equal \( q^*(t) \).

\(^{63}\)
We next verify that the given strategies and beliefs satisfy fair pricing and yield the payoff function \( u^* (\cdot) \) for the issuer. For any possible choice \((q, \bar{p})\) of the issuer, let \( T(q, \bar{p}) \) denote the set \( \{ t \mid (q, \bar{p}) = \text{equal to} (q^*(t), \bar{p}^*(t)) \} \) of types who make this choice in equilibrium. If \( T(q, \bar{p}) \) is a singleton \( \{t\} \), then \( p^*(q, \bar{p}) = f(t) \) by (1) as the price caps \( \bar{p}^*(t) \) are nonbinding by construction. Else let the types \( s < t \) lie in \( T(q, \bar{p}) \), whence \( q^*(s) = q^*(t) = q \) so, by RLP, \( u^*(s) = (1-\delta)qf(s) \geq q(f(t)-\delta f(s)) \) or, equivalently, \( q(f(t)-f(s)) \leq 0 \). Since \( f(t) \geq f(s) \) by ASSUMPTION A, if \( q_i > 0 \) then \( f(t) = f(s) \). Since this is true for all types in \( T(q, \bar{p}) \), \( q_i > 0 \) implies \( p^*(q, \bar{p}) = f(t) \) for all \( t \in T(q, \bar{p}) \) by (1): the equilibrium is fairly priced. Thus \( q^*(t)p^*(q^*(t), \bar{p}^*(t)) = q^*(t)f(t) \), whence the payoff of a type-\( t \) issuer is \( u^*(t) = (1-\delta)q^*(t)f(t) \) as claimed.

We now turn to Payoff Maximization. We first show that no type wishes to mimic any other type. For suppose type \( s \) mimics type \( t \). Since the equilibrium is fairly priced, type \( s \) does not gain from this deviation as long as the following inequality holds.

\[
IC(s, t): \quad q^*(t)(f(t) - \delta f(s)) \leq u^*(s) = (1-\delta)q^*(s)f(s).
\]

To verify \( IC(s, t) \) for all \( s, t \), it suffices to show that if \( IC(s, s') \) holds for all types \( s \) and \( s' \) that are strictly less than \( t \) (which is clearly so when \( t = 0 \)), then \( IC(s, s') \) also holds for all \( s, s' \leq t \). For \( s' < t \), \( IC(s', t) \) must hold by RLP. **Reference source not found.**, while \( IC(t, s') \) holds by the following lemma with \( p = f(s') \) and \( q = q^*(s') \); the lemma’s assumptions hold since \( f(s') \leq f(t) \) by ASSUMPTION A and since \( IC(s, s') \) holds for all \( s < t \) by induction.

**Lemma.** Let \( p \in \mathbb{R}^n_+ \) and let \( q \in \mathbb{R}^n_+ \) satisfy \( q \leq a \). Suppose \( q(p-f(t)) \leq 0 \) and, for all \( s < t \), \( q(p-\delta f(s)) \leq u^*(s) \). Then \( q(p-\delta f(t)) \leq u^*(t) \).

**Proof of Lemma:** Note that \( q(p-\delta f(t)) = q(p-f(t)) + (1-\delta)qf(t) \leq (1-\delta)qf(t) \). Now if \( q \) lies in \( A_t \) (defined in RLP), \( (1-\delta)qf(t) \leq u^*(t) \) so we are done. Else let \( \alpha \) be the largest scalar such that \( \alpha q \) lies in \( A_t \). Then \( \alpha < 1 \) and, for some \( s < t \), \( \alpha q(f(t)-\delta f(s)) = u^*(s) \geq q(p-\delta f(s)) \). Rearranging, and using the fact that \( \alpha < 1 \) and the definition of \( u^* \), we obtain
\[ q(p - \delta f(t)) \leq (1 - \delta)\alpha qf(t) + \delta(1 - \alpha)q(f(s) - f(t)) \leq (1 - \delta)\alpha qf(t) \leq u^*(t) \]

as claimed. 

Accordingly, no type will imitate any other type. We now show that no type will deviate to any choice \((q, \bar{p})\) that lies off the equilibrium path: for which \(T(q, \bar{p}) = \emptyset\). As shown in the text, if type \(\tau^*(q)\) does not gain from deviating to \((q, \bar{p})\), then no other type gains from deviating to \((q, \bar{p})\). It thus suffices to show that type \(\tau^* = \tau^*(q)\) will not deviate to \((q, \bar{p})\). Given beliefs \(\mu^*\), the price vector following the deviation is \(p^*(q, \bar{p}) = \bar{p} \wedge f(\tau^*) \leq f(\tau^*)\) by (1). Thus, for type \(\tau^*\) not to gain by deviating to \((q, \bar{p})\), it suffices that \((1 - \delta)qf(\tau^*) \leq u^*(\tau^*)\). By RLP, to prove this inequality it suffices to verify that \(q\) is in \(A_{\tau^*}\). For suppose not: \(q\) is not in \(A_{\tau^*}\). Let \(\alpha\) be the largest scalar such that \(\alpha q\) is in \(A_{\tau^*}\). Then \(\alpha < 1\) and by continuity we must have \(\alpha q [f(\tau^*) - \delta f(s)] = u^*(s)\) for some \(s < \tau^*\). Therefore, by (5),

\[
\alpha q[f(\tau^*) - \delta f(s)] = u^*(s) = u^*(s) + \delta qf(s) - \delta qf(s) > u^*(\tau^*) + \delta qf(\tau^*) - \delta qf(s).
\]

(The inequality is strict by (5) and the fact that Error! Reference source not found. \(s < \tau^*\)). This can be rearranged to yield

\[
u^*(\tau^*) < (1 - \delta)\alpha qf(\tau^*) + \delta q(1 - \alpha)(f(s) - f(\tau^*)) \leq (1 - \delta)\alpha qf(\tau^*),\]

where the last inequality follows since \(\alpha < 1\) and \(f(s) \leq f(\tau^*)\). But this implies that \(\alpha q\), which lies in \(A_{\tau^*}\), is strictly better than \(q^*(\tau^*)\) is for type \(\tau^*(q)\) – a contradiction. Having now verified Payoff Maximization, we conclude that the profile \((q^*, \bar{p}^*, p^*, \mu^*)\) is an Asset Sale Equilibrium.

To finish the proof of Proposition 1, it remains to show that every intuitive equilibrium of the asset sale game is fairly priced, has the same outcome \(u = u^*\), and has an asset sale function \(q^*()\) that solves Error! Reference source not found. RLP. We showed above that \((q^*, \bar{p}^*, p^*, \mu^*)\) is an equilibrium. Now consider any other intuitive equilibrium with outcome \(u\). First we show that it is fairly priced. Suppose that with positive probability,
type \( t \) makes asset sale decision \((q, \overline{p})\). Assume investors respond with price vector \( p \). First say \( qp < qf(t) \): the issue is underpriced. Let \( s \) be the largest type such that \( qf(s) < qf(t) \).\(^{64}\) Then define \( \lambda \in [0,1) \) such that

\[
\frac{q(p - \delta f(t))}{q(f(t) - \delta f(t))} < \lambda < \frac{q(p - \delta f(s))}{q(f(t) - \delta f(s))}.
\]

(\( \lambda \) must exist as the second ratio is increasing in \(-qf(s)\).) Consider the feasible deviation \((\lambda q, f(t))\) for type \( t \). First, by ASSUMPTION A, for all \( s' \leq s \), \( qf(s') \leq qf(s) < qf(t) \) and so from (16), \( \lambda q[f(t) - \delta f(s')] < q[p - \delta f(s')] \leq u(s') \), where the last inequality follows from the incentive constraint for type \( s' \). Thus, no type \( s' \leq s \) has an incentive to deviate to \((\lambda q, f(t))\).

On the other hand, from (16), \( \lambda q[f(t) - \delta f(t)] > q[p - \delta f(t)] = u(t) \), so type \( t \) could gain from the deviation if the realized price \( p \) equals \( f(t) \). But then the Intuitive Criterion implies that \( \mu(s' \mid \lambda q, f(t)) = 0 \) for all \( s' \leq s \). That is, investor beliefs must put weight only on types \( t' \) such that \( qf(t') \geq qf(t) \). By ASSUMPTION A, for these types \( f(t') \geq f(t) \) for all securities \( i \) such that \( q_i > 0 \). Hence, for each such security \( i \),

\[
p_i(\lambda q, f(t)) = f_i(t) \wedge \sum_{t'} f_i(t') \mu(t' \mid \lambda q, f(t)) = f_i(t).
\]

Accordingly, type \( t \) gains from the deviation to \((\lambda q, f(t))\) - a contradiction. Thus, it must be the case that \( qp \geq qf(t) \) for all types \( t \) that make sale decision \((q, \overline{p})\) in equilibrium; that is, there is no equilibrium underpricing. But from (1),

\[
qp = q[\overline{p} \wedge \sum_i f_i(t) \mu(t \mid q, \overline{p})] \leq \sum_i qf_i(t) \mu(t \mid q, \overline{p}).
\]

That is, \( qp \) is bounded above by a convex combination of \( qf(t) \) for all types \( t \) that make sale decision \((q, \overline{p})\) in equilibrium. But as just shown, \( qp \) is also bounded below by \( qf(t) \) for any such \( t \). Thus, \( qp \) must equal \( qf(t) \) for all types \( t \) that ever choose \((q, \overline{p})\): without underpricing there can be no overpricing.

---

\(^{64}\) If there is no such type \( s \), then for all types \( t' \), \( f_i(t) \geq f_i(t') \) whenever \( q_i \) is positive. Thus, if type \( t \) deviates to \((q, f(t))\), investors will respond with a price vector \( p' \geq f_i(t) \), which is better than \( p \) for \( t \).
Further, if \( t, t' \) make sale decision \( (q, \overline{p}) \) in equilibrium and \( t > t' \), then \( f(t) \geq f(t') \) by ASSUMPTION A. Since \( qf(t) = qf(t') \), it must be that \( f(t) = f(t') \) if \( q_i > 0 \). Hence, \( p_i(q, \overline{p}) = f_i(t) \) if \( q_i > 0 \), whence the equilibrium is fairly priced.

It remains to show that in any intuitive equilibrium, the outcome \( u \) equals \( u^* \) and the equilibrium asset sale function \( q(\cdot) \) solves Error! Reference source not found. RLP. Let \( q^*(\cdot) \) solve RLPError! Reference source not found.. As shown above, any intuitive equilibrium is fairly priced, whence \( u \leq u^* \). Let \( t \) be the smallest type such that either (a) \( u(t) < u^*(t) \) or (b) \( q(t) \) does not solve RLPError! Reference source not found.. As \( u(s) = u^*(s) \) for all \( s < t \), the payoff \( u(t) \) equals its maximum value of \( u^*(t) \) if and only if \( q(t) \) solves RLPError! Reference source not found.. Hence, conditions (a) and (b) are equivalent: they both must hold for type \( t \). Since \( u(t) < u^*(t) \), it follows that \( u(t) + \delta q^*(t)f(t) \) is less than \( u^*(t) + \delta q^*(t)f(t) \), which equals \( q^*(t)f(t) \) by Error! Reference source not found. RLP. Also by RLPError! Reference source not found., for all \( s < t \), \( q^*(t)f(t) \) does not exceed \( u^*(s) + \delta q^*(t)f(s) \), which equals \( u(s) + \delta q^*(t)f(s) \) by hypothesis. Hence, there exists a price cap vector \( \overline{p} \) close to but less than \( f(t) \) such that for any type \( s < t \),

\[
  u(t) + \delta q^*(t)f(t) < q^*(t)\overline{p} < q^*(t)f(t) \leq u(s) + \delta q^*(t)f(s).
\]

But then intuitive beliefs put no weight on types \( s < t \) if \( (q^*(t), \overline{p}) \) is observed. Hence, \( p(q^*(t), \overline{p}) = \overline{p} \). But then because \( u(t) < q^*(t)(\overline{p} - \delta f(t)) \), type \( t \) could gain by deviating to \( (q^*(t), \overline{p}) \) – a contradiction. It follows that there is no such minimum type \( t \): for all types \( t \), \( u(t) = u^*(t) \) as claimed and \( q(\cdot) \) solves RLP Error! Reference source not found.

\[ \star \]

**PROOF OF PROPOSITION 5.** First, \( q^*(0) = a \) by Error! Reference source not found. RLP and ASSUMPTION B. For \( t > 0 \), since \( f(t) - \delta f(t-1) > 0 \) by ASSUMPTION A and ASSUMPTION B, equation (7) has a unique solution in \( \hat{A} \). Equation (7) is simply the incentive constraint
in Error! Reference source not found. for \( s = t-1 \). It remains to show that this constraint binds when \( s = t-1 \). Suppose instead that

\[
q^*(t)[f(t) - \delta f(t-1)] < u^*(t-1) = (1-\delta)q^*(t-1)f(t-1).
\]

From the definition of \( q^*(t-1) \) in Error! Reference source not found., for any \( s < t-1 \),

\[
q^*(t-1) (f(t-1) - \delta f(s)) \leq u^*(s).
\]

Combining these two yields:

\[
q^*(t)[f(t) - \delta f(t-1)] - (1-\delta)q^*(t-1)f(t-1) + q^*(t-1) (f(t-1) - \delta f(s)) < u^*(s)
\]

or equivalently,

\[
q^*(t) (f(t) - \delta f(s)) + \delta (q^*(t-1) - q^*(t)) (f(t-1) - f(s)) < u^*(s).
\]

From Proposition 1, \( q^*(t-1) \geq q^*(t) \), and from Assumption A, \( f(t-1) \geq f(s) \). Thus,

\[
q^*(t) (f(t) - \delta f(s)) < u^*(s),
\]

which implies that none of the incentive constraints in Error! Reference source not found. bind for \( q^*(t) \). This contradicts the optimality of \( q^*(t) \) unless \( q^*(t) f(t) = a f(t) \). But by the initial supposition,

\[
q^*(t) f(t) < u^*(t-1) + \delta q^*(t) f(t-1) \leq a f(t-1) \leq a f(t).
\]

Hence, the incentive constraint for \( t-1 \) must bind.  

**Proof of Proposition 6.** The proposition follows from Proposition 1 as long as we can show that if \( j \) is junior to \( i \) then it is more informationally sensitive; that for any \( t > s \),

\[
f_j(t) / f_j(s) > f_i(t) / f_i(s).
\]

Fix types \( t > s \). As both informational sensitivity and priority are invariant to an arbitrary rescaling of each security’s payoff, w.l.o.g. we can let \( f_j(t) = f_i(t) = 1 \). Hence, we need to show that for \( s < t \), \( f_j(s) < f_i(s) \).

Because securities are prioritized, \( F_j(Y) = h(Y) F_i(Y) \) for some nondecreasing function \( h \) that is not almost surely a constant on the set \( F_i(Y) > 0 \). From our earlier normalization,
\[0 = f_i(t) - f_i(t) = E[F_i(Y_t) - F_i(Y_t)] = E[(h(Y_t) - 1)F_i(Y_t)] = E[(h(Y_t) - 1)F_i(Y_t) | F_i(Y_t) > 0].\]

Thus, because \( h \) is nondegenerate on the set \( F_i(Y_t) > 0 \), it must be strictly below 1 and strictly above 1 with positive probability on this set.

Next we make use of the following lemma.

**Lemma:** Suppose \( Y \) satisfies HRO. Then there exists a random variable \( Z \) with the same support as, and independent of, \( Y \), such that \( Y \) and \( Y \wedge Z \) have the same distributions.

**Proof:** Let \( \bar{y} \) denote the essential supremum of \( y \): 
\[ \bar{y} = \inf \left\{ y : \Pr(Y \geq y) = 0 \right\}. \]

Fix \( s < t \), and let 
\[ R(y) = \frac{\Pr(Y_s \geq y)}{\Pr(Y_t \geq y)} \text{ for } y < \bar{y} \text{ and zero for } y > \bar{y}, \]
with \( R(\bar{y}) \) defined so that \( R \) is left-continuous. Note that \( R(0) = 1 \), and \( R \) is strictly decreasing on the support of \( Y \) and constant elsewhere. Thus we can define a new, independent random variable \( Z \) with distribution \( \Pr(Z \geq y) = R(y) \) and note that \( Z \) has the same support as \( Y \).\(^{65}\) Finally,
\[
\Pr(Y_i \wedge Z \geq y) = \Pr(Y_i \geq y, Z \geq y) = \Pr(Y_i \geq y) \Pr(Z \geq y) = \Pr(Y_i \geq y) \frac{\Pr(Y_s \geq y)}{\Pr(Y_t \geq y)}
\]
\[= \Pr(Y_s \geq y), \]

as claimed. \( \star \)

Using the lemma, together with the monotonicity of the securities, we have  
\[ f_i(s) = E\left[F_i(Y_s) \mid s\right] = E\left[F_i(Y_t) \wedge Z\right], \]

and similarly for \( f_j \). If \( h(z) < 1 \), then
\[
E\left[F_j(Y_i \wedge z)\right] = E\left[h(Y_i \wedge z)F_i(Y_i \wedge z)\right] \leq E\left[F_i(Y_i \wedge z)\right],
\]

where the inequality is strict unless \( F_i(z) = 0 \). Conversely, if \( h(z) \geq 1 \) then

\(^{65}\) See e.g. Theorem 12.4 of Billingsley (1986).
\[ E[F_j(Y_i \wedge Z)] = E[F_j(Y_i) - (F_j(Y_i) - F_j(z))^+] \]
\[ = 1 - E\left[(h(Y_i)F_j(Y_i) - h(z)F_j(z))^+\right] \]
\[ \leq 1 - E\left[(h(z)F_j(Y_i) - h(z)F_j(z))^+\right] \]
\[ \leq 1 - E\left[(F_j(Y_i) - F_j(z))^+\right] = E[F_j(Y_i \wedge z)]. \]

Thus, \( f_j(s) = E[F_j(Y_i \wedge Z)] < E[F_j(Y_i \wedge Z)] = f_j(s) \) where the inequality is strict because there is a positive probability that \( F_j(z) > 0 \) and \( h(Z) < 1 \).

**Proof of Proposition 7.** Pricing and Beliefs ensure that \( \tilde{e} \) satisfies Competitive Pricing and Rational Updating. As for Payoff Maximization, the payoff of an issuer of type \( t \) from taking an action \((I, P, \rho)\) that is unexpected in \( \tilde{e} \) equals her payoff from taking this action in \( E \), since, by Pricing and Beliefs, investors respond with the same price vector. As for expected actions in \( \tilde{e} \), we rely on the following claim, whose proof appears below.

**Claim.** Let \((\tilde{I}, P, \rho)\) be expected in \( \tilde{e} \), and let \( \Sigma \) denote the set of actions \((\tilde{I}, P', \rho)\) that are taken in \( E \) by any type whose prescribed action in \( \tilde{e} \) is \((I, P, \rho)\). Then an issuer of any given type gets the same payoff in \( \tilde{e} \) from taking the action \((\tilde{I}, P, \rho)\) as she gets in \( E \) from taking any action in \( \Sigma \).

By this Claim, no action in \( \tilde{e} \) affords an issuer of a given type more than her equilibrium payoff in \( E \); and taking her prescribed action in \( \tilde{e} \) gives her the same payoff that she gets, in equilibrium, in \( E \). Thus she will take her prescribed action in \( \tilde{e} \) : Payoff Maximization holds, whence \( \tilde{e} \) is an equilibrium as claimed.

**Proof of Claim.** Let \((\tilde{I}, P', \rho)\) be any action in \( \Sigma \). Note first that \((\tilde{I}, P', \rho)\) has the same aggregate payout function \( W(Y) = P'(\tilde{I}^1(Y)) \) as \((I, P, \rho)\) and thus the same aggregate expected payout conditional on the issuer’s type, \( E[W(Y) | t] \).

Hence the Claim holds if \((\tilde{I}, P', \rho)\) raises the same revenue in \( E \) as \((\tilde{I}, P, \rho)\) raises in \( \tilde{e} \). We first show that all actions in \( \Sigma \) raise the same revenue in \( E \):
LEMMA. In $E$, following the equilibrium interim asset choice $\hat{I}$, if there are two pactions that are expected given $\hat{I}$ and that have the same revenue cap and aggregate payout function, they must raise the same revenue.

PROOF OF LEMMA. Suppose not: there are types $t$ and $t'$ (possibly equal) who, after choosing $\hat{I}$ and seeing their types, choose pactions $(P, \rho)$ and $(P', \rho)$, respectively, that have the same revenue cap and aggregate payout function, such that $(P, \rho)$ yields less revenue than $(P', \rho)$:

$$\rho \wedge \sum_{j=1}^{n(p)} p^j(I, P, \rho) < \rho \wedge \sum_{j=1}^{n(p)} p^j(I, P', \rho).$$

Since the aggregate payout functions are identical, their conditional expectations are the same conditional on the issuer’s type being $t$: $E[W_p^j(Y) | t] = E[W_{p'}^j(Y) | t]$. But then by (9), type $t'$’s payoff $U(\hat{I}, P', \rho | t, \rho)$ from $(P', \rho)$ exceeds her payoff $U(\hat{I}, P, \rho | t, \rho)$ from $(P, \rho)$ which she is therefore unwilling to choose - a contradiction.

By this Lemma, each action in $\Sigma$ raises the same issuance revenue $R_\Sigma$ in $E$. And if, in $E$, the issuer chooses action $(\hat{I}, P', \rho)$ in $\Sigma$ then, by (8) and (10), investors’ willingness to pay $WTP_E(\hat{I}, P', \rho) = \sum_{j=1}^{n(p)} p^j(\hat{I}, P', \rho)$ for her ex-post assets equals their estimate $\sum_j E[W(Y) | t] \mu(t | \hat{I}, P', \rho)$ of the aggregate payout given what the action $(\hat{I}, P', \rho)$ reveals about the issuer’s type. The issuance revenue $R_\Sigma$ in $E$ is then the minimum of $WTP_E(\hat{I}, P', \rho)$ and the cap $\rho$. Since, by the above Lemma, this issuance revenue is the same for any action $(\hat{I}, P', \rho)$ in $\Sigma$, it follows that either (i) $WTP_E(\hat{I}, P', \rho) > \rho$ for every such action or (ii) $WTP_E(\hat{I}, P', \rho) \leq \rho$.
for every such action. Moreover, in case (ii), $WTP_\tilde{E}(\tilde{i}, P', \rho)$ must take a common value $WTP_\tilde{E}(\Sigma)$ for any action $(\tilde{i}, P', \rho)$ in $\Sigma$.

Likewise, if the issuer chooses action $(\tilde{i}, P, \rho)$ in $\tilde{E}$ then the amount $\tilde{p}^1(\tilde{i}, P, \rho)$ that investors are willing to pay for her ex-post security is just their estimate $\sum_t E[W(Y) | t] \tilde{\mu}(t | \tilde{i}, P, \rho)$ of this security’s payout conditional on the action $(\tilde{i}, P, \rho)$; the resulting issuance revenue is the minimum of $\tilde{p}^1(\tilde{i}, P, \rho)$ and the cap $\rho$.

But for each $t$, by Beliefs, $\tilde{\mu}(t | \tilde{i}, P, \rho)$ can be written as $\frac{g(t)1_{s \in (P, \rho)l}}{\sum_s g(s)1_{s \in (P, \rho)l}}$. And the sets $T(P', \rho | \tilde{i})$ of types who, in $E$, take actions $(\tilde{i}, P', \rho) \in \Sigma$, is a partition of the set $\tilde{T}(P, \rho | \tilde{i})$ of types who take the action $(\tilde{i}, P, \rho)$ in $\tilde{E}$. Thus we can rewrite $\tilde{\mu}(t | \tilde{i}, P, \rho)$ as a weighted sum $\sum_{(i, P', \rho) \in \Sigma} \mu(t | \tilde{i}, P', \rho) w_{(i, P', \rho)}^{t}$ where the weight

$$w_{(i, P', \rho)}^{t} = \frac{\sum_s g(s)1_{s \in T(P', \rho)l}}{\sum_s g(s)1_{s \in T(P, \rho)l}}$$

is the probability that the issuer would have chosen $(\tilde{i}, P', \rho)$ in $E$ conditional on her having chosen $(\tilde{i}, P, \rho)$ in $\tilde{E}$. Hence we can rewrite $\tilde{p}^1(\tilde{i}, P, \rho)$ as $\sum_{(i, P', \rho) \in \Sigma} w_{(i, P', \rho)}^{t} E[W(Y) | t] \mu(t | \tilde{i}, P', \rho)$ or, equivalently, as $\sum_{(i, P', \rho) \in \Sigma} w_{(i, P', \rho)}^{t} R_{\tilde{E}}(\tilde{i}, P', \rho)$. Since, the weights $w_{(i, P', \rho)}^{t}$ sum to one, the revenue $\rho \wedge R_{\tilde{E}}(\tilde{i}, P, \rho)$ raised by the action $(\tilde{i}, P, \rho)$ in $\tilde{E}$ equals $\rho$ in case (i) above (where the revenue raised in $E$ by each $(\tilde{i}, P', \rho)$ in $\Sigma$ is also $\rho$) and the common value $WTP_\tilde{E}(\Sigma)$ in case (ii) (when the revenue raised in $E$ by each $(\tilde{i}, P', \rho)$ in $\Sigma$ is also $WTP_\tilde{E}(\Sigma)$). This proves the Claim.
It remains to show that if $E$ is intuitive, so is $\tilde{E}$. Suppose $E$ is intuitive. Let $u(\cdot | I)$ and $\tilde{u}(\cdot | I)$ denote the equilibrium payoffs in $E$ and $\tilde{E}$, respectively, of an issuer of type $t$ who chooses the interim asset vector $I$ and then follows her continuation strategy in the given equilibrium. Consider any action $(I, P, \rho)$ for which there are types $t$ and $s$ satisfying:

$$\rho < \tilde{u}(t | I) + \delta E \left[ W_p^t(Y) | t \right] \quad \text{and} \quad \rho \geq \tilde{u}(s | I) + \delta E \left[ W_p^t(Y) | s \right].$$

(17)

First assume $(I, P, \rho)$ is unexpected in $\tilde{E}$. As previously noted, the payoff of an issuer of type $t$ from taking the action $(I, P, \rho)$ in $\tilde{E}$ equals her payoff from taking this action in $E$. It follows that $\tilde{u}(v | I) = u(v | I)$ for $v = s, t$. Hence $\rho < u(t | I) + \delta E \left[ W_p^t(Y) | t \right]$ and $\rho \geq u(s | I) + \delta E \left[ W_p^t(Y) | s \right]$ whence, since $E$ is intuitive, $\mu(t | I, P, \rho)$ is zero. But since $(I, P, \rho)$ is unexpected in $\tilde{E}$, Beliefs implies that $\mu(t | I, P, \rho)$ equals $\tilde{\mu}(t | I, P, \rho)$ which thus also is zero as required.

Now suppose instead that $(I, P, \rho)$ is expected in $\tilde{E}$: $\tilde{T}(P, \rho | I)$ is nonempty. By (17), $\tilde{u}(t | I)$ exceeds $\rho - \delta E \left[ W_p^t(Y) | t \right]$ which is the highest payoff type $t$ can expect from $(I, P, \rho)$. Hence $t$ is not in $\tilde{T}(P, \rho | I)$ whence, by Beliefs, $\tilde{\mu}(t | I, P, \rho)$ is zero. We conclude that $\tilde{E}$ is intuitive as claimed. ♦

**Proof of Proposition 8.** Define the following two sets.

1. $I_{\text{AS}}$ is the set of intuitive Asset Sale Equilibria $(q(\cdot), \overline{p}(\cdot), p(\cdot, \cdot), \mu(\cdot, \cdot, \cdot))$ of the asset sale game (which we will denote $G_{\text{AS}}$) with portfolio $(F^*, a)$ in which investor beliefs following any issuance choice $(q, \overline{p})$ depend only on the quantity choices $q$ and the issuer’s maximum issuance revenue $\overline{p} = \overline{p} q$. Changing notation slightly, we now denote these beliefs as $\mu(\cdot | q, \overline{p}q)$ rather than $\mu(\cdot | q, \overline{p})$.

2. $I_{\text{SD}}$ is the set of intuitive Ex Post Security Design Equilibria $\left( (S, \overline{p})_{i=0}^T, \hat{p}(\cdot, \cdot), \hat{\mu}(\cdot, \cdot, \cdot) \right)$ of the ex post security design game (which we will denote $G_{\text{SD}}$).
These two sets are equivalent in the following sense

**Lemma.**

1. For any equilibrium $e$ in $I_{AS}$, there is an equilibrium $\hat{e}$ in $I_{SD}$ with the same outcome. If type $t$ sells the quantities $q(t)$ in $e$, then this type issues the security $S_i(Y) = q(t) F^*(Y)$ in $\hat{e}$.

2. For any equilibrium $\hat{e}$ in $I_{SD}$, there is a set of equilibria $e$ in $I_{AS}$ with the same outcome. If type $t$ issues the security $S_i$ in $\hat{e}$, then this type sells the quantities $q^S_i$ in any such equilibrium $e$.

**Proof of Lemma.** Consider any equilibrium $e$ in $I_{AS}$. Its outcome is $u(t) = q(t) \left[ p(q(t), \bar{p}(t)) - \delta f^*(t) \right]$ and investors’ price function is $p(q, \bar{p}) = \bar{p} \wedge \sum f^*(t) \mu(t \mid q, \bar{p}q)$. We build an equivalent Security Design Equilibrium $\hat{e}$ in $I_{SD}$ as follows. Type $t$ issues the monotone security $S_i(Y) = q(t) F^*(Y)$ with maximum revenue $\bar{p}_i$ equal to $\bar{p}(t) q(t)$. This implies, in particular, that $q^S_i = q(t) \left[ F^*(y_i) - F^*(y_{i-1}) \right] / (y_i - y_{i-1}) = q_i(t)$; collecting terms, $q^S = q(t)$. Given any security design choice $(S, \bar{p})$, investors respond with beliefs $\hat{\mu}(\cdot \mid S, \bar{p}) = \mu(\cdot \mid q^S, \bar{p})$ and associated price $\hat{p}(S, \bar{p}) = \max \left\{ \bar{p}_i, \sum_i E[S(Y) \mid t] \hat{\mu}(t \mid S, \bar{p}) \right\}$, whence Competitive Pricing holds in $\hat{e}$. A type $t$ issuer’s payoff from the choice $(S, \bar{p})$ is

$$\hat{p}(S, \bar{p}) - \delta E[S(Y) \mid t] = \max \left\{ \bar{p}^S, \sum_i f^*(t) \mu(t \mid q^S, \bar{p}) \right\} - \delta q^S f^*(t),$$

which is identical to her payoff $q^S \left[ p(q^S, \bar{p}) - \delta f^*(t) \right]$ from the issuance choice $(q^S, \bar{p})$ in $e$ where $\bar{p}$ is any $n$-vector of asset price caps that gives the same maximum revenue: $\bar{p} = \bar{p} q^S$. Moreover, all such asset price cap vectors give the same payoff in $e$ because – by assumption - investor beliefs and thus issuance revenue depend only on the quantity vector $q$ and maximum revenue $q \bar{p}$. Hence, since $(q(t), \bar{p}(t))$ maximizes
the payoff of type $t$ in $\hat{e}$, it follows that $(S_t, \overline{\rho}_t)$ maximizes the payoff of type $t$ in $\hat{e}$.

Payoff Maximization holds in $\hat{e}$. By restricting to choices made in equilibrium, the preceding also implies that the outcome $\hat{u}(t) = \hat{p}(S_t, \rho_t) - \delta E[S_t(Y)|t]$ of $\hat{e}$ equals the outcome $u(t) = q(t)[p(q(t), \overline{p}(t)) - \delta f^{*}(t)]$ of $e$. By construction, $\hat{\mu}(\cdot | S_t, \overline{\rho}_t) = \mu(\cdot | q^S, \overline{\rho}_t) = \mu(\cdot | q(t), \overline{p}(t))$ and the latter is given by Bayes’s Rule whenever possible. Since the issuer’s behavior is also the same in the two cases, Rational Updating holds: $\hat{e}$ is a Security Design Equilibrium. Finally, consider any security design choice $(S, \overline{\rho})$ in $\hat{e}$ such that, for some type $t$,

$$\overline{\rho} \geq \hat{u}(t) + \delta E[S(Y)|t] = u(t) + \delta q^S f^{*}(t).$$

Then following the issuance choice $(q^S, \overline{p})$ in $e$, investors put zero weight on any type $s$ for which $\overline{\rho} < u(s) + \delta q^S f^{*}(s) = \hat{u}(s) + \delta E[S(Y)|s]$. Hence,

$$\hat{\mu}(s | S_t, \overline{\rho}_t) = \mu(s | q^S, \overline{\rho}_t) = 0: \hat{e} \text{ is intuitive.}$$

Consider any equilibrium $\hat{e}$ in $I_{SD}$. Its outcome is $\hat{u}(t) = \hat{p}(S, \overline{\rho}_t) - \delta E[S_t(Y)|t]$ and the investors’ price function is $\hat{p}(S, \overline{\rho}) = \max \{\overline{\rho}, \sum_t E[S(Y)|t] \hat{\mu}(t | S, \overline{\rho})\}$. We build an equivalent Asset Sale Equilibrium $e$ in $I_{AS}$ as follows. (As the price cap vector will not be unique, this defines a set of equilibria $e$, as indicated in the statement of this proposition.) Type $t$ chooses the quantity vector $q(t) = q^S$ and any price cap vector $\overline{p}(t)$ that satisfies $\overline{p}(t)q(t) = \overline{\rho}_t$. In response to any asset sale choice $(q, \overline{p})$, investors’ posterior beliefs are $\mu(\cdot | q, \overline{p}) = \hat{\mu}(\cdot | qF^*, \overline{p}q)$. The associated price vector is $p(q, \overline{p}) = \overline{p} \wedge \sum_t f^{*}(t) \mu(t | q, \overline{p})$, which implies Competitive Pricing in $e$. The price functions in $e$ and $\hat{e}$ are thus related by $p(q, \overline{p})q = \hat{p}(qF^*, \overline{p}q)$. Hence, a type $t$ issuer’s payoff $q[p(q, \overline{p}) - \delta f^{*}(t)]$ in $e$ from the issuance choice $(q, \overline{p})$ equals her

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66 The security $qF^*(Y)$ in $G_{SD}$ promises the same realized payoff to investors as the quantity vector $q$ in $G_{AS}$. 

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payoff \( \hat{p}(qF^*, \bar{p}q) - \delta E[qF^*(Y)|t] \) in \( \hat{e} \), from the security design \((qF^*, \bar{p}q)\). Thus, since \((S, \bar{p}, \bar{r})\) maximizes the payoff of type \( t \) in \( \hat{e} \), it follows that \((qS, \bar{p})\) maximizes the payoff of type \( t \) in \( e \), where \( \bar{p} \) is any price cap vector that satisfies \( \bar{p}qS = \bar{r} \): Payoff Maximization holds in \( e \). By restricting to choices made in equilibrium, the preceding also implies that the outcome \( u(t) = q(t)[p(q(t), \bar{p}(t)) - \delta f^*(t)] \) of \( e \) equals the outcome \( \hat{u}(t) = \hat{p}(S, \rho_t) - \delta E[S(Y)|t] \) of \( \hat{e} \). By construction, 
\[
\mu(S, \bar{p}) = \hat{\mu}(S, \bar{p}) = \hat{\mu}(S, \bar{r}) \text{ and the latter is given by Bayes’s Rule whenever possible. Since the issuer’s behavior is also the same in the two settings, Rational Updating holds: } e \text{ is an Asset Sale Equilibrium. Finally, consider any issuance choice } (q, \bar{p}) \text{ in } e \text{ such that, for some type } t, 
\]
\[
\bar{p}q \geq u(t) + \delta qf^*(t) = \hat{u}(t) + \delta E[S(Y)|t] 
\]
where \( S \) denotes the security \( qF^* \). Then, following the corresponding issuance choice \((S, \bar{p}q)\) in \( \hat{e} \), investors put zero weight on any type \( s \) for which \( \bar{p}q < \hat{u}(s) + \delta E[S(Y)|s] = u(s) + \delta qf^*(s) \). Hence, 
\[
\mu(s | q, \bar{p}) = \hat{\mu}(s | qF^*, \bar{p}q) = 0: e \text{ is intuitive.} \]

Now let \( e_{AS}^* \) denote the intuitive equilibrium \( e^* \) of the Asset Sale game of section 1, specialized to the case in which the endowment \((a, f)\) equals the endowment \((a^*, f^*)\) of \( G_{AS} \). We claim that \( e_{AS}^* \) lies in \( I_{AS} \). Why? In \( e_{AS}^* \), each type \( t \) can choose any price cap vector that exceeds \( f^*(t) \) and investors ignore these caps. Thus, beliefs on the equilibrium path do not depend on the price cap vector. Following any out-of-equilibrium choice \((q, \bar{p})\), investors’ beliefs are given by (4) and (5), which are also independent of the price cap vector. So investor beliefs in \( e_{AS}^* \) do not depend on the price cap vector at all. Thus, as \( e_{AS}^* \) is intuitive by PROPOSITION 1, it must lie in \( I_{AS} \).

By the above Lemma, then, there exists an equilibrium \( e_{SD}^* \) in \( I_{SD} \) with the same outcome as \( e_{AS}^* \), in which each type \( t \) issues the security \( S_t^*(Y) = q^*(t)F^*(Y) \) where \( q^*(t) \) is chosen by type \( t \) in \( e_{AS}^* \) and solves RLP. Finally, FOSD together with the fact that \( F^* \) is monotone
implies that ASSUMPTION A holds. Hence, by PROPOSITION 1, the quantity choice function in any equilibrium in $I_{AS}$ must be a solution $q^*(\cdot)$ to RLP. Thus, by the above Lemma, in any equilibrium in $I_{SD}$, the issuer’s optimal security $S^*_i(Y)$ must equal $q^*(t)F^*(Y)$ where $q^*(t)$ solves RLP. •

**PROOF OF PROPOSITION 9.** From PROPOSITION 8, $S^*_i(Y) = q^*(t)F^*(Y)$. By PROPOSITION 6, IIS holds so that from PROPOSITION 4, $q^*_i(t) = 1$ for $i < h(t)$ and $q^*_i(t) = 0$ for $i > h(t)$. Therefore $S^*_i(y_i) = q^*(t)F^*(y_i) = y_i$ for $i < h(t)$ and

$$S^*_i(y_i) = q^*(t)F^*(y_i) = y_{h(t)-1}^* + q^*_{h(t)}(t)(y_{h(t)} - y_{h(t)-1}^*)$$

for $i \geq h(t)$. Thus, $S^*_i(Y) = \min(D^*_i, Y)$ where $D^*_i = y_{h(t)-1}^* + q^*_{h(t)}(t)(y_{h(t)} - y_{h(t)-1}^*)$. Finally, since $q^*$ and $h$ are decreasing in $t$ by PROPOSITION 4, $D^*_i$ is decreasing in $t$. That $D^*_i$ satisfies (12) follows from PROPOSITION 5. Finally, as the face value function is strictly decreasing, the equilibrium is fully revealing; hence, $u^*(t) = (1-\delta)v(D,t)$. Off-equilibrium, suppose debt $D \in (D^*_s, D^*_t)$ is issued. The price $p$ that would make type $t$ willing to deviate satisfies

$$p = u^*(t) + \delta v(D,t) = (1-\delta)v(D^*_t, t) + \delta v(D,t).$$

For any type $s < t$, the incentive constraint for $s$ implies

$$v(D^*_t, t) - \delta v(D^*_s, s) \leq u^*(s).$$

Combining these, we have

$$p + \delta \left[ v(D^*_t, t) - v(D, t) - \left( v(D^*_t, s) - v(D, s) \right) \right] \leq u^*(s) + \delta v(D, s),$$

where the terms in brackets is positive: the value of junior debt with face value $D^*_t - D$ is higher for type $t$ than for type $s < t$. Therefore, types $s < t$ would not find it profitable to deviate given price $p$. Thus, the beliefs specified in Error! Reference source not found. must concentrate on $t$, and so the market price given deviation to $D$ is $v(D, t)$. •