ABSTRACT. We consider the problem faced by a privately informed issuer holding a portfolio of securities that can be sold to raise cash. If good news about one security is good news for all the securities held, then a unique equilibrium satisfies the Intuitive Criterion. If, in addition, the securities can be ordered in terms of their sensitivity to the issuer’s private information, then the issuer sells the least information-sensitive securities first. This result confirms the Pecking-Order Hypothesis. We also show that splitting a given security into smaller tranches increases the issuer’s payoff. Finally, we study applications of these results to ex post security design under asymmetric information. A unique equilibrium survives the Intuitive Criterion when signals and shocks are discrete. By taking limits we obtain an equilibrium of the continuous model, which is given by a simple differential equation. Moreover, the issuer's expected profits in the discrete model converge uniformly to her profits in the continuous model.
1. Introduction

In this paper we consider the problem faced by an issuer who holds a portfolio of assets and who desires to liquidate that portfolio to raise cash. With symmetric information, the issuer would be able to sell the assets at their fair market value and liquidation would be efficient and costless. When the issuer has private information regarding the value of the assets and has discretion over the type and quantity of assets to sell, this is no longer the case. The issuer now faces a “lemons” problem (Akerlof 1970): the more she sells, the more pessimistic buyers become regarding her private information and thus the lower a price they will offer per asset. This downwards sloping demand curve leads the seller to retain some of her assets and thus not to realize the full potential gains from trade.

Leland and Pyle (1979) apply this idea to the equity issuance decision of an entrepreneur, and show that the entrepreneur will retain equity (and be inefficiently diversified) in order to signal positive information about future profits. Myers and Majluf (1984) show that a manager who acts in the interests of initial shareholders may also refrain from issuing equity (and hence underinvest) in order to signal that her firm is undervalued.

DeMarzo and Duffie (1999) consider the problem faced by an issuer of an asset-backed security. The issuer first designs a security. She then sees a signal of the quality of her underlying assets and decides what proportion of her security to sell. There is a unique separating equilibrium, in which the market price is decreasing in the quantity sold. Moreover, the issuer can reduce her signalling costs by choosing a security whose payout is relatively insensitive to her private information. Under certain weak assumptions, the optimal such security is standard debt.

In all of these models, the issuer sells a single asset. Thus, the sale decision is a simple quantity choice: what proportion of the asset to sell? When the issuer holds a portfolio of assets, the sale decision is multi-dimensional: the issuer must choose a quantity of each asset in the portfolio to sell. A number of new questions arise. Does “liquidity” differ endogenously across securities, so that the issuer prefers to sell some securities over others? How is the market price of one security affected by the issuer’s sale decision for another security? Does the issuer prefer to sell her securities separately or to sell her entire portfolio as a single pool?

The issuer’s control of both the type and the quantity of the securities sold is an important feature of many applied settings. In the corporate context, both debt and equity may be issued, as well as other hybrid securities. In the setting of asset-backed securities, assets are generally first pooled into a special purpose vehicle (such as a real estate mortgage investment conduit, or REMIC) which then issues several classes (tranches) of securities. More generally, financial intermediaries often hold debt, equity and derivative securities that they liquidate when cash is needed for new investment opportunities.

We make certain assumptions that simplify the multi-dimensional issuance problem. First, the issuer is risk neutral and prefers to hold cash over long term securities. Risk neutrality means that risk sharing will not be an incentive for trade. This is reasonable, for instance, when the issuer is a large financial intermediary that sells assets in order to invest the cash in other markets or to expand its capital base (perhaps due to regulatory
requirements). Or the issuer may be a large firm that sells assets in order to raise cash for worthwhile investment opportunities.

We also assume that the issuer’s private information can be ordered so that information which is good news (in the sense of first order stochastic dominance) about one asset in her portfolio is good news about the other assets as well. This is also natural in many settings. For instance, private information about industry profits or local macroeconomic conditions will tend to move the default risk of different bonds in the same direction. Or the issuer’s assets may be monotone securities (such as debt, equity, or senior/subordinated tranches) that are secured by a common asset pool or by pools with positively correlated returns.

These two key assumptions allow for a sharp characterization of the equilibrium. First, we show in Section 2 of the paper that there is a unique signaling equilibrium that satisfies the Cho-Kreps (1987) intuitive criterion, the weakest standard refinement in the signaling literature. Interestingly, the strength of this refinement in this setting hinges on the ability of the issuer to underprice securities and to ration their allocation, even though no underpricing is observed in equilibrium. We also show that this unique equilibrium can be efficiently computed as a recursive linear program.

In Sections 3 and 4, we specialize the setting further to consider informationally ordered securities. Securities are informationally ordered if they can be ranked from least to most informationally sensitive. This holds, for example, when the information satisfies the hazard rate ordering property (which is weaker than the commonly assumed monotone likelihood rate property) and the securities are prioritized by seniority (as with debt/equity and senior/subordinated tranches of asset backed securities). In this setting, the predictions of the Pecking Order Hypothesis (Myers 1984; Myers and Majluf 1984) hold: the issuer will not sell its junior securities until all of its more senior securities have been sold.

Thus far, the set of securities the issuer can sell has been taken as exogenous. Sections 5 and 6 use our prior results to explore the security design problem. Section 5 considers the ex ante security design problem first studied by DeMarzo and Duffie (1999). In their model, the issuer designs a single security whose payout is secured by a given asset portfolio. We first simplify this problem by restricting the issuer to equity securities. We then show that if the returns of the issuer’s assets are not perfectly correlated, she can increase her securitization profits by selling the assets separately rather than as a pool.

In the ex ante security design problem of DeMarzo and Duffie (1999), the issuer designs her security, sees her signal, and decides what proportion of the security to retain, in that order. However, issuers often have private information about their assets before they design a security. Section 6 studies this ex post security design problem: the issuer sees her signal and then designs her security. This multidimensional signalling problem is both more complex and more realistic than the one dimensional quantity choice considered in the prior literature.

We show that with discrete signals and shocks, this problem can be reduced to the portfolio liquidation problem considered earlier. If the issuer’s information satisfies the hazard rate ordering property and she is restricted to monotone securities, a unique equilibrium survives the Intuitive Criterion. As in DeMarzo and Duffie (1999), the issuer
chooses a standard debt contract. However, she signals high quality not by retaining more shares, but by lowering the security's face value.

The intuition for this difference is as follows. Retaining more shares lowers the payout to investors in all states by the same proportion. (By "state" we mean a particular realization of the value of the underlying assets.) Lowering the face value also lowers the payout by the same proportion, but only in states in which the security is not in default. These states are more likely to occur when the issuer has high quality assets. Thus, a lower face value is a more efficient way for an issuer to signal high quality. Since the Intuitive Criterion selects the equilibrium in which signalling is done the most efficiently, it favors the use of a lower face value to signal quality.

The foregoing results assume that signals and shocks are discrete. In many applications, they are continuous. We show that in the limit as signals and shocks become continuous, the issuer’s equilibrium face value function converges to a simple differential equation. There exists a unique solution to this differential equation, which is an equilibrium of the continuous model. Moreover, this equilibrium is the limit of the equilibria that survive the Intuitive Criterion in the discrete model. This is important since the Intuitive Criterion cannot be used directly to select among equilibria in models with continuous signals (Cho and Kreps 1987).

A researcher may wish to embed the security design problem into a model in which the joint distribution of signals and shocks depends on prior actions chosen by the issuer as she accumulates her asset portfolio. We show that as the gaps between signals and shocks shrink to zero, the issuer's expected profits in the discrete model converge to her expected profits in the discrete model, uniformly in the joint distribution of signals and shocks (and thus in the issuer’s prior actions). This result may be used to show that the issuer’s optimal prior actions in the discrete case can be approximated by her optimal prior actions in the continuous case.

Manelli (1996, 1997) studies a general sequence of finite signalling games (which have finite signal and message spaces) that converges to a continuous signalling game that, like ours, has compact signal and message spaces. Manelli (1996) shows that any sequence of equilibria of the finite games has a convergent subsequence that converges to some equilibrium of the continuous game. Similarly, Manelli (1997) shows that if a sequence of equilibria of the finite games, each of which satisfies the Never a Weak Best Response criterion of Kohlberg and Mertens (1986), converges to some equilibrium of the continuous game, then this limiting equilibrium satisfies the same criterion. Since Manelli’s finite games have finite message space while the discrete models in our section 6 have an infinite message space (the set of monotone securities), Manelli’s results do not apply to our setting.

Nachman and Noe (1994) consider the ex post security design problem of an issuer who needs to raise a fixed amount of capital to fund an investment opportunity. They also find that standard debt is optimal under certain assumptions. However, since the issuer’s target securitization revenue is fixed, there is pooling: the issuer sells a standard debt

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1 Their assumptions include Conditional Stochastic Dominance, which is equivalent to our Hazard Rate Ordering property (Nachman and Noe 1994, n. 13, p. 19), and the D1 refinement of Banks and Sobel (1987), which is stronger than the Intuitive Criterion.
security whose face value does not depend on her signal. In our model, rather than having a fixed funding target, the issuer seeks to maximize her securitization profits. This yields a separating equilibrium in which the issuer varies her security’s face value in order to signal the quality of her underlying assets.

Our security design results have important implications for the design of asset-backed securities by financial intermediaries. These implications are explored in DeMarzo (2005), who develops a dynamic model of informed intermediation. He shows that the process of intermediation leads to the pooling and tranching of assets for resale to investors, with specific predictions regarding the nature of the assets to be pooled, the optimal size of pools, and the design of the tranches which are ultimately sold to investors.

In another application, Frankel and Jin (2013) study the effects of securitization on lending competition. In their model, loan resale markets encourage banks to lend remotely in order to avoid a lemons problem in the loan resale market. Using our results, they show that this effect is smaller if banks can design optimal securities with their loans as collateral.

2. The Asset Sale Game

A risk-neutral issuer holds a portfolio of \( n \) securities. Each security \( i \) is described by the random variable \( F_i \) representing its future payoff. The issuer has private information \( t \in \{0, \ldots, T\} \) regarding these payoffs, which denotes the issuer’s type. Conditional on this information, security \( i \) has an expected payoff of \( f(\alpha) = E[F_i | t] \). Let \( F \in \mathbb{R}^n \) be the column vector of security payoffs. Similarly, let \( f(\alpha) = E[F | t] \) be the column vector of expected payoffs.

The issuer’s initial portfolio is represented by the row vector \( a \in \mathbb{R}^n_+ \), where \( a_i > 0 \) represents the number of shares held of security \( i \). The issuer’s portfolio sale decision is represented by a row vector \( q \in \mathbb{R}^n_+ \), such that \( 0 \leq q \leq a_i \). That is, for each security \( i \), \( q_i \leq a_i \) represents the number of shares of security \( i \) sold by the issuer. The issuer may also set a maximum price for each security, \( \overline{p}_i \in [0, \infty] \). That is, the issuer commits to ration the security if the market clearing price for security \( i \) exceeds \( \overline{p}_i \). This allows the issuer the option of intentionally underpricing the security. Let \( \overline{p} \) be the column vector of maximum prices.\(^3\)

After the issuer announces the portfolio sale decision \( (q, \overline{p}) \), the securities are sold for some price to investors. This will be the market-clearing price unless the price cap set by the issuer is binding. Let \( p \in \mathbb{R}^n_+ \) be the column vector of realized prices; i.e., \( p_i \) is the price received for security \( i \). Obviously, \( p \leq \overline{p} \).

\(^2\) For vectors \( x \) and \( y \), \( x \leq y \) is used to denote \( x_i \leq y_i \) for all \( i \).

\(^3\) Of course, one could also allow the issuer to set reserve or minimum prices for the securities. However, since investors would refuse to buy overpriced securities, extending the strategy space in this way would play no role in equilibrium. (Note that in other auction environments, a reserve price is useful to extract additional surplus from buyers. This model differs in that investors are perfectly competitive. Hence they earn no surplus even absent a reserve price.)
Recall that the issuer is risk-neutral. The issuer also has a cost of capital represented by a positive discount factor \( \delta < 1 \). That is, the issuer is indifferent between receiving \( \delta \) in cash and holding securities with an expected payoff of 1. Equivalently, the issuer has access to positive NPV projects with return \( r = 1/\delta - 1 \). Hence, the payoff to the issuer given asset sale \( q \) at price \( p \) is given by

\[
U(t, q, p) \equiv q p + \delta (a - q) E[F | t] = \delta a f(t) + q (p - \delta f(t)).
\]

Investors are also assumed to be risk-neutral and to have a discount factor \( \lambda > \delta \). Without loss of generality let \( \lambda = 1 \); equivalently, the market rate of interest is normalized to zero. Investors do not observe the information \( t \), but share a common prior \( \pi(t) > 0 \) regarding the probability that the issuer learns \( t \). Based on the portfolio sale decision of the issuer, these prior beliefs are updated to some posterior beliefs \( \mu(t | q, p) \). The demand schedule of investors for security \( i \) is thus perfectly elastic at the price

\[
\sum_i f_i(t) \mu(t | q, p).
\]

Given the maximum prices \( \bar{p} \), the realized prices for the securities is therefore given by

\[
p = \bar{p} \wedge \sum_i f_i(t) \mu(t | q, p),
\]

where the notation \( x = a \wedge b \) represents the component-wise minimum, \( x_i = \min(a_i, b_i) \).

Given this setup, the following definition is standard.

**DEFINITION.** A sequential equilibrium for the asset sale game is a sale decision, a price response and beliefs, \( (q(\cdot), \bar{p}(\cdot), p(\cdot, \cdot), \mu(\cdot | \cdot, \cdot)) \) such that,

1. For all \( t \), \( (q(t), \bar{p}(t)) \) solves: \( \max_{q', \bar{p}'} U(t, q', p(q', \bar{p}')) \) subject to \( 0 \leq q' \leq a \),
2. For all \( (q, \bar{p}) \), \( p(q, \bar{p}) = \bar{p} \wedge \sum_i f_i(t) \mu(t | q, \bar{p}) \),
3. \( \mu(t | q, \bar{p}) \) follows Bayes’ rule when applicable.

We also introduce the following natural terminology:

**DEFINITION.** The equilibrium outcome for the issuer is given by

\[
u(t) = q(t) \left[ p(q(t), \bar{p}(t)) - \delta f(t) \right],
\]

and the equilibrium is said to be fairly priced if for all \( i \) and \( t \), \( q_i(t) > 0 \) implies

\[
p_i(q(t), \bar{p}(t)) = f_i(t).
\]

\(^4\) The fact that \( \lambda > \delta \) provides a motive for trade; that is, in the absence of informational asymmetries, the efficient allocation requires that the issuer sell the portfolio to the investors. Without such a motive, this setting would satisfy the conditions of the No Trade Theorem (see Milgrom and Stokey (1982)).
Note that \( u(t) \) ignores the component \( \delta a f(t) \) of the issuer’s payoff, since it is independent of the outcome of the game. Thus \( u(t) \) represents the additional surplus type \( t \) can recover through an asset sale. The notion of the equilibrium being fairly priced is similar to the idea of a fully-revealing equilibrium in this setting. It implies that no traded security is mispriced.

2.1. A Fairly Priced Equilibrium

In this section, we construct a fairly priced sequential equilibrium for the asset sale game. In this equilibrium, there is no underpricing by the issuer, so the “price cap” \( \bar{p} \) will not bind. The important component of the issuer’s decision is therefore the quantity choice \( q \). The following important assumption regarding the issuer’s private information makes it possible to characterize this choice in a tractable way:

**Assumption A.** For \( t > s \), \( f(t) \geq f(s) \). In addition, \( f(0) \geq 0 \) and \( a f(0) > 0 \).

This assumption states that increasing \( t \) is (weakly) good news regarding the expected payoff of each security, and that the portfolio is valuable even with the worst possible news.\(^5\) Obviously, if the number of securities \( n = 1 \), this assumption is not restrictive since the types can be reordered. For \( n > 1 \), this assumption is restrictive, and implies that the information is common across the securities. For example, the issuer may hold a portfolio of stocks from a given sector and have information about that sector as a whole, or a portfolio of bonds and have information about interest rates. Alternatively, the portfolio may consist of multiple securities backed by a common pool of assets, such as the debt and equity of a single firm, or the mortgage-backed security tranches of a mortgage pool, with the issuer having private information about the value of the assets.

Under this assumption, we demonstrate the existence of a fairly-priced sequential equilibrium by constructing the equilibrium inductively according to the following recursive linear program:

Given \( u^*(s) \) for \( s < t \), define

\[
\begin{align*}
    u^*(t) &= \max_q (1-\delta) q f(t) \\
    \text{subject to } u^*(s) &\geq q (f(t) - \delta f(s)) \text{ for all } s < t, \quad \text{(2)} \\
    0 &\leq q \leq a.
\end{align*}
\]

and let \( q^*(t) \) be a solution to (2) for each \( t \).\(^6\)

The intuition for the above problem is as follows. Suppose each type were to receive a fair price in equilibrium. Then \( q f(t) \) is the amount of cash raised by selling \( q \), and \( (1-\delta) q f(t) \) is the net gain in terms of the recovery of the assets’ holding cost implied by the discount factor \( \delta \). Thus each type attempts to maximize the recovery of the holding cost. The constraint implies that no worse type would gain by mimicking a better type. The

\(^5\) Note that this does not rule out zero (or even negative) payoffs \( F \) for the securities.

\(^6\) If the solution to (2) is multi-valued, then \( q(t) \) can also be allowed to be a mixed strategy on the solution set. For convenience, I will often speak in terms of pure strategies, though all of the results hold in the general case.
gain from mimicking is the holding cost recovery $(1-\delta) q f(s)$ plus the gain from mispricing $q (f(t) - f(s))$, or a total gain of $q (f(t) - \delta f(s))$.

Note, however, that the problem only includes the constraint that worse types do not choose to mimic better types. In equilibrium, it is also necessary that better types do not want to mimic worse types. Including those constraints would not allow for a recursive solution, however. Thus, we ignore them for now, and show later (Proposition 2) that they also hold.

First, we show that a solution $u^*$ exists and is decreasing.

**Proposition 1.** $u^*$ is well-defined, strictly positive, and weakly decreasing in $t$.

**Proof:** See appendix. ✷

The intuition for the fact that $u^*$ is decreasing is straightforward. $u^*$ is proportional to $q^*(t) f(t)$, the amount of cash raised by type $t$. Since if type $s < t$ mimics type $t$ it will enjoy a mispricing gain of $f(t) - f(s)$, this must be offset by the fact that less cash is raised and increased holding costs are borne.7

The equilibrium being constructed is one in which each type $t$ receives the fair price $f(t)$ for the securities. Thus, the maximum price $\bar{p}$ does not bind, and so can be set arbitrarily as long as $\bar{p}(t) \geq f(t)$.

Next we define investor beliefs $\mu^*$. For $(q, \bar{p})$ in the support of the issuer’s strategy, $\mu^*$ is defined simply according to Bayes’ rule. For $(q, \bar{p})$ outside the support, there is no restriction on beliefs in a sequential equilibrium. One obvious choice of beliefs is $\mu(0 | q, \bar{p}) = 1$; i.e., off the equilibrium path, investors believe the issuer has the worst possible news.

The problem with such beliefs is that they may not be “reasonable” according to standard refinement concepts we will introduce later. In anticipation of that, consider the following. Suppose type $t'$ deviates to $q$ and receives price $p$. Such a deviation is profitable if $q (p - \delta f(t')) > u^*(t')$, or equivalently,

$$q p > u^*(t') + \delta q f(t').$$

Thus, $u^*(t') + \delta q f(t')$ represents the minimum proceeds from the issue such that type $t'$ would have an incentive to deviate. This motivates the following specification of off-equilibrium beliefs:

$$\mu^*(\tau^*(q) | q, \bar{p}) = 1, \text{ where } \tau^*(q) = \min\{t \mid t \in \arg\min_{t' \neq t} u^*(t') + \delta q f(t')\}. \quad (3)$$

That is, when investors observe an unanticipated sale decision, they believe the issuer’s type is the worst type of the set of types with the greatest incentive to deviate to quantity choice $q$.9

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7 This tradeoff of mispricing versus holding/retention costs is shared by many other models including Leland and Pyle (1977), Myers and Majluf (1984), and DeMarzo and Duffie (1999). In all of these models, the best types signal their quality by raising less external capital.

8 Again, a mixed strategy can be allowed here. See footnote 6.
Finally, given the beliefs $\mu^*$, investor’s price response $p^*(q, \bar{p})$ is defined from $\mu^*$ according to (1). We now show that $(q^*, \bar{p}^*, p^*, \mu^*)$ is indeed an equilibrium.

**Proposition 2.** The profile $(q^*, \bar{p}^*, p^*, \mu^*)$ is a fairly-priced sequential equilibrium of the asset sale game, with outcome $u^*(t) = (1-\delta) q^*(t) f(t)$. In this equilibrium, the price function $p^*$ is weakly decreasing in $q$.

**Proof:** See appendix. ♦

The proof of this result is fairly straightforward. As mentioned above, the main detail to resolve is that the incentive constraints for good types to mimic bad types are not violated. For that, the proof uses a technique related to the approach of Cho and Sobel (1990).

### 2.2. Refinements and Uniqueness

While $(q^*, \bar{p}^*, p^*, \mu^*)$ defines a sequential equilibrium, sequential equilibria are in general not unique. This section considers the reasonableness of focusing on this particular equilibrium. We show that subject to rather weak refinement restrictions, this equilibrium is unique in terms of outcomes. Interestingly, though this equilibrium does not involve underpricing by the issuer, the ability of the issuer to underprice plays an important role in establishing uniqueness.

First, we show that among all fairly-priced equilibria, the one constructed is optimal in a strong sense.

**Proposition 3.** The outcome $u^*$ Pareto dominates the outcome of any other fairly-priced equilibrium.

**Proof:** Note that in a fairly priced equilibrium $(q, \bar{p}, p, \mu)$, the payoff to type $t$ is $u(t) = (1-\delta) q(t) f(t)$. The proof then follows by induction. Suppose $u(s) \leq u^*(s)$ for all $s < t$. Then the IC constraint for type $s < t$ in the candidate equilibrium is

$$q(t) (f(t) - \delta f(s)) \leq u(s) \leq u^*(s).$$

But this implies that $q$ is feasible in (2) for type $t$, so that $u(t) \leq u^*(t)$. ♦

Refinements of sequential equilibrium focus on the reasonableness of beliefs when the issuer makes an out of equilibrium asset sale decision $(q, \bar{p})$. Of the refinements introduced in the literature, the weakest is the “intuitive criterion” of Cho and Kreps (1987). This refinement argues that if a deviation is observed such that type $t$ could not benefit from that deviation no matter what the resultant posterior beliefs of the uninformed, then it is unreasonable for the uninformed to believe that type $t$ would make that deviation.

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9 The refinement-savvy reader will recognize these beliefs as motivated by the D1 (Divinity) refinement of Cho and Sobel (1990).
In the context of the asset sale game, suppose type $s$ makes the deviation $(q, \overline{p})$ and the market responds with prices $p$. Type $s$ weakly gains from this deviation if and only if $q(p - \delta f(s)) \geq u(s)$, or equivalently,

$$q \ p \geq u(s) + \delta \ qf(s).$$

Suppose the deviation $(q, \overline{p})$ is such that $q \ \overline{p} \ < u(s) + \delta qf(s)$. Then since the market price $p \leq \overline{p}$, type $s$ would always be worse off by deviating to $(q, \overline{p})$ no matter what posterior beliefs investors have. Hence, it seems unreasonable for investors to believe that type $s$ made the deviation $(q, \overline{p})$.

The formal definition of the refinement for this game is given below. It states that beliefs should be concentrated on those types that could possibly gain from the deviation, as long as some such type exists.

**Definition.** A sequential equilibrium $(q, \overline{p}, p, \mu)$ is **intuitive** if it also satisfies

$$q \ \overline{p} \ < u(s) + \delta qf(s) \ \text{implies} \ \mu(s \mid q, \overline{p}) = 0$$

whenever $q \ \overline{p} \geq u(t) + \delta qf(t)$ for some type $t$.

We now show the following key result:

**Proposition 4.** The sequential equilibrium $(q^*, \overline{p}^*, p^*, \mu^*)$ is intuitive. Moreover, every intuitive sequential equilibrium of the asset sale game is fairly priced and has the same outcome $u = u^*$.

**Proof:** See appendix. ♦

The fact that $(q^*, \overline{p}^*, p^*, \mu^*)$ is intuitive follows immediately from the definition of beliefs in (3); investors believe the deviation was made by the type with the greatest incentive to deviate. Thus, the equilibrium is “robust” to this refinement.

Moreover, the proposition states that this equilibrium is unique in satisfying this refinement, at least in terms of outcomes. For the main idea of the proof, suppose in equilibrium type $t$ sells quantity $q$ and receives price $p$ such that $q_i > 0$ and $p_i < f_i(t)$. That is, security $i$ is underpriced. Then consider the deviation to $q_i' < q_i$ such that

$$q_i' \ (f_i(t) - \delta f_i(t)) = q_i \ (p_i - \delta f_i(t)).$$

That is, raise the price of security $i$ to $f_i(t)$ and lower the quantity to $q_i'$ to keep $t$'s surplus unchanged. Then for any other type $s$ with $f_i(s) < f_i(t)$,

$$q_i' \ (f_i(t) - \delta f_i(s)) < q_i \ (p_i - \delta f_i(s)),$$

since type $s$ suffers the decrease in quantity for a larger surplus. Thus, type $s$ would not benefit from this deviation. Hence, the deviation to $q_i'$ and $\overline{p_i} = f_i(t)$ would only be made by types $s$ with $f_i(s) \geq f_i(t)$, and so investors with intuitive beliefs will be willing to

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10 Obviously, the equilibrium could not be unique in terms of strategies. For example, any choice of $\overline{p}(t) \geq f(t)$ is consistent.
pay the maximum price \( f(t) \). Thus, underpricing cannot occur in an intuitive equilibrium. By investor rationality, overpricing is therefore also not possible. Thus, every intuitive equilibrium is fairly priced, and the remainder of the proof is straightforward.

Thus, \((q^*, \tilde{p}^*, \mu^*)\) characterizes the outcomes of the asset sale game subject to the weak requirement of Cho and Krep’s intuitive criterion.\(^{11}\) Having established the robustness of the equilibrium, we now use it to analyze further properties of the asset sale game.

### 2.3. A Numerical Example

We have shown thus far that given a set of securities satisfying ASSUMPTION A, the equilibrium can be characterized as the solution to (2). Note that (2) defines a recursive linear programming problem, and so can be solved efficiently. We conclude this section with a brief numerical example:

**Example 1.** Suppose the issuer holds 1 share of each of 2 securities, with expected payoffs of each (conditional on \( t \)) given below:

While the overall sensitivity to the information \( t \) for each security is similar (a 10% increase in value as \( t \) increases from 0 to 200), security 2 is more sensitive to \( t \) for \( t < 50 \) and vice versa for \( t > 50 \). Intuitively, one might then expect that in the solution to \( q^* \), the issuer sells more of security 1 for low \( t \), and more of security 2 for high \( t \). This is verified in the figure below, which depicts the equilibrium strategies for \( \delta = .9 \). Note that for \( t \leq 50 \), the issuer signals high quality by retaining more of security 2. For \( t > 50 \), the qualitative nature of the solution changes abruptly, and the issuer signals high quality by retaining greater amounts of security 1.

\(^{11}\) Other standard refinements in the literature (forward induction, D1, strategic stability) are stronger than the intuitive criterion, and this equilibrium would therefore be unique subject to them as well.
In the following section, we formalize the intuition from this example regarding the impact of relative information sensitivity on the equilibrium asset sale decision.

3. Informationally Ordered Securities

In this section we introduce an ordering of securities based on their sensitivity to the issuer’s information. For such securities that can be ordered, the equilibrium of the asset sale game has a simple form: the issuer sells the least informationally sensitive securities first. We also show conditions under which debt and equity type securities satisfy this ordering. This result generalizes the intuition of the Myers (1984) and Myers and Majluf (1984) Pecking-Order Hypothesis.

Suppose that information \( t \) versus \( t' \) induces a higher percentage change in the expected payoff of security \( i \) versus security \( j \). If this is true for all \( t \) and \( t' \), then security \( i \) is more sensitive to the issuer’s private information than security \( j \). For this idea to be applied, percentage changes must be well-defined. Hence the following is introduced to strengthening of our initial assumption that \( af(0) > 0 \):
ASSUMPTION B. For all $i, f_i(0) > 0$.\(^{12}\)

This leads to the following:

DEFINITION. Security $i$ is more information sensitive at $t$ than security $j$ if, for all $s < t$, $f_i(t)/f_i(s) > f_j(t)/f_j(s)$\(^{13}\). Security $i$ is more information sensitive than security $j$ if the above holds for all $t$.

Intuitively, one would expect that the asymmetric information problem would be least severe for those securities that are least sensitive to the issuer’s private information. Hence, the market for these securities will be most liquid, and these securities should be sold first. This intuition can be made precise as follows:

PROPOSITION 5. Suppose security $i$ is more information sensitive at $t$ than security $j$. Then $q^*(t) > 0$ implies $q_j^*(t) = a_j$. Thus, if security $i$ is more information sensitive than security $j$, then security $i$ is not issued unless the issuer has fully liquidated holdings of security $j$.

PROOF: Recall that $q^*$ solves (2) for each $t$. Constructing the Lagrangian, this implies that $q^*(t)$ maximizes

$$q f(t) - \sum_{s < t} \lambda(s) q (f(t) - \delta f(s)).$$

The derivative with respect to $q_i$ can then be written as

$$\delta \sum_{s < t} \lambda(s) f_i(s) - k f_i(t),$$

where $k = (\sum_{s < t} \lambda(s)) - 1$. Thus, $q^*(t) > 0$ implies

$$\sum_{s < t} \lambda(s) f_i(s)/f_i(t) \geq k/\delta.$$  

Since $f_i(s)/f_i(t) < f_j(s)/f_j(t)$, then $\sum_{s < t} \lambda(s) f_j(s)/f_j(t) > k/\delta$, and thus $q^*_j = a_j$.

A further characterization of equilibrium is possible under the additional assumption that all the securities can be completely ordered according to informational sensitivity.

DEFINITION. The securities have increasing information sensitivity (IIS) if for all $i > j$, security $i$ is more information sensitive than security $j$.

If IIS is satisfied, then security 1 is the least information sensitive in the sense that it has the smallest percentage change in value of any security for a given change in information. Conversely, security $n$ is the most information sensitive.

Under IIS, PROPOSITION 5 immediately implies that the issuer will choose to sell all of the less information sensitive securities and retain all of the more information sensitive securities, with the exact cutoff, or hurdle class, determined by the issuer’s type.

---

12 Note again that $f_i(0) = E[F_i | t = 0 > 0$ is a weak restriction that does not rule out securities with zero payoffs in some states. It requires that whatever the issuer’s information, all securities retain some option value.

13 If the relationship between $f$ and $t$ is extended to the interval $[0,T]$, this assumption is equivalent to

$$\partial \ln f_i / \partial t > \partial \ln f_j / \partial t.$$
Moreover, as is shown below, this hurdle class is decreasing in the issuer’s type. Intuitively, an issuer with good information can signal that to the market and receive a high price for the securities only by retaining a large portion of the portfolio.

**Proposition 6.** Suppose the securities have increasing information sensitivity. Then for each type \( t \), there exists a cutoff or hurdle class \( c(t) \) such that \( q^*_i(t) = a_i \) for \( i < c(t) \) and \( q^*_i(t) = 0 \) for \( i > c(t) \). Furthermore, for \( t > s \), \( q^*_i(t) \leq q^*_i(s) \) and \( c(t) \leq c(s) \).

**Proof:** The existence of a hurdle class \( c(t) \) follows as an immediate corollary of IIS and **Proposition 5**. To see that \( q^* \) is decreasing, note that the existence of a hurdle class implies that \( q^* \) is ordered; that is, either \( q^*_t(t) \leq q^*_s(s) \) or \( q^*_t(t) > q^*_s(s) \). Also recall from **Proposition 1** that \( u^*_t(t) = (1-\delta) q^*_t(t) f(t) \leq u^*_s(s) = (1-\delta) q^*_s(s) f(s) \). Since \( f(t) \geq f(s) \) by **Assumption A**, it must be that \( q^*_t(t) \leq q^*_s(s) \).

The previous result implies that the optimal issuance decision \( q^*(t) \) can be restricted to the set

\[
C = \{ q \mid \text{for some integer } c, q_i = a_i, i < c, \text{ and } q_i = 0, i > c \},
\]

of quantities having a cutoff or hurdle class. This reduces the issuer’s decision to a one-dimensional quantity choice. Moreover, as is shown below, in this case only the local incentive compatibility constraints bind in (2). This allows the equilibrium to be characterized via a simple difference equation.

**Proposition 7.** If the securities have increasing information sensitivity, then \( q^*(0) = a \), and for \( t > 0 \), \( q^*(t) \) is the unique element of \( C \) such that

\[
q^*(t) (f(t) - \delta f(t-1)) = (1-\delta) q^*(t-1) f(t-1).
\]

**Proof:** See appendix.

**Proposition** Thus, in the IIS case, equilibria can be easily characterized. In essence, even though the signal space available to the issuer is multi-dimensional, the game collapses into one-dimensional signaling game for which standard solution techniques are available. In the next Section, we demonstrate an application in which one might expect securities to satisfy IIS.

### 4. An Application: Asset-Backed Securities

In this section we apply the results of the general model to the case of an issuer holding a portfolio of securities backed by a common set of assets generating cash flow \( Y \). Simple examples of this include debt and equity securities of an individual firm, or various CMO securities backed by a single mortgage pool. We show first that if the payoffs of the individual securities are monotone in \( Y \) and the issuer’s private information satisfies First Order Stochastic Dominance, then the equilibrium results of Section 2 can be applied. We then consider the special case of prioritized securities (such as debt and equity), under the stronger assumption that the issuer’s information satisfies the Hazard Rate Ordering. In this case the results of Section 3 are used to demonstrate that the issuer’s
optimal strategy will exhibit a lexicographic preference for issuing senior versus junior securities. This result confirms the Pecking Order Hypothesis.

Consider securities backed by assets with cash flows $Y$. Let the payoff of each security $i$ be measurable with respect to $Y$ so that the payoff can be written $F_i(Y)$.

The issuer has private information regarding the underlying cash flows $Y$. It is natural to consider the case in which information is ordered so that higher $t$ corresponds to “good news” about $Y$. The weakest such ordering is given by the following:

**Definition.** The issuer’s information satisfies *First Order Stochastic Dominance (FOSD)* if $Pr(Y \geq y \mid t)$ is weakly increasing in $t$ for all $y$.

Given information $t$, the expected payoff of each security is given by $f_i(t) = E[ F_i(Y) \mid t ]$. The following result is therefore immediate.

**Proposition 8.** Suppose $F$ is weakly increasing in the cash flows $Y$. If the issuer’s information satisfies First Order Stochastic Dominance, then $f(t)$ is weakly increasing in $t$. Thus, if $f(0) > 0$, assumption A is satisfied and the equilibrium asset sale is determined by (2).

**Proof:** Since higher $t$ increases $Y$ in the sense of FOSD, the expectation of any increasing function of $Y$ is increasing in $t$. ♦

**Proposition 8** provides immediately for a rich set of applications for the results of Section 2. For example, standard debt, equity, and convertible-type securities on a corporation’s assets are all monotone in the value of the underlying assets. Also, many natural specifications of the issuer’s information satisfy FOSD.

### 4.1. Prioritized Securities

In this section, we strengthen the assumption on the nature of the issuer’s securities and information in order to apply the stronger results of Section 3. Interestingly, the assumptions needed are standard ones in finance. In particular, we assume the securities are prioritized (such as debt and equity), and that the issuer’s information implies a hazard rate ordering (a stronger assumption than FOSD, but much weaker than the commonly used monotone likelihood ratio property). Under these conditions, we show that the issuer will choose to issue the securities in the order of their priority, from highest to lowest. This is consistent with the Pecking Order Hypothesis of Myers (1984) with regard to capital structure.

A set of securities is said to be prioritized if they can be ordered by seniority so that no security receives cash flows until all more senior securities have been fully paid. This is equivalent to the following:

**Definition.** The set of securities $\{F_i\}$ is prioritized if there exists $d \in \mathbb{R}^{n-1}$, such that

$$F_i = \min( d_i, \max(Y - \sum_{j<i} d_j, 0) )$$

for $i < n$, and $F_n = \max(Y - \sum d_j, 0)$. 
Note that $F_n$ corresponds to a residual equity claim on the assets $Y$, whereas $F_1, \ldots, F_{n-1}$ represent senior to progressively more junior debt.

The issuer has private information $t$ about the asset payoff $Y$. We introduce the following notion of information ordering:

**DEFINITION.** The issuer’s information satisfies the *Hazard Rate Ordering (HRO)* if for all $t > s$,

$$
\frac{\Pr( Y \geq y | t )}{\Pr( Y \geq y | s )}
$$

is increasing in $y$, for $y$ in the support of $Y$.$^{14}$

Note that while HRO is stronger than FOSD, it is much weaker than the Monotone Likelihood Ratio Property (MLRP), which is commonly assumed in signaling environments.$^{15}$

We can now state the following main result:

**PROPOSITION 9.** Suppose the securities are prioritized, $f(0) > 0$, $Y$ is continuously distributed, and the issuer’s information satisfies HRO. Then the securities have increasing information sensitivity, and the issuer will sell its entire holdings of all more senior securities before selling a junior security.

**PROOF:** See appendix. ♦

The intuition for the proof follows from the fact that we can write

$$
f_i(t) = E[ F_i | t ] = \int \min( d_i, \max( Y - D_{i-1}, 0 ) ) dG(y|t) = \int_{D_{i-1}}^{Y} \Pr(Y \geq y | t) dy,
$$

where $G$ is the conditional distribution of $Y$ and the last inequality follows from integration by parts. It is then easy to show that HRO implies IIS.


Thus far, the set of securities held by the issuer has been taken as exogenous. In many applications this assumption is entirely appropriate -- the issuer may only have the option of selling existing securities from its portfolio. But in some circumstances, the issuer can choose the design of the securities. For example, investment banks holding mortgage

$^{14}$ The definition given above is slightly different from the standard hazard rate ordering, defined as

$$
h(y) = g(y|t) / (1 - G(y|t))
$$

decreasing in $t$,

where $g$ and $G$ represent the conditional density and distribution of $Y$. The definition provided above is a generalization of the standard definition that does not rely on the existence of a density for $Y$. This is useful for examples in which $Y$ is discretely valued. To see that the definitions are equivalent, note that if a density exists, then for $t > s$,

$$
\frac{g(y|t)}{g(x|t)} = \frac{\Pr(Y \geq y|t)}{\Pr(Y \geq x|t)} \left[ h_s(y) - h_s(x) \right] > 0.
$$

$^{15}$ MLRP is typically defined as: $g(y|t)/g(x|t)$ is increasing in $t$ for $y > x$. In fact, MLRP implies HRO, which implies FOSD. See, e.g., Ross (1983).
pools often split these pools into distinct tranches prior to selling them to investors. Similarly, a corporation decides what securities to make available to investors; it is not restricted to debt or equity. In this section we consider the implications of the preceding analysis for optimal security design, and show that the issuer benefits by splitting existing securities into smaller sub-pieces prior to observing her signal. Splitting securities gives the issuer additional flexibility in the issuance decision, allowing better types to signal their quality more efficiently. This result may explain some of the gains associated with tranching mortgage pools into CMO’s.

Suppose that before learning her information \( t \), the issuer has the opportunity to split all or some of the securities into “sub”-securities, or tranches. For example, it might be possible to split security \( F_1 \) into securities \( F_{1a} \) and \( F_{1b} \) such that \( F_1 = F_{1a} + F_{1b} \). Intuitively, splitting a security in this fashion gives the issuer more flexibility in the issuance decision (securities \( 1a \) and \( 1b \) do not need to be issued in the same proportions), and this flexibility may be valuable.\(^{16}\)

First, we define the notion of splitting more generally:

**Definition.** The portfolio \( \hat{a} \in \mathbb{R}_+^n \) of the \( \hat{n} \) securities represented by the random payoffs \( \hat{F} \in \mathbb{R}_+^n \) is a splitting of the portfolio \( a \) of the \( n \) securities represented by \( F \) if there exists a non-negative matrix \( \Lambda \in \mathbb{R}_+^{n \times \hat{n}} \) such that

\[
\Lambda \hat{F} = F \quad \text{and} \quad a \Lambda = \hat{a}.
\]

For example, in the simple case above in which one security is split into two, \( \Lambda = [1 \ 1] \).

The general definition above allows for an arbitrary partitioning of the cash flows of the original securities into a finer set.

Given the characterization of the issuance game in Section 2, the following result is immediate:

**Proposition 10.** Suppose the portfolio \( \hat{a} \) of the securities \( \hat{F} \) is a splitting of the portfolio \( a \) of the securities \( F \), and that for either case assumption \( A \) is satisfied. Let \( \hat{u}^* \) and \( u^* \) represent the outcomes of the issuance game in each case. Then \( \hat{u}^* \geq u^* \).

**Proof:** Suppose that for all \( s < t \), \( \hat{u}^*(s) \geq u^*(s) \). Then consider the problem (2) for \( \hat{u}(t) \). Since \( \hat{F} \) is a splitting, \( \Lambda \hat{F} = F \) for some non-negative \( \Lambda \). Define \( \hat{q} = q^*(t) \Lambda \). Since \( \Lambda \) is non-negative, \( 0 \leq \hat{q} = q^*(t) \Lambda \leq a \Lambda = \hat{a} \). Also, since \( \Lambda \hat{f} = f \),

\[
\hat{q} \left( \hat{f}(t) - \delta \hat{f}(s) \right) = q^*(t) \Lambda \left( \hat{f}(t) - \delta \hat{f}(s) \right) = q^*(t) \left( f(t) - \delta f(s) \right) \leq u^*(s) \leq \hat{u}^*(s).
\]

\(^{16}\) It is not automatic that increased flexibility benefits the issuer, since in a strategic setting there are often gains to commitment. Indeed, in this setting the seller could gain by committing ex ante to quantities (e.g., committing to sell all securities before any information is learned). Another interpretation of the results in this section is that while committing to quantities can be helpful, committing to ratios (of the quantity of one security to another) never is.
Thus, \( \hat{q} \) is feasible for type \( t \). Hence, 
\[
\hat{u}^*(t) \geq (1-\delta) \hat{q} \hat{f}(t) = (1-\delta)q^*(t)f(t) = u^*(t).
\]

Hence, splitting securities always weakly increases the issuer’s payoff. Generically, the improvement will be strict, since it will generally not be optimal to issue the split securities in the same proportions as they were found in the “un-split” security. This is demonstrated below.

**Example 2.** Consider the case of the issuer with two securities in Example 1. Suppose the issuer instead only had a single security with conditional payoffs \( f = f_1 + f_2 \). By Proposition 10, the issuer with the single security has a lower equilibrium payoff than the issuer with two separate securities. In this case, the issuer with a single security is forced to issue \( f_1 \) and \( f_2 \) in equal proportions, whereas the issuer with the split securities finds it optimal to vary the proportions as shown in Example 1. We plot below the payoffs in each case:

![Graph showing payoffs](image)

**Proposition 10** extends the results of DeMarzo and Duffie (1999). They consider a case in which the issuer chooses a single security to issue prior to learning the information \( t \). They show that it is optimal for the issuer to tranche the underlying assets and only issue a subset of the available cash flows, and that under some conditions the optimal security is a debt tranche. The results here show that the issuer can do even better by creating multiple tranches, and splitting the cash flows even further.\(^{17}\)

\(^{17}\) Indeed, the results here suggest that there is no limit to the splitting that should occur. Of course, transactions and marketing costs (not modeled here) are likely limiting factors in practice. Also, the benefit to splitting stems from the differences in information sensitivity between the new pieces. If the cash flows allocated to a particular security all have the same information sensitivity, there would be no gain from further splitting that security.

Suppose that after observing the signal $t$, the issuer can design a security to be sold to investors. What security would she optimally choose, if not restricted to some set of existing assets? The answer to this question is complicated by the fact that the issuer’s security design will likely reveal information. That is, the security design itself becomes a signal. In this section, we described the signaling game that results, and show how the techniques in this paper can be applied to solve the game.

Consider an issuer with a given asset portfolio that will pay an unknown future cash flow $Y$. The issuer can issue a single security whose payoff is contingent on $Y$. We can represent the security by a function $S : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, where $S(y)$ is the payment to the security holders if $Y$ equals $y$. We restrict to monotone securities: those whose payments to both the holder and the issuer are nondecreasing in $Y$. The set of monotone securities is denoted

$$M = \{S : S(0) = 0 \text{ and, for all } y \geq x, \text{ both } S(y) \geq S(x) \text{ and } y - S(y) \geq x - S(x)\}.$$  

This restriction is common in the security design literature. One justification is to assume that the issuer has free disposal over $Y$, and can also borrow cash short-term to inflate $Y$. By free disposal, the payment to the issuer must weakly increase in $Y$. And if the payoff to the security holders were falling in $Y$ over some range, the issuer could borrow to inflate $Y$, pay the security holders less, and then repay the loan.

Given information $t$, if the issuer sells security $S$ for price $p$, the issuer’s payoff is given by

$$p + \delta E[ Y - S(Y) | t ] = \delta E[ Y | t ] + p - \delta E[ S(Y) | t ].$$

Note that since the component $\delta E[ Y | t ]$ of the issuer’s payoff is independent of the strategic variables $S$ and $p$, it can be ignored in the analysis of the signaling game.

Given this setup, an equilibrium security design is defined as follows:

**Definition.** A sequential equilibrium for the security design game is a security design, a price cap, a price response and beliefs $(S_1, \ldots, S_r, \overline{p}(\cdot), p(\cdot, \cdot), \mu(\cdot | \cdot, \cdot))$ such that

1. For all $t$, $(S_r, \overline{p}(t))$ solves: $\max_{S, \overline{p}} p(S, \overline{p}) - \delta E[S(Y) | t]$ subject to $S \in M$,

2. For all $S \in M$, $p(S, \overline{p}) = \overline{p} \land \sum_t E[S(Y) | t] \mu(t | S, \overline{p})$,

3. $\mu(t | S, \overline{p})$ follows Bayes’ rule when applicable.

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18 See, for example, Matthews (2001), Hart and Moore (1995), Nachman and Noe (1994), and DeMarzo and Duffie (1999).
In general, the solution to the security design problem is difficult to characterize. Also, the signal space now includes the space of functions of $Y$, leading to technical difficulties in applying standard refinements. In this section we introduce several assumptions that allow the techniques of Sections 2 and 3 to be applied to solve the security design game.

First we make a discreteness assumption. This assumption is not restrictive but is technically convenient. We relax it below.

**ASSUMPTION C.** The cash flows $Y$ are discrete; that is, $Y \in \{y_0, \ldots, y_n\}$ with $y_0 = 0$.

Define the following portfolio of prioritized securities for $i = 1, \ldots, n$:

$$F_i^* (Y) = \min (y_i - y_{i-1}, \max (Y - y_{i-1}, 0)), \text{ and } a_i^* = 1.$$  

Note that $a^* F^* = Y$. That is, the above portfolio splits $Y$ into $n$ monotone securities. Moreover, as the following result shows, any monotone security design can be reinterpreted as combination of the securities $F^*$.

**PROPOSITION 11.** If $S \in M$ then $S(Y) = q F^*(Y)$ for some $q$ such that $0 \leq q \leq a^*$. Conversely, if $0 \leq q \leq a^*$, then there exists $S \in M$ such that $q F^*(Y) = S(Y)$.

**PROOF:** Suppose $S \in M$. Then define $q_i = [S(y_i) - S(y_{i-1})] / (y_i - y_{i-1})$. Clearly, $q F^*(Y) = S(Y)$, and $q_i \geq 0$ since $S$ is monotone. Finally, $S \in M$ also implies $S(y_i) - S(y_{i-1}) \leq y_i - y_{i-1}$, so that $q_i \leq 1 = a^*_i$.

Next suppose $0 \leq q \leq a^*$. Define $S(y) = q F^*(y)$. Then, $S(0) = q F^*(0) = 0$, and $S$ is weakly increasing since $F^*$ is. Also, since $a^* F^*(y)$ has slope of at most one, $S$ also increases with slope of at most one. Therefore, $S \in M$.  

This result implies that the optimal security design problem is equivalent to the optimal issuance problem faced by an issuer holding the portfolio $a^*$ of securities $F^*$. Thus, the techniques of Section 2 can be applied:

**PROPOSITION 12.** Suppose the issuer’s information satisfies FOSD. Then the optimal monotone security design $S^*_i (Y) = q^*_i (t) F^*_i (Y)$, where $q^*_i (t)$ is the solution to the issuance game with portfolio $a^*_i$ and securities $F^*_i$.

**PROOF:** By **PROPOSITION 11** there is a payoff equivalence between securities $S \in M$ and issuance choices $q$ for the securities $F^*$. The definitions of equilibria for the two games therefore coincide. Finally, FOSD together with the fact that $F^*$ is monotone implies that **ASSUMPTION A** holds. Hence $q^*$ characterizes the equilibrium.

**6.1. The Optimality of Debt**

By the previous results, the optimal security design problem can be transformed into an optimal issuance decision. Moreover, in Section 3 of the paper, we identified conditions

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19 A similar construction appears in Hart and Moore (1995), although they exploit it for a different purpose.
under which the issuance game has a simple characterization in terms of a hurdle class. In this section, we show that by strengthening the assumption on the information structure, that simple characterization can be applied in this case to yield debt as the optimal security design.

The first step is the following:

**Proposition 13.** Suppose the issuer’s information satisfies the hazard rate ordering. Then $F^*$ has increasing information sensitivity.

**Proof:** Note that $f^*_i(t) = E[F^*_i(Y) \mid t] = (y_i - y_{i-1}) \Pr(Y \geq y_i \mid t)$. Therefore $f^*_i(t)/f^*_i(s) = \Pr(Y \geq y_i \mid t)/\Pr(Y \geq y_i \mid s)$ is increasing in $i$ for $t > s$ by HRO, so IIS holds.

This leads to the main result. Let $V(D,t)$ denote the expected payout $E[\min\{D,Y\} \mid t]$ of a standard debt contract with face value $D$. Consider the following initial value problem.

**Discrete (DP).** The condition

$$V\left(D^*(t+1),t+1\right) - \delta V\left(D^*(t+1),t\right) = (1-\delta)V\left(D^*(t),t\right)$$

with $D^* : \{0,1,\ldots,T\} \to \mathbb{R}$, together with the initial value $D^*(0) = y_n$.

In (5), the payoff to type $t - 1$ from mimicking type $t$ is equated to the equilibrium payoff of type $t - 1$. If it holds, type $t - 1$ is just willing not to mimic type $t$.

**Proposition 14.** Suppose the issuer’s information satisfies the hazard rate ordering and the issuer is restricted to monotone securities. Then there is a unique equilibrium that satisfies the Intuitive Criterion, with the following properties.

1. The optimal security design is standard debt; that is, there exists $D^*(t)$ such that $S^*_i(Y) = \min\{D^*(t),Y\}$. Moreover, the equilibrium face value $D^*(t)$ is strictly decreasing in the signal $t$. The issuer’s expected securitization profits conditional on $t$ are the gains from trade, $(1-\delta)V\left(D^*(t),t\right)$.

2. The equilibrium face value function $D^*$ satisfies the initial value problem DP.

3. For all signals $t$, the equilibrium face value $D^*(t)$ is positive and the security’s price equals its expected payout $V(D^*(t),t)$. 


**PROOF:** From **Proposition 12**, \( S_i^* (Y) = q^* (t) F^* (Y) \). By **Proposition 13**, IIS holds so that from **Proposition 6**, \( q_i^* (t) = 1 \) for \( i < h(t) \) and \( q_i^* (t) = 0 \) for \( i > h(t) \). Therefore

\[
S_i^* (y_i) = q^* (t) F^* (y_i) = y_i \quad \text{for} \quad i < h(t)
\]

and

\[
S_i^* (y_i) = q^* (t) F^* (y_i) = y_{h(t)-1} + q_h^* (t) (y_{h(t)} - y_{h(t)-1})
\]

for \( i \geq h(t) \).

Thus, \( S_i^* (Y) = \min \left( D^* (t), Y \right) \) where \( D^* (t) = y_{h(t)-1} + q_h^* (t) (y_{h(t)} - y_{h(t)-1}) \). Finally, since \( q^* \) and \( h \) are decreasing in \( t \) by **Proposition 6**, \( D^* (t) \) is decreasing in \( t \). This shows part 1. Part 2 follows from **Proposition 7**. As for part 3, the issuer will never choose a negative face value.\(^{20}\) Now suppose that for some signal \( t \), the issuer chooses a face value of zero. As \( D^* \) is strictly decreasing, \( D^* (T) = 0 \) and, for all \( t < T \), \( D^* (t) > 0 \). Now alter the model to permit the new signal \( T + 1 \), such that HRO is still satisfied. This change does not affect the face values \( D^* (t) \) chosen for signals \( t \leq T \) by part 2. Thus, \( D^* (T + 1) < D^* (T) = 0 \), which is impossible. Finally, as the face value function is strictly decreasing, the equilibrium is fully revealing; hence, since investors are rational and risk-neutral, the security’s price must equal its conditional expected payout.\(^{\star} \)

This result generalizes a number of results in the security design literature. First, DeMarzo and Duffie (1999) show that if the issuer designs her security before learning her signal \( t \), then standard debt is the optimal security design. Here we have given conditions under which standard debt is also optimal if the issuer sees \( t \) before designing her security. However, in the current model the debt’s face value acts as a signal to investors, unlike in DeMarzo and Duffie (1999) where the proportion retained plays this role.

Second, Nachman and Noe (1994) also demonstrate debt as an optimal ex post security design when an issuer must raise a fixed amount of cash in order to fund a worthwhile project. The all-or-nothing nature of the financing problem leads to a pooling equilibrium and equilibrium mispricing. Here, the issuer’s flexibility in the amount of cash raised allows for a fairly priced equilibrium in which the issuer’s information is revealed through the choice of face value.

### 6.2. A Continuous Solution

The preceding results assume discrete distributions of the signal \( t \) and final asset value \( Y \). We now consider the continuous case.

If we define \( v(D,t) = E[ \min(D,Y) \mid t ] \) and let \( \Delta \) be the increment between types, the incentive compatibility condition (5) can be written as

\[
v(D^*(t+\Delta),t+\Delta) - \delta \ v(D^*(t+\Delta),t) = (1-\delta) \ v(D^*(t),t) \]

\(^{20}\) With a negative face value, the security is a risk-free loan from the issuer to the investors. However, the gains from trade from such a loan are negative as investors are more patient than the issuer and are not liquidity constrained. Thus, such a loan will not be made.
Taking the derivative with respect to $\Delta$ at $\Delta = 0$ yields the following initial value problem:

**Continuous Initial Value Problem (CP).**  The differential equation

$$
\frac{\partial}{\partial \tau} D^* = \frac{-v_2(D^*, \tau)}{(1-\delta) v_1(D^*, \tau)} = \frac{-\frac{\partial}{\partial \tau} E \left[ \min(D^*, Y) | t \right]}{(1-\delta) \Pr(Y > D^* | t)}. 
$$

(6)

with $D^*: [0, T] \rightarrow \mathbb{R}$, together with the initial value

$$
D^*(0) = \bar{y}_0, \quad (7)
$$

where $\bar{y}_0$ is the essential supremum of the support of $Y$ conditional on $t = 0$.

The boundary condition, equation (7), is given by the fact the lowest type $t = 0$ is unconstrained in (2), and so chooses to sell $Y$ entirely.

We now have the following result.

If there exists a unique solution to the above differential equation, it is a separating equilibrium for the continuous case. More precisely:

**Proposition 15.**  Assume HRO and the existence of a unique function $D^*$ that solves CP.  Assume, moreover, that $D^*$ is continuous, positive, and decreasing.  Then there is a sequential equilibrium of the continuous model in which, on seeing each signal $t$, the issuer announces a security whose payout function is $S^*(Y) = \min\{D^*(t), Y\}$.

**Proof:**  Appendix.

The preceding two propositions raise several important questions.  First, does there exist a unique solution to CP?  We show that there does, under certain weak assumptions.  Second, does the solution to DP converge to the solution to CP in the limit as the signal and final asset value distributions become continuous?  Under our assumptions, it does.  This is important since there may be equilibria other than CP in the continuous case.  The Intuitive Criterion does not select among these equilibria as it assumes a discrete signal distribution.  Our convergence result shows that the solution to CP approximates the unique equilibrium in the discrete case when the gaps between signals and final asset values are small.

Finally, a researcher may wish to embed the security design problem into a model in which the joint distribution of signals and shocks depends on the actions chosen by the issuer and other model participants in a prior "pregame" period.  In this case, it may be more tractable or convenient to assume that the lender chooses her action to maximize her expected profits in the continuous model rather than their counterpart in the discrete model.  We show that as the gaps between signals and shocks shrink to zero, the gap between profits in the discrete and continuous model also shrinks to zero, uniformly in the joint distribution of signals and shocks (and thus in the pregame action profile).  This
result might be used to show that the issuer’s optimal action in the discrete case converges to her optimal action in the continuous case.\(^{21}\)

In order to prove these results, we must specify a continuous model as well as a sequence of discrete models \(i = 1,2,...\) that converges to the continuous one. In our continuous model, which we refer to as model \(\infty\), the issuer’s signal \(t\) lies in \([0,1]\) and has an unconditional distribution \(G\). Arbitrarily low signals occur with positive probability: for any \(t > 0\), \(G(t) > 0\). The final asset value \(Y\) is the product of a scale parameter \(y\) and a shock \(z\). Let \(H(z | t)\) denote the conditional distribution of the shock \(z\) given the signal \(t\). We assume that \(G\) and \(H\) are continuously differentiable in their arguments and have no atoms.

We make the following technical assumptions. First, the density \(g\) of the signal distribution function \(G\) is bounded and Lipschitz continuous:

\[
LIPSCHITZ-G (L-G). \text{ There are constants } k_0,k_1 \in (0,\infty) \text{ such that for all signals } t \text{ and } t' \text{ in } [0,1], \ g(t) \leq k_0 \text{ and } |g(t) - g(t')| \leq k_1|t - t'|. 
\]

Second, the conditional distribution function \(H\) is Lipschitz continuous and also has some minimum sensitivity to its arguments. More precisely:\(^{22}\)

\[
LIPSCHITZ-H (L-H). \text{ There are constants } k_2,k_3 \in (0,\infty) \text{ such that for all } z \text{ and } t \text{ in } [0,1], \text{ the derivative } \partial H(z|t)/\partial z \text{ is in } (k_2,k_3) \text{ and } -\partial H(z|t)/\partial t \text{ lies in } [k_2z(1-z),k_3].
\]

The first condition in \(L-H\) implies that even if the signal is zero, the shock has a positive expectation: \(E[z|t=0] > 0\). As for the second, as \(H(0,t)\) and \(H(1,t)\) are identically zero and one, respectively, we cannot require that \(\partial H(z|t)/\partial t\) be sensitive to the signal \(t\) for all shocks \(z\). However, we can require that as \(z\) moves away from zero or one, this sensitivity rises at least linearly in \(z\). That is the effect of the factor \(z(1-z)\) in the lower bound on \(-\partial H(z|t)/\partial t\).

We also assume that both partial derivatives of the conditional density \(H\) are Lipschitz continuous in the signal \(t\):

\[
LIPSCHITZ PARTIAL DERIVATIVES (LPD). \text{ There is a constant } k_4 \text{ in } (0,\infty) \text{ such that for all } z, t', \text{ and } t'' \text{ all in } [0,1],
\]

\[
\max \left\{ \left| \frac{\partial H(z|t')}{\partial z} - \frac{\partial H(z|t'')}{\partial z} \right|, \left| \frac{\partial H(z|t)}{\partial t} \right|_{t=t'} - \left| \frac{\partial H(z|t)}{\partial t} \right|_{t=t''} \right\} < k_4 |t' - t''|. 
\]

Finally, we restate the hazard rate ordering property for \(H\):

\[
HAZARD RATE ORDERING (HRO). \text{ For any two signals } t' > t'', \left( 1 - H(z | t') \right) / \left( 1 - H(z | t'') \right) \text{ is increasing in the shock } z \in [0,1].
\]

\(^{21}\) This will generally require additional assumptions, such as the strict quasiconcavity of the issuer’s expected total profits in her pregame action.

\(^{22}\) A higher signal \(t\) is good news about the shock and thus lowers \(H(z|t)\). Accordingly, we state the bounds in terms of \(-\partial H(z|t)/\partial t\) which is nonnegative.
We now define a sequence of discrete models \( i = 1, 2, \ldots \) that converge to model \( \infty \). Let \( (N_i)_{i=1}^{\infty} \) and \( (N_i')_{i=1}^{\infty} \) be two increasing sequences of positive integers. In model \( i \), the gaps between adjacent signals \( t \) and shocks \( z \) are \( \Delta_i = 1/N_i \) and \( \Delta_i' = 1/N_i' \), respectively. More precisely, \( t \) lies in \( S_i = \{0, \Delta_i, \ldots, 1 - \Delta_i, 1\} \) and \( z \) lies in \( S_i' = \{0, \Delta_i', \ldots, 1 - \Delta_i', 1\} \). By construction, both gaps \( \Delta_i \) and \( \Delta_i' \) converge to zero as \( i \) goes to infinity.

In model \( i \), the distribution of the signal \( t \) is given by the restriction of the continuous distribution function \( G \) to the finite set \( S_i \). Likewise, the conditional distribution of the shock \( z \) given the signal \( t \) is the restriction of the continuous conditional distribution function \( H \) to shocks \( z \) in \( S_i' \) and signals \( t \) in \( S_i \). Thus, in model \( i \), the probability that the signal does not exceed some \( t' \) in \( S_i \) is \( G(t') \), while the probability, conditional on \( t = t' \), that the shock \( z \) does not exceed some \( z' \) in \( S_i' \) is \( H(z'|t) \).

We next adapt the expected payout function \( V \) to our framework. Let \( E_i \) and \( E_\infty \) denote the expectation operator in models \( i \) and \( \infty \). For any \( D \in \mathcal{R} \), let

\[
v^{H_{yi}}(D, t) = E_i \left[ \min(D, yz) \mid t \right]
= \sum_{c=1}^{1/\Delta_i} \min(D, yc\Delta_i') \left[ H(c\Delta_i' \mid t) - H((c-1)\Delta_i' \mid t) \right]
\]

and

\[
v^{H_{yi}}(D, t) = E_\infty \left[ \min(D, yz) \mid t \right]
= \int_{z=0}^{1} \min(y, Dz) dH(z \mid t)
\]

denote the expected payout of a standard debt contract with face value \( D \) in models \( i \) and \( \infty \), respectively, conditional on the issuer's signal having the realization \( t \), where \( t \) belongs to \( S_i \) in (8) and to \([0,1]\) in (9). Integrating by parts yields the following useful expression for the the expected payout in model \( \infty \):

\[
v^{H_{yi}}(D, t) = D - y \int_{z=0}^{D/y} H(z \mid t) dt
\]

The analogue to DP in model \( i \) is:

**DISCRETE INITIAL VALUE PROBLEM WITH PARAMETERS** \( G \) and \( y \) (DP\(^{Gyi}\)). The incentive compatibility condition

\[
v^{H_{yi}}(D + \Delta_i, t + \Delta_i) - \delta v^{H_{yi}}(D + \Delta_i, t) = (1 - \delta) v^{H_{yi}}(D, t)
\]

with \( D^{H_{yi}}: S_i \to \mathcal{R} \), together with the initial value \( D^{H_{yi}}(0) = y \).

The analogue to CPD in model \( \infty \) is:
**Continuous Initial Value Problem with Parameters $G$ and $y$ (CP$^Gy$).** The differential equation

$$
\frac{dD^Hy}{dt} = f^Hy\left(D^Hy, t\right) = -\frac{1}{1-\delta} \frac{v_2^Hy\left(D^Hy, t\right)}{v_1^Hy\left(D^Hy, t\right)}
$$

(12)

with $D^Hy : [0,1] \to \mathbb{R}$, together with the initial value $D^Hy(0) = y$.

The function $f^Hy$ in CP$^Hy$ can also be written in two other ways:

$$
f^Hy\left(D^Hy, t\right) = -\frac{1}{1-\delta} \frac{\partial E^w\left[\min\left[D^Hy, \varepsilon t\right]\right]}{\partial t} = \frac{y}{1-\delta} \int_{s=0}^{D^Hy/y} \frac{\partial H\left[s|t\right]}{\partial t} dz,
$$

(13)

using (9) and (10), respectively.\(^{23}\)

We define the following additional functions, which may be useful in applications. In model $i$, let $p^H_i(t)$ denote $v^H_i(D^H_i(t), t)$: the equilibrium expected payout of a standard debt security, conditional on the signal $t$. Since investors are risk neutral, $p^H_i(t)$ also equals the price of this security. Let $\pi^H_i(t)$ denote $(1 - \delta)p^H_i(t)$: the expected gains from trade from such a contract conditional on the signal $t$. Finally, let $E\pi^{G^H_i} = E[\pi^H_i(t)]$ denote the unconditional expected gains from trade. Analogously, in the continuous model let $p^H(t)$ denote $v^H(D^H(t), t)$, let $\pi^H(t)$ denote $(1 - \delta)p^H(t)$, and let $E\pi^{G^H}$ denote $E[\pi^H(t)]$.

By Proposition 14, $p^H_i(t)$ and $D^H_i(t)$ are, respectively, the price and face value of the issuer's security in model $i$, conditional on the signal $t$, in the unique sequential equilibrium that satisfies the Intuitive Criterion. Moreover, since competition among investors drives their payoffs to zero, the conditional expected gains from trade $\pi^H_i(t)$ also equal the issuer's conditional expected securitization profits from selling this security, while $E\pi^{G^H_i}$ denotes her unconditional expected securitization profits.

Finally, let $\Pi^{G^H_i}(t)$ denote the issuer's conditional (on $t$) expected total profits in model $i$: the sum of her securitization profits $\pi^{G^H_i}(t)$ and her expected gross portfolio return $E[yz|t]$. Let $E\Pi^{G^H_i} = E[\Pi^{G^H_i}(t)]$ denote her unconditional expected total profits. This last quantity is especially important in applications: if there is a pregame period, the issuer will act so as to maximize the sum of $E\Pi^{G^H_i}$ and any pregame payoff. We also define analogous quantities in the continuous model: $\Pi^{G^H}(t) = \pi^{G^H}(t) + E[\pi^H(t)]$ and $E\Pi^{G^H} = E\left[\Pi^{G^H}(t)\right]$.

In order to state our convergence result, we must extend these discrete functions to the set of all signals $t$ in the unit interval. We do so in a simple way: for any signal $t$ in $[0,1],

\[^{23}\] To see this, integrate by parts:

$$
\int_{z=0}^{D} H\left(z \mid t\right) dz = H\left(0 \mid t\right) D - \int_{s=0}^{D} zdH\left(0 \mid t\right) = \int_{z=0}^{D} \left(D - z\right) dH\left(z \mid t\right) = E\left[\max\left(0, D - z\right) \mid t\right] = D - E\left[\min\left(D, z\right) \mid t\right],
$$

whence

$$
-\int_{z=0}^{D} \frac{\partial H\left[z|t\right]}{\partial t} dz = \frac{\partial E\left[\min\left(D, z\right)\right]}{\partial t} \text{ as claimed.}
$$
we evaluate the function at the highest signal in $S_i$ that does not exceed $t$. That is, for any signal $t$ in $[0,1]$, let $\tau_i^\ast$ denote $\Delta_i[t/\Delta_i]$: the greatest multiple of $\Delta_i$ that does not exceed $t$.  

We define $D^{H_{i}}(t)$ to be $D^{H_{i}}(\tau_i^\ast)$ and extend $p^{H_{i}}$ and $\pi^{H_{i}}$ in the same way.

Our convergence result is as follows.

**Proposition 16.** Fix constants $k_0$, $k_1$, $k_2$, $k_3$, $k_4$, and $\overline{y}$, all in $(0,\infty)$. Let $\mathbf{G}$ be the set of distribution functions $G$ that satisfy Lipschitz-$G$ with constants $k_0$ and $k_1$. Let $\mathbf{H}$ be the set of conditional distribution functions $H$ that satisfy Lipschitz-$H$ with constants $k_2$ and $k_3$, Lipschitz Partial Derivatives with constant $k_4$, and Hazard Rate Ordering. For any distribution function $G$ in $\mathbf{G}$, conditional distribution function $H$ in $\mathbf{H}$, and parameter $y$ in $(0,\overline{y}]$:

1. There exists a unique function $D^{H_{y}}$ that satisfies $CP^{H_{y}}$. This function is decreasing and differentiable, and takes values in $(0,y]$. The associated price and profit functions, $p^{H_{y}}$ and $\pi^{H_{y}}$, are continuous and decreasing in the signal $t$ as well.

2. For each discrete model $i=1,2,...$, there exists a unique, decreasing function $D^{H_{i}}$ that satisfies $DP^{H_{i}}$.

3. The sequences of face value functions, price functions, conditional and unconditional expected securitization profit functions, and conditional and unconditional expected total profit functions in model $i$ converge to their continuous counterparts as $i$ grows, uniformly in the distributions $G$ and $H$, the parameter $y$, and (except in the case of the unconditional expected profit functions which do not depend on the signal) the signal $t \in [0,1]$. More precisely, for all $\varepsilon > 0$ there is an $i^\ast$ such that for all models $i > i^\ast$, $G$ in $\mathbf{G}$, $H$ in $\mathbf{H}$, $y$ in $(0,\overline{y}]$, and $t$ in $[0,1]$, $|\omega^{H_{i}}(t)-\omega^{H_{y}}(t)| < \varepsilon$ for each $\omega = D, p, \pi, \Pi$, and $|E\omega^{H_{i}}-E\omega^{H_{y}}| < \varepsilon$ for $\omega = \pi$ and $\Pi$.

4. All of the functions defined above are homogeneous of degree one in the parameter $y$: $\omega^{H_{y}} = y\omega^{H_{1}}$ and $\omega^{H_{i}} = y\omega^{H_{1}}$ for each $\omega = D, p, \pi, \Pi, E\pi, E\Pi, \Pi$.

By part 4, the issuer’s expected securitization profits are linear in the parameter $y$. This property can be useful if the issuer has pregame choices that influence $y$.

A rough outline of the proof of parts 1-3 is as follows. The Picard-Lindelöf theorem is the usual tool for proving the existence and uniqueness of the solution to a differential equation. However, we cannot apply this theorem directly because the function $f$ defined in (12) is not Lipschitz continuous in $D^{H_{y}}$: it approaches negative infinity as $D^{H_{y}}$ approaches its initial value of one. This difficulty also prevents us from using standard techniques to show that the discrete solution $D^{H_{i}}$ converges to $D^{H_{y}}$ as the gaps between signals and shocks shrink to zero (as $i \to \infty$).

---

24 For any real number $x$, $[x]$ denotes the greatest integer that does not exceed $x$. 

26
We sidestep this difficulty in the following way. For any constant $k > 0$, we define upper and lower bounds $f_k$ and $\tilde{f}_k$, respectively, on $f$, which are Lipschitz continuous in $D^{Hy}$ with Lipschitz constant $k$. When the function $f$ in equation (12) is replaced by each of these functions, we obtain (by the Picard-Lindelöf theorem) unique solutions $\tilde{D}_k$ and $D_k$, which are upper and lower bounds, respectively, on any solution $D^{Hy}$ to $CP^{Hy}$. Moreover, as the Lipschitz constant $k$ goes to infinity, these upper and lower bounds approach the same limit, which satisfies the differential equation (12). Hence, there exists a unique solution $D^{Hy}$ to (12), which is this limit.

An extension of this technique lets us show the convergence of the discrete solution $D^{Hy_i}$ to the continuous solution $D^{Hy}$, as follows. For any model $i = 1, 2, \ldots$ and constant $k > 0$, we define analogous upper and lower bounds $\tilde{D}_k^i$ and $D_k^i$ on any solution $D^{Hy_i}$ to $DP^{Hy_i}$. These bounds are Lipschitz continuous with constant $k$. Moreover, as $i$ goes to infinity, the upper (lower) bound on $D^{Hy_i}$ converges to the upper (lower) bound, defined above, on $D^{Hy}$. But the gap $D^{Hy_i} - D^{Hy}$ lies between $\bar{D}_k^i - \tilde{D}_k = \left(\bar{D}_k^i - \bar{D}_k\right) + \left(\bar{D}_k - \tilde{D}_k\right)$ and $\tilde{D}_k^i - D_k = \left(\tilde{D}_k^i - \tilde{D}_k\right) + \left(\tilde{D}_k - D_k\right)$, inclusive. As noted, all four quantities in parentheses go to zero as $i$ and $k$ go to infinity. This implies that $D^{Hy_i} - D^{Hy}$ must go to zero as well: $D^{Hy_i}$ converges to $D^{Hy}$.

We end with a numerical example.

**EXAMPLE 3.** Suppose the cash flows $Y$ are lognormally distributed conditional on the issuer’s information. Let the issuer’s information be the drift of $Y$, so that

$$Y = 100e^{t-\frac{1}{2}\sigma^2 + \alpha Z}$$

where $Z$ is standard normal. Suppose $t \in [0, .25]$ and $\delta = 0.95$. We derive the optimal face value of the debt as a function of $t$ for different volatilities $\sigma$ of $Y$:
Note that the debt choice is decreasing in the issuer’s private information of the drift $t$. On the other hand, the debt choice is not monotone in the (publicly known) volatility $\sigma$. This is difficult to interpret, however, since the quality of the debt depends upon both face value and volatility. A better comparison is to compute the amount of cash raised by the issuer, $E[\min(D^*,Y) \mid t]$, for these same three cases. Recall that the issuer’s payoff is given by $(1-\delta) E[\min(D^*,Y) \mid t]$; that is, the issuer recovers the holding cost $(1-\delta)$ on the amount of cash raised by the issue.
Note that the amount of cash raised (and thus the issuer’s payoff) is decreasing in the volatility of the cash flows. Thus, the signaling problem induces an implied risk aversion for the issuer, even though all agents in the model are risk neutral. This also has important consequences for the issuer’s initial preferences over the types of assets to hold. This is explored further in DeMarzo (2005).

7. Conclusion

In this paper we consider the problem faced by an informed issuer holding a portfolio of securities that can be sold to raise cash. We characterize the unique equilibrium for this multi-dimensional signaling problem for the setting in which the issuer has information that has a monotone impact on all securities.

Special cases of this result are then considered. When securities can be ordered according to their information sensitivity, we show that a “pecking order” result applies: the least information sensitive securities are sold first. When information satisfies the hazard rate ordering and securities are prioritized by seniority, we show that information sensitivity coincides with seniority. Thus, in equilibrium, the issuer will first issue the most senior securities, then the next most senior, etc., down to a hurdle class which will be partially sold. All more junior securities will be retained by the issuer.

Finally, the model also has implications for the optimal design of securities. In general, “splitting” a security into smaller tranches before becoming informed increases the issuer’s payoff. We also show that if the issuer can choose any monotone security design after becoming informed, then the optimal design is standard debt if the issuer’s
information satisfies the hazard rate ordering. This is shown in the case of discrete signals and shocks. By taking limits, we show that the issuer’s optimal face value in the continuous case is given by a simple differential equation. Moreover, the issuer's expected profits in the discrete model converge uniformly to her profits in the continuous model.

The results in this paper have a number of important applications. The results regarding optimal portfolio liquidation have implications for the ex-ante design of a portfolio. Investors will be willing to pay a premium to hold securities in their portfolio which improve the liquidation payoff they will receive. The ex-ante investor demand functions can then be used to derive equilibrium “liquidity premia” for different types of securities.

Building on the security design results of Sections Error! Reference source not found. and 6, DeMarzo (2005) shows that even if all market participants are risk neutral, the issuer has an induced risk aversion because a less risky security is also less sensitive to the issuer’s information. Under certain circumstances, this leads the issuer to pool her assets rather than splitting them and issue tranches of the resulting portfolio.

8. References


9. Appendix

PROOF OF PROPOSITION 1: First, $u^*(0) = (1-\delta) a f(0) > 0$ by assumption A. Next suppose $u^*(s)$ is well-defined and strictly positive for $s < t$. Then for type $t$, $q = 0$ is strictly feasible. Since the set of feasible $q$ is non-empty and compact, $u^*(t)$ and $q^*(t)$ exist. Also, since there exists $\varepsilon > 0$ sufficiently small so that $\varepsilon a f(t) \geq (1-\delta) \varepsilon a f(0) > 0$.

To see that $u^*(t) \leq u^*(s)$ for $s < t$, note that since $f(t) \geq f(s)$ by assumption A, if $q$ is feasible for $t$ then $q$ is also feasible for $s$. Therefore,

$$(1-\delta) q^*(t) f(s) \leq u^*(s) = (1-\delta) q^*(s) f(s).$$

Combining this with the constraint in (2) implies

$$q^*(t) f(t) \leq u^*(s) + \delta q^*(t) f(s) \leq u^*(s) + \delta q^*(s) f(s) = q^*(s) f(s),$$

so that $u^*(t) \leq u^*(s)$.

PROOF OF PROPOSITION 2:

First we verify fair pricing. Let $T(q, \bar{p}) = \{t \mid (q, \bar{p}) \text{ is in the support of } (q^*(t), \bar{p}^*(t))\}$. If $T(q, \bar{p}) = \{t\}$, then obviously $p^*(q, \bar{p}) = f(t)$. Next suppose $s, t \in T(q, \bar{p})$ and $s < t$. Then from (2), $u^*(s) = (1-\delta) q f(s) \geq q (f(t) - \delta f(s))$, or equivalently, $q (f(t) - f(s)) \leq 0$. Since $f(t) \geq f(s)$ by assumption A, if $q_i > 0$ then $f_i(t) = f_i(s)$. Since this is true for all types in $T(q, \bar{p})$, then $q_i > 0$ implies $p_i^*(q, \bar{p}) = f_i(t)$ for all $t \in T(q, \bar{p})$. Hence the equilibrium is fairly priced.
Second, by fair pricing,  
\( q^* (t) p^* (q^* (t), \overline{p}^* (t)) = q^* (t) f (t) \). Thus the equilibrium outcome is  
\( (1-\delta) q^* (t) f (t) = u^* (t) \).

It thus remains only to verify the incentive constraints for each type. Suppose type  \( s \) mimics type  \( t \). Since the equilibrium is fairly priced, type  \( s \) does not gain from this deviation as long as the following inequality holds:

\[
IC(s, t): \quad q^* (t) ( f (t) - \delta f (s) ) \leq u^* (s) = (1-\delta) q^* (s) f (s).
\]

It is necessary to show that  \( IC(s, t) \) holds for all  \( s, t \). Suppose  \( IC(s, s') \) holds for all  \( s, s' < t \). Clearly this is true for  \( t = 1 \); we now show by induction it is true for all  \( t \). For  \( s' < t \),  \( IC(s', t) \) is the constraint in (2) and thus is satisfied. Now consider  \( IC(t, s') \). We use the following result (proved below):

**Lemma.** Suppose  \( q (p - f(t)) \leq 0 \) and  \( q (p - \delta f(s)) \leq u^* (s) \) for all  \( s < t \). Then  \( q (p - \delta f(t)) \leq u^* (t) \).

Apply the lemma with  \( p = f(s') \) and  \( q = q^* (s') \). Then the conditions of the lemma are satisfied since  \( f(s') \leq f(t) \) and  \( IC(s, s') \) holds for all  \( s < t \) by induction. Hence  \( IC(t, s') \) holds as well.

Finally, we need to demonstrate that no type deviates to a  \( (q, \overline{p}) \) off the equilibrium path; i.e., to a  \( (q, \overline{p}) \) such that  \( T(q, \overline{p}) = \emptyset \). Given the beliefs  \( \mu^* \), the market price following such a deviation is  
\( p^* (q, \overline{p}) = \overline{p} \land f (\tau^* (q)) \leq f (\tau^* (q)) \). Thus, type  \( s \) does not gain by deviating if the following holds:

\[
IC(s, \tau^* (q), q): \quad q (f (\tau^* (q)) - \delta f (s)) \leq u^* (s).
\]

This is equivalent to  \( q f (\tau^* (q)) \leq u^* (s) + \delta q f (s) \). But by the definition of  \( \tau^* (q) \),

\[
\min_s u^* (s) + \delta q f (s) = u^* (\tau^* (q)) + \delta q f (\tau^* (q)).
\]

Thus, it is sufficient to check that type  \( \tau^* (q) \) does not wish to deviate to  \( (q, \overline{p}) \). (This is natural since  \( \tau^* (q) \) was initially defined as the type with the greatest incentive to deviate to  \( q \).)

To show that  \( \tau^* = \tau^* (q) \) does not deviate to  \( q \), note that the  \( IC(\tau^*, \tau^*, q) \) reduces to  \( (1-\delta) q f (\tau^*) \leq u^* (\tau^*) \). That is, since  \( \tau^* \) receives fair pricing given  \( q \), the deviation is profitable only if it saves more of the holding cost. Note that, by the optimality of  \( q^* (\tau^*) \),  \( IC(\tau^*, \tau^*, q) \) is satisfied if  \( q \) is feasible in the problem (2) for  \( u^* (\tau^*) \).

Suppose  \( q \) is not feasible for  \( \tau^* \). Let  \( \alpha \) be the largest scalar such that  \( \alpha q \) is feasible for  \( \tau^* \). Then  \( \alpha < 1 \) and by continuity  \( IC(s, \tau^*, \alpha q) \) must bind for some  \( s < \tau^* \). Therefore, using the definition of  \( \tau^* \),

\[
\alpha q (f (\tau^*) - \delta f (s)) = u^* (s) = u^* (s) + \delta q f (s) - \delta q f (s) > u^* (\tau^*) + \delta q f (\tau^*) - \delta q f (s).
\]

This can be rearranged to yield,

\[
u^* (\tau^*) < (1-\delta) \alpha q f (\tau^*) + \delta q (1-\alpha) (f (s) - f (\tau^*)) \leq (1-\delta) \alpha q f (\tau^*),
\]
where the last inequality follows since $\alpha < 1$ and $f(s) \leq f(\tau^*)$. But this implies that $\alpha q$ is feasible and superior for $\tau^*$ in (2), which contradicts the definition of $u^*(\tau^*)$.

Thus, $q$ is feasible for $\tau^*$ and hence IC($s$, $\tau^*(q)$, $q$) holds for all $s$. This completes the proof that $(q^*, \bar{p}^*, p^*, \mu^*)$ is a sequential equilibrium. The fact that $p^*$ is weakly decreasing in $q$ follows from the fact that $\tau^*$ is weakly decreasing.

**Proof of Lemma:** Note that

\[ q \left( p - \delta f(t) \right) = q \left( p - f(t) \right) + (1-\delta) q f(t) \leq (1-\delta) q f(t). \]

Thus if $q$ is feasible in (2) for $t$, $(1-\delta) q f(t) \leq u^*(t)$ and we are done. If $q$ is not feasible for $t$, then let $\alpha$ be the largest scalar such that $\alpha q$ is feasible. Then $\alpha < 1$ and by continuity for some $s < t$

\[ \alpha q \left( f(t) - \delta f(s) \right) = u^*(s) \geq q \left( p - \delta f(s) \right). \]

Rearranging, and using the fact that $\alpha < 1$ and the definition of $u^*$, we get

\[ q \left( p - \delta f(t) \right) \leq (1-\alpha) q f(t) + \delta (1-\alpha) q \left( f(s) - f(t) \right) \leq (1-\alpha) q f(t) \leq u^*(t). \]

**Proof of Proposition 4:** First we show that $(q^*, \bar{p}^*, p^*, \mu^*)$ is intuitive. Suppose that

\[ u(t) + \delta q f(t) \leq q \bar{p} < u(s) + \delta q f(s). \]

Then from (3), $s \neq \tau^*(q)$ and $\mu(s\mid q, \bar{p}) = 0$.

Next we show that any intuitive equilibrium is fairly priced. Consider any intuitive equilibrium with outcome $u$ and suppose type $t$ makes asset sale decision $(q, \bar{p})$ and receives price $p$ with positive probability in equilibrium.

Suppose $q p < q f(t)$ so that the issue is underpriced. Let $s$ be the largest type such that $q f(s) < q f(t)$. Then define $\lambda \in [0,1)$ such that

\[ \frac{q \left( p - \delta f(t) \right)}{q \left( f(t) - \delta f(t) \right)} < \lambda < \frac{q \left( p - \delta f(s) \right)}{q \left( f(t) - \delta f(s) \right)}. \]

Note that such a $\lambda$ exists (the RHS is increasing in $-q f(s)$). Consider the feasible deviation $(\lambda, q, f(t))$ for type $t$. First, by Assumption A, for all $s' \leq s$, $q f(s') \leq q f(t) < q f(t)$ and thus

\[ \lambda \ q \left( f(t) - \delta f(s') \right) < q \left( p - \delta f(s') \right) \leq u(s'), \]

where the last inequality follows from the incentive constraint for type $s'$. Thus, no type $s' \leq s$ has an incentive to deviate to $(\lambda, q, f(t))$.

On the other hand,

\[ \lambda \ q \left( f(t) - \delta f(t) \right) > q \left( p - \delta f(t) \right) = u(t), \]  \hspace{1cm} (14)

so type $t$ could gain from the deviation if the realized price $p = f(t)$. But then the intuitive criterion implies that
\[
\mu(s' \mid \lambda, q, f(t)) = 0 \text{ for all } s' \leq s.
\]
That is, investor beliefs must put weight only on types \( t' \) such that \( q \cdot f(t') \geq q \cdot f(t) \). By ASSUMPTION A, for these types \( f(t') \geq f(t) \) for all securities \( i \) such that \( q_i > 0 \). Hence,
\[
p(\lambda, q, f(t)) = f(t) \wedge \sum_r f(t') \mu(t' \mid \lambda, q, f(t)) = f(t).
\]
But then, from (14), this contradicts the incentive constraint for type \( t \).

Thus, it must be the case that \( q \cdot p \geq q \cdot f(t) \) for all types \( t \) that make sale decision \((q, \overline{p})\) in equilibrium; that is, there is no equilibrium underpricing. But from (1),
\[
q \cdot p = q(\overline{p} \wedge \sum_r f(t')\mu(t' \mid q, \overline{p})) \leq \sum_r q \cdot f(t')\mu(t' \mid q, \overline{p}).
\]
That is, \( q \cdot p \) is a convex combination of \( q \cdot f(t) \) for all types \( t \) that make sale decision \((q, \overline{p})\) in equilibrium. Therefore, \( q \cdot p = q \cdot f(t) \); i.e., by investor rationality without underpricing there can be no equilibrium overpricing.

Further, if \( t, t' \) make sale decision \((q, \overline{p})\) in equilibrium and \( t > t' \), then \( f(t) \geq f(t') \) by ASSUMPTION A. Since \( q \cdot f(t) = q \cdot f(t') \), it must be that \( f(t) = f(t') \) if \( q_i > 0 \). Hence, \( p_i(q, \overline{p}) = f_i(t) \) if \( q_i > 0 \) and the equilibrium is fairly priced.

Finally, it must be shown that the equilibrium outcome is \( u = u^* \). From PROPOSITION 3, \( u \leq u^* \). Let \( t \) be the smallest type such that \( u(t) < u^*(t) \). This implies
\[
u(t) + \delta q^*(t) f(t) < u^*(t) + \delta q^*(t) f(t) = q^*(t) f(t).
\]
Also, from condition (2), for all \( s < t \),
\[
q^*(t) f(t) \leq u^*(s) + \delta q^*(t) f(s) = u(s) + \delta q^*(t) f(s).
\]
Hence, there exists \( \overline{p} \) sufficiently close to but less than \( f(t) \) such that for all \( s < t \),
\[
u(t) + \delta q^*(t) f(t) < q^*(t) \overline{p} < u(s) + \delta q^*(t) f(s).
\]
But then, intuitive beliefs put no weight on types \( s < t \) if \((q^*(t), \overline{p})\) is observed. Hence, \( p(q^*(t), \overline{p}) = \overline{p} \), which contradicts incentive compatibility for type \( t \). Thus \( u(t) = u^*(t) \). •

**PROOF OF PROPOSITION 7:** First, \( q^*(0) \) follows immediately from (2) and ASSUMPTION B. For \( t > 0 \), since \( f(t) - \delta f(t-1) > 0 \) by ASSUMPTION A and ASSUMPTION B, equation (4) has a unique solution in \( C \). Note that (4) is equivalent to the incentive constraint in (2) for \( s = t-1 \). Thus it remains to show that this incentive constraint binds at the solution to (2). Suppose instead that
\[
q^*(t) (f(t) - \delta f(t-1)) < u^*(t-1) = (1-\delta) q^*(t-1) f(t-1).
\]
Next note that from the definition of \( q^*(t-1) \) in (2), for any \( s < t-1 \),
\[
q^*(t-1) (f(t-1) - \delta f(s)) \leq u^*(s).
\]
Combining these two yields
\( q^*(t) \left( f(t) - \delta f(s) \right) + \delta \left( q^*(t-1) - q^*(t) \right) \left( f(t-1) - f(s) \right) < u^*(s). \)

From **Proposition 6**, \( q^*(t-1) \geq q^*(t) \), and from **Assumption A**, \( f(t-1) \geq f(s) \). Thus,

\[ q^*(t) \left( f(t) - \delta f(s) \right) < u^*(s), \]

which implies that none of the incentive constraints in (2) bind for \( q^*(t) \). This contradicts the optimality of \( q^*(t) \) unless \( q^*(t) f(t) = a f(t) \). But by the initial supposition,

\[ q^*(t) f(t) < u^*(t-1) + \delta q^*(t) f(t-1) \leq a f(t-1) \leq a f(t). \]

Hence, the incentive constraint for \( t-1 \) must bind. 

**Proof of Proposition 9:** Define \( D_0 = 0, D_i = \sum_{j=i}^n d_i, \) and \( D_n = \infty \). Then

\[ f_i(t) = \mathbb{E}[ F_i | t ] = \int \min( d_i, \max( Y - D_{i-1}, 0 ) ) \, dG(Y|t), \]

where \( G \) is the conditional distribution of \( Y \) and the last equality follows from integration by parts. Therefore, IIS is equivalent to

\[ f(t)/f(s) = \int_{D_{i-1}}^{D_i} \Pr(Y \geq y | t) \, dy / \int_{D_{i-1}}^{D_i} \Pr(Y \geq y | s) \, dy \text{ increasing in } i. \]

This is an immediate consequence of HRO and the following lemma:

**Lemma.** Suppose \( a(y)/b(y) \) is increasing in \( y \). Then \( \int_c^d a(y)dy/\int_c^d b(y)dy \) is increasing in \( c \) and \( d \).

Given IIS, the remainder of the proposition follows from **Proposition 6**.

**Proof of Lemma:** Note that

\[ \int_c^d a(y)dy/\int_c^d b(y)dy = \int_c^d \frac{a(y)}{b(y)} \frac{b(y)}{b(y)}dy = \left( \frac{a(c)}{b(c) ! b(d)} \right). \]

Thus,

\[
\frac{\partial}{\partial d} \left[ \frac{\int_c^d a(y)dy}{\int_c^d b(y)dy} \right] = \frac{b(d)}{b(y)} \left( \frac{a(d)}{b(d)} - \frac{\int_c^d a(y)dy}{\int_c^d b(y)dy} \right) > 0.
\]

A similar calculation holds for \( c \). \( \diamond \) \( \diamond \)
PROOF OF PROPOSITION 15: Let the range of \( D^* \) be \([D, y]\). Define \( U(t, \hat{t}, D) \) to be the payoff \( E \left[ \min(D, Y) | \hat{t} \right] - \delta E \left[ \min(D, Y) | t \right] \) of an issuer with signal \( t \) when the market believes the signal is \( \hat{t} \) and simple debt with face value \( D \) is issued. Then

\[
\frac{U_D}{U_{\hat{t}}} = \frac{\Pr(Y > D | \hat{t}) - \delta \Pr(Y > D | t)}{\delta \frac{\partial}{\partial t} E \left[ \min(D, Y) | \hat{t} \right]}
\]

is decreasing in \( \hat{t} \) by FOSD. Hence the single-crossing property holds. Standard arguments then show that this issuer will not imitate any other type: announce simple debt with face value \( D^* (t') \) for any \( t' \neq t \). See Mailath (1987).

It remains to consider deviations to simple debt with face values not in the range of \( D^* \) as well as deviations to other types of monotone securities. We may assume that investors respond to any such deviation with the the most pessimistic beliefs: that the signal is zero.

First consider a deviation of a type \( t \) to simple debt with a face value \( D \) that is not in the range \([D, y]\) of \( D^* \). Assume this deviation makes type \( t \) strictly better off than sticking to the equilibrium. First, suppose \( D > y \). This security has the same payout for any \( Y \) as simple debt with face value \( y \), and both lead investors to believe that \( t = 0 \). By the preceding result, no type strictly prefers such a deviation. Now consider \( D < D \). Investor beliefs cannot be more optimistic than those that result from simple debt with face value \( D \), since these are the beliefs that the signal is one. Moreover, there are gains from trade, so a higher face value is more profitable for the issuer (holding investor beliefs constant). Hence, any \( D < D \) is worse for the issuer than \( D = D \).

Finally, we consider deviations to a general monotone security \( \hat{S} \). The issuer’s securitization profits from \( \hat{S} \) equal the price assigned by investors less the discounted expected security payout:

\[
\hat{\Pi}_t = E \left[ \hat{S}(Y) | 0 \right] - \delta E \left[ \hat{S}(Y) | t \right] = \int_{y=0}^{\hat{Y}} \hat{S}(Y) \frac{d[H(Y | 0) - \delta H(Y | t)]}{\partial t} \]

where \( \delta \) denotes the conditional distribution of \( Y \) given \( t \). The security \( \hat{S} \) is monotonic iff \( \hat{S}(0) = 0 \) and, for all \( Y \), the control variable \( c \left( Y \right) = \hat{S}(Y) \) lies in \([0, 1] \). For all \( Y \in [0, \hat{Y}] \) define the Hamiltonian

\[
\mathcal{H} = \lambda(Y)c(Y) + \hat{S}(Y) \left[ H'(Y | 0) - \delta H'(Y | t) \right] + \mu_0(Y)c(Y) + \mu_1(Y)
\]

where \( \mu_0, \mu_1 \geq 0 \) are Lagrange multipliers that capture the constraints on \( c \) and \( \lambda(Y) \) is a costate variable. By Pontryagin’s maximization principle (Pontryagin et al 1962), the optimal control \( c \) must maximize the Hamiltonian \( \mathcal{H} \) while the multipliers \( \mu_0, \mu_1 \geq 0 \) must minimize it. As \( \mathcal{H} \) is linear in \( c \), this implies the first order condition

\[
0 = \frac{\partial}{\partial c(Y)} \mathcal{H} = \lambda(Y) + \mu_0(Y) - \mu_1(Y) \tag{1}
\]

as well as the Kuhn-Tucker (complementary slackness) conditions:

\[
\mu_0(Y)c(Y) = 0 \text{ and } \mu_1(Y)[1 - c(Y)] = 0. \tag{2}
\]
Additionally, for all \( Y \), the costate equation

\[
\lambda' (Y) = -\mathcal{H}_S^c = -H' (Y|0) + \delta H' (Y|t)
\]  

must be satisfied. Finally, as the final state \( \widetilde{S}(Y_0) \) is not fixed, the terminal costate must be zero:

\[
\lambda (Y_0) = 0.
\]  

Solving (3) subject to (4), we obtain \( \lambda (Y) = [1 - H (Y|0)] \left[ 1 - \delta \frac{1 - H(Y|t)}{1 - H(Y|0)} \right] \). By HRO, \( \frac{1 - H(Y|t)}{1 - H(Y|0)} \) is increasing in \( Y \) since \( t > 0 \). Hence, by (1), (2), and the nonnegativity of \( \mu_0 \) and \( \mu_1 \), there exists an \( D \in \mathcal{R} \) such that \( c (Y) = 1 \) for all \( Y < D \) and \( c (Y) = 0 \) for all \( Y > D \): the optimal security is simple debt with face value \( D \) and thus, as shown above, \( D = D^* (t) \).

Q.E.D.

**PROPOSITION 15**

**PROOF OF PROPOSITION 16:** Without loss of generality, we restrict to face values \( D \) that do not exceed the maximum final asset value \( y \). An integration by parts formula for the function \( v_{Hyi} \) defined in (8) is as follows.

**CLAIM 1** For any face value \( D \in [0,y] \) and signal \( t \),

\[
v_{Hyi} (D,t) = D - y \int_{z=0}^{\frac{D}{y^t}} H \left( \Delta_i \left[ \frac{z}{\Delta_i} \right] |t| \right) dz.
\]

**PROOF OF CLAIM 1.** One can easily verify (using \( H (1|t) = 1 \) and \( H (0|t) = 0 \)) that for \( D \in [0,y] \),

\[
v_{Hyi} (D, t) = \min \{ D, y \} - \sum_{c=1}^{\left[ \frac{1}{\Delta_i} \right]} H (c \Delta_i|t) \left[ \min \{ D, y(c+1) \Delta_i \} - \min \{ D, yc \Delta_i \} \right]
\]

by (8). As \( c \) is an integer, \( c \leq x \) iff \( c \leq \lfloor x \rfloor \). Thus,

\[
\min \{ D, y(c+1) \Delta_i \} - \min \{ D, yc \Delta_i \} = \begin{cases} 
\frac{y \Delta_i}{y^t} & \text{if } c \leq \left[ \frac{D}{y^t} \right] - 1 \\
D - yc \Delta_i & \text{if } c = \left[ \frac{D}{y^t} \right] \\
0 & \text{if } c > \left[ \frac{D}{y^t} \right]
\end{cases}
\]

whence, as \( D \leq y \),

\[
v_{Hyi} (D, t) = D - y \Delta_i \left[ H \left( \left[ \frac{D}{y^t} \right] \Delta_i|t| \right) \left( \frac{D}{y^t} - \left[ \frac{D}{y^t} \right] \right) \right] 1 \left( \left[ \frac{D}{y^t} \right] \leq \left[ \frac{1}{\Delta_i} \right] - 1 \right).
\]

---

\(^1\)Picking a higher face value is equivalent to picking \( y \) since the underlying assets cannot be worth more than \( y \).
If \( z \in [c \Delta_i', (c + 1) \Delta_i'] \), then \( |z/\Delta_i'| = c \). So
\[
\left[ \frac{D(y \Delta_i')}{(y \Delta_i')} \right]_{c=1}^{-1} \sum_{c=1}^{[D/(y \Delta_i')]} H(c \Delta_i'|t) = \sum_{c=1}^{[D/(y \Delta_i')]} \int_{z=c \Delta_i'}^{(c+1) \Delta_i'} H \left( \Delta_i' \left[ \frac{z}{\Delta_i'} \right] |t \right) dz
\]
\[
= \frac{1}{\Delta_i'} \int_{z=\Delta_i'}^{\Delta_i'} H \left( \Delta_i' \left[ \frac{z}{\Delta_i'} \right] |t \right) 1 \left( \left[ \frac{z}{\Delta_i'} \right] < \frac{1}{\Delta_i'} \right) dz \quad (7)
\]
since \( 1 \left( \left[ \frac{z}{\Delta_i'} \right] < \frac{1}{\Delta_i'} \right) \) equals one for all \( z < \Delta_i' \left[ \frac{D}{y \Delta_i'} \right] \). Moreover,
\[
H \left( \left[ \frac{D}{y \Delta_i'} \right] \Delta_i'|t \right) \left( \frac{D}{y \Delta_i'} - \left[ \frac{D}{y \Delta_i'} \right] \right) 1 \left( \left[ \frac{D}{y \Delta_i'} \right] \leq \frac{1}{\Delta_i'} - 1 \right)
\]
\[
= \frac{1}{\Delta_i'} \int_{z=\Delta_i'}^{\Delta_i'} H \left( \left[ \frac{D}{y \Delta_i'} \right] \Delta_i'|t \right) 1 \left( \left[ \frac{D}{y \Delta_i'} \right] \leq \frac{1}{\Delta_i'} - 1 \right) dz \quad (8)
\]
\[
= \frac{1}{\Delta_i'} \int_{z=\Delta_i'}^{\Delta_i'} H \left( \left[ \frac{D}{y \Delta_i'} \right] \Delta_i'|t \right) 1 \left( \left[ \frac{z}{\Delta_i'} \right] < \frac{1}{\Delta_i'} \right) dz \quad (9)
\]
where the last equality holds for two reasons. First, \( \left[ \frac{z}{\Delta_i'} \right] = \left[ \frac{D}{y \Delta_i'} \right] \) in the interval of integration in line (8). Second, since \( \left[ \frac{z}{\Delta_i'} \right] \) and \( \frac{1}{\Delta_i'} \) are both integers, \( \left[ \frac{z}{\Delta_i'} \right] < \frac{1}{\Delta_i'} \) if and only if \( \left[ \frac{z}{\Delta_i'} \right] \leq \frac{1}{\Delta_i'} - 1 \). Combining (6), (7), and (9), we obtain
\[
v^{Hy_i}(D, t) = D - y \int_{z=\Delta_i'}^{\Delta_i'} H \left( \Delta_i' \left[ \frac{z}{\Delta_i'} \right] |t \right) 1 \left( \left[ \frac{z}{\Delta_i'} \right] < \frac{1}{\Delta_i'} \right) dz.
\]
As \( 1 \left( \left[ \frac{z}{\Delta_i'} \right] < \frac{1}{\Delta_i'} \right) \) equals one except possibly at the upper endpoint of the integral, it can be omitted. Finally, the lower limit of integration can be reduced to zero since \( H(0|t) = 0 \).
Q.E.D.-Claim 1

The next claim shows that the first derivatives with respect to \( D \) and \( t \) of the expected payout \( v^{Hy_i}(D, t) \) in model \( i \) are bounded above and below (parts 1 and 2) and converge uniformly to the respective first derivatives of the expected payout \( v^{Hy}(D, t) \) in the continuous case (parts 3 and 4):

**Claim 2** Assume L-H and BSD. Define
\[
\Omega^{Hy_i}(D', t', D', t'') = v^{Hy_i}(D', t') - v^{Hy_i}(D', t'') - [v^{Hy}(D', t') - v^{Hy}(D', t'')].
\]
1. For all \( D \in (y \Delta_i', y) \), and all \( t', t'' \) in \( S \), such that \( t' > t'' \),
\[
\max \left\{ 0, k_2 \left( \frac{3y - 2D'}{6y^2} \right) \right\} < \frac{v^{Hy_i}(D, t') - v^{Hy_i}(D, t'')}{t' - t''} < y k_3 \min \left\{ \frac{D}{y}, 1 - \Delta_i' \right\},
\]
where \( D' = D - 2y \Delta_i' \).
2. For all $D', D''$ in $[0, y]$ such that $D' > D''$ and for all $t \in S_i$,
\[
k_3 \left( 1 - \frac{D''}{y} + \Delta_i \right) > \frac{v^{Hyi}(D', t) - v^{Hyi}(D'', t)}{D' - D''} > k_2 \left( 1 - \Delta_i' \left[ \frac{D'}{y \Delta_i'} \right] + \Delta_i' \right) \geq k_2 \left( 1 - \frac{D'}{y} \right).
\]

3. For all $\varepsilon > 0$ there exists an $i^*$ such that if $i > i^*$, then for all $G \in \mathcal{G}$, $H \in \mathcal{H}$, $y \in (0, \overline{y}]$, $D$ in $[0, \overline{y}]$, and $t' < t''$ in $S_i$ such that $t' > t''$, $|\Omega^{Hyi}(t', t'', D, D)| < \varepsilon (t' - t'')$.

4. For all $\varepsilon > 0$ there exists an $i^*$ such that if $i > i^*$, then for all $G \in \mathcal{G}$, $H \in \mathcal{H}$, $y \in (0, \overline{y}]$, $t \in S_i$, and $D', D''$ in $[0, y]$ such that $D' > D''$, $|\Omega^{Hyi}(t, t', D', D'')| < \varepsilon (D' - D'')$.

**Proof of Claim 2.** Part 1. By (5),
\[
v^{Hyi}(D, t') - v^{Hyi}(D, t'') = y \int_{z=0}^{D} H \left( \Delta_i' \left[ \frac{z}{\Delta_i'} \right] |t''\right) - H \left( \Delta_i' \left[ \frac{z}{\Delta_i'} \right] |t'\right) \right) dz
\]
which by L-H is less than $yk_3 (t' - t'') \left( \frac{D}{y} - \Delta_i' \right)$ and at least
\[
yk_2 (t' - t'') \left( \frac{D}{y} - \Delta_i' \right) \int_{z=0}^{D} \left[ \frac{z}{\Delta_i'} \right] - \left[ \frac{z}{\Delta_i'} \right] \right) dz,
\]
which is zero if $D < y\Delta_i'$ and positive otherwise. Let $c = \left[ \frac{D}{y \Delta_i'} \right]$ where $D' = D - 2y\Delta_i'$. Recall $N_i' = 1/\Delta_i'$. Hence, for $D > y\Delta_i'$, the integral in (10) is at least
\[
\int_{z=0}^{c\Delta_i'} \left[ \frac{z}{\Delta_i'} \right] \left( N_i' - \left[ \frac{z}{\Delta_i'} \right] \right) dz = \Delta_i' \sum_{n=1}^{c-1} n (N_i' - n) = \Delta_i' \sum_{n=1}^{c-1} \frac{n (3N_i' - 2c + 1)}{6}
\]
\[
\geq \Delta_i' \left( \frac{D'}{y \Delta_i'} + 1 \right) \frac{D'}{y \Delta_i'} \left( 3N_i' - 2c + 1 \right) \frac{6}{6y^3 (\Delta_i')^2} \text{ as } c \geq \frac{D'}{y \Delta_i' + 1}
\]
\[
> \frac{(D')^2 (3y - 2D')}{6y^3 (\Delta_i')^2} \text{ as } D' \leq y (1 - 2\Delta_i').
\]

This proves the result.

Part 2. By L-H and (8),
\[
v^{Hyi}(D', t) - v^{Hyi}(D'', t)
\]
\[
= \sum_{c=1}^{1/\Delta_i'} \left[ \min \{D', yc \Delta_i'\} - \min \{D'', yc \Delta_i'\} \right] \left[ H (c \Delta_i'|t) - H ((c - 1) \Delta_i'|t) \right]
\]

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but
\[
\min \{ D', yc\Delta'_i \} - \min \{ D'', yc\Delta'_i \} = \begin{cases} 
0 & \text{if } c \leq \frac{D''}{y\Delta'_i} \\
yc\Delta'_i - D'' & \text{if } \frac{D''}{y\Delta'_i} < c < \frac{D'}{y\Delta'_i} \\
D' - D'' & \text{if } c \geq \frac{D'}{y\Delta'_i}
\end{cases}
\]

so
\[
v^{Hy_i} (D', t) - v^{Hy_i} (D'', t) \geq \sum_{c=\left\lceil \frac{D''}{y\Delta'_i} \right\rceil}^{1/\Delta'_i} (D' - D'') \left[ H \left( c\Delta'_i | t \right) - H \left( (c - 1)\Delta'_i | t \right) \right] \]
\[
> (D' - D'') k_2 \Delta'_i \left( \frac{1}{\Delta'_i} - \left\lceil \frac{D''}{y\Delta'_i} \right\rceil + 1 \right) \]
\[
\geq (D' - D'') k_2 \left( 1 - D''/y + \Delta_i \right).
\]

Moreover, $v^{Hy_i} (D', t) - v^{Hy_i} (D'', t)$ is at most
\[
\sum_{c=\left\lceil \frac{D''}{y\Delta'_i} \right\rceil}^{1/\Delta'_i} (D' - D'') \left[ H \left( c\Delta'_i | t \right) - H \left( (c - 1)\Delta'_i | t \right) \right] \]
\[
< (D' - D'') k_3 \Delta'_i \left( \frac{1}{\Delta'_i} - \left\lceil \frac{D''}{y\Delta'_i} \right\rceil + 1 \right) \leq (D' - D'') k_3 \left( 1 - D''/y + \Delta_i \right).
\]

Part 3. Let $i$ be large enough that $\bar{y}k_4 \Delta'_i < \varepsilon$. Then
\[
| \Delta (t', t'', D, D) | = \left| \sum_{c=1}^{1/\Delta'_i} \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} \left| \min (D, yc\Delta'_i) - \min (D, yz) \right| d \left[ H \left( z | t' \right) - H \left( z | t'' \right) \right] \right|
\]
\[
\leq \sum_{c=1}^{1/\Delta'_i} \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} \left| \min (D, yc\Delta'_i) - \min (D, yz) \right| d \left[ H \left( z | t' \right) - H \left( z | t'' \right) \right] \]
\[
\leq \sum_{c=1}^{1/\Delta'_i} yk_4 \Delta'_i \left| t' - t'' \right| = yk_4 \Delta'_i \left| t' - t'' \right| < \varepsilon \left| t' - t'' \right|
\]

since $| \min (D, yc\Delta'_i) - \min (D, yz) | < y\Delta'_i$ and by BSD, $\left| \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} d \left[ H \left( z | t' \right) - H \left( z | t'' \right) \right] \right| \leq k_4 \Delta'_i \left| t' - t'' \right|$.  

Part 4. As $y > 0$, \[
\Omega^{Hy_i} (t, t, D', D'') = \left\{ \sum_{c=1}^{1/\Delta'_i} \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} \eta (yc\Delta'_i, yz, D'', D') \ d H \left( z | t \right) \right\} \]
where $\eta (\xi', \xi'', \xi_0, \xi_1) = \max \{ \xi_0, \min \{ \xi_1, \xi' \} \} - \max \{ \xi_0, \min \{ \xi_1, \xi'' \} \}$. 

40
Remark 3 \( \eta ( \zeta', \zeta'', \zeta_0, \zeta_1 ) \) lies in \([0, \zeta' - \zeta'']\) if \( \zeta e' \geq \zeta e'' \) and is zero if either

\[
\max \{ \zeta', \zeta'' \} \leq \zeta e_0
\]

or \( \min \{ \zeta', \zeta'' \} \geq \zeta e_1. \)

Since \( z \) lies in \([(c-1) \Delta'_i, c \Delta'_i] \), the integrand \( \eta (y c \Delta'_i, y z, D'', D') \) is zero if either \( c \leq \frac{D''}{y \Delta'_i} \)
or \( c \geq \frac{D'}{y \Delta'_i} + 1 \). Hence,

\[
\sum_{c=1}^{1/\Delta'_i} \int_{z=(c-1)\Delta'_i}^{c \Delta'_i} \eta (y c \Delta'_i, y z, D'', D') \, dH (z|t)
\]

\[
= \sum_{c=\frac{D'}{y \Delta'_i}}^{\frac{D''}{y \Delta'_i}+1} \int_{z=(c-1)\Delta'_i}^{c \Delta'_i} \eta (y c \Delta'_i, y z, D'', D') \, dH (z|t).
\]

Since \( c \Delta'_i \geq z \) in each integral, the integrands \( \eta (y c \Delta'_i, y z, D'', D') \) are all nonnegative, so we may dispense with the absolute value signs. The first summand, which corresponds to \( c = \left[ D''/y \Delta'_i \right] \), is

\[
\int_{z=\left(\left[\frac{D''}{y \Delta'_i}\right]-1\right) \Delta'_i}^{\left[\frac{D''}{y \Delta'_i}\right] \Delta'_i} \eta \left( y \left[\frac{D''}{y \Delta'_i}\right] \Delta'_i, y z, D'', D' \right) dH (z|t)
\]

\[
= \left[ \eta \left( y \left[\frac{D''}{y \Delta'_i}\right] \Delta'_i, D'', D', D' \right) H \left( \frac{D''}{y} | t \right) - H \left( \left[\frac{D''}{y \Delta'_i}\right] - 1 \right) \Delta'_i | t \right] \right]
\]

\[
+ \int_{z=D''/y}^{\left[\frac{D''}{y \Delta'_i}\right] \Delta'_i} \eta \left( y \left[\frac{D''}{y \Delta'_i}\right] \Delta'_i, y z, D'', D' \right) dH (z|t)
\]

\[
\leq \left[ y k_3 \left( \Delta'_i \right)^2 \left( \left[\frac{D''}{y \Delta'_i}\right] - \frac{D''}{y \Delta'_i} \right) \left[\frac{D''}{y \Delta'_i} - \left(\left[\frac{D''}{y \Delta'_i}\right] - 1 \right) \Delta'_i \right] \right]
\]

by L.-H. There are now two cases.

Case 1: \( \left[\frac{D'}{y \Delta'_i} + 1\right] \leq \frac{1}{\Delta'_i} \). The last summand in (11), which corresponds to \( c = \left[ D'/y \Delta'_i + 1 \right] \),
is

\[
\int_{z=(y'y^I+1)-1}^{\Delta_i^I} \eta \left( y \cdot \left[ \frac{D'}{y \Delta_i^I} + 1 \right] \Delta_i^I, yz, D'', D' \right) dH(z|t)
\]

\[
= \int_{z=(y'y^I+1)-1}^{\Delta_i^I} \eta \left( y \cdot \left[ \frac{D'}{y \Delta_i^I} + 1 \right] \Delta_i^I, yz, D'', D' \right) dH(z|t)
\]

\[
+ \eta \left( y \cdot \left[ \frac{D'}{y \Delta_i^I} + 1 \right] \Delta_i^I, D', D'', D' \right) \left[ H \left( \left[ \frac{D'}{y \Delta_i^I} + 1 \right] \Delta_i^I | t \right) - H \left( \frac{D'}{y | t} \right) \right]
\]

\[
\leq yk_3 \left( \Delta_i^I \right)^2 \left[ \frac{D'}{y \Delta_i^I} + 1 - \left[ \frac{D'}{y \Delta_i^I} \right] - 1 \right].
\]

The remainder of the sum in (11) is

\[
\sum_{c=\left[ \frac{D'}{y \Delta_i^I} + 1 \right] + 1}^{\left[ \frac{D'}{y \Delta_i^I} + 1 \right] - 1} \int_{\Delta_i^I}^{c \Delta_i^I} \eta \left( yc \Delta_i^I, yz, D'', D' \right) dH(z|t) \leq yk_3 \left( \Delta_i^I \right)^2 \left[ \frac{1}{\Delta_i^I} - \left[ \frac{D''}{y \Delta_i^I} \right] - 1 \right].
\]

Collecting terms, \(|\Omega^{Hiy}(t,t,D',D'')| \leq k_3 \Delta_i^I (D' - D''). \)

Now take \(i^*\) large enough that \(k_3 \Delta_i^I < \varepsilon.\)

Case 2: \(\left[ \frac{D'}{y \Delta_i^I} + 1 \right] > \frac{1}{\Delta_i^I} \) or, equivalently, \(\left[ \frac{D'}{y \Delta_i^I} \right] > \frac{1}{\Delta_i^I} - 1.\)

The final sum on the right hand side of (11) then corresponds to \(c = 1/\Delta_i^I.\)

Moreover,

\[
\sum_{c=\left[ \frac{D'}{y \Delta_i^I} + 1 \right] + 1}^{\left[ \frac{D'}{y \Delta_i^I} + 1 \right] - 1} \int_{\Delta_i^I}^{c \Delta_i^I} \eta \left( yc \Delta_i^I, yz, D'', D' \right) dH(z|t) \leq yk_3 \left( \Delta_i^I \right)^2 \left[ \frac{1}{\Delta_i^I} - \left[ \frac{D''}{y \Delta_i^I} \right] - 1 \right].
\]

Thus,

\[
|\Omega^{Hiy}(t,t,D',D'')| \leq yk_3 \left( \Delta_i^I \right)^2 \left[ \frac{1}{\Delta_i^I} - 1 - \frac{D''}{y \Delta_i^I} \right] < yk_3 \left( \Delta_i^I \right)^2 \left[ \left[ \frac{D'}{y \Delta_i^I} \right] - \frac{D''}{y \Delta_i^I} \right]
\]

\[
\leq yk_3 \left( \Delta_i^I \right)^2 \left[ \frac{D'}{y \Delta_i^I} \right] = k_3 \Delta_i^I (D' - D'')
\]

as before. Q.E.D.

Claim 2

\(^2\) By Remark 3, line (12) is zero as \(y \left[ \frac{D'}{y \Delta_i^I} + 1 \right] \Delta_i^I = D' = y \Delta_i^I \left( \left[ \frac{D'}{y \Delta_i^I} + 1 \right] - \frac{D'}{y \Delta_i^I} \right) > 0. \) The inequality in line (13) then follows from Lipschitz Continuity.
For all $D, D' \in [0, y]$ and all $t \in S_i \setminus \{1\}$, define the difference quotients of $v^{Hy_i}(D, t)$ with respect to $D$ and $t$:

$$
\Delta_1^{Hy_i}(D, D', t) = \frac{v^{Hy_i}(D, t) - v^{Hy_i}(D', t)}{D - D'} \quad \text{and} \quad (14)
$$

$$
\Delta_2^{Hy_i}(D, t) = \frac{v^{Hy_i}(D, t + \Delta_i) - v^{Hy_i}(D, t)}{\Delta_i} \quad \text{and} \quad (15)
$$

By parts 1 and 2 of Claim 2, if $D < D'$, then

$$
\Delta_1^{Hy_i}(D, D', t) \in \left( k_2 \left( 1 - \Delta_i \left[ \frac{D'}{y\Delta_i} \right] + \Delta_i \right), k_3 \left( 1 - \frac{D}{y} + \Delta_i \right) \right) \subset (0, \infty) \quad (16)
$$

while if $D > y\Delta_i$,

$$
\Delta_2^{Hy_i}(D, t) \in \left( 0, k_3 \min \left\{ D, y \left( 1 - \Delta_i \right) \right\} \right). \quad (17)
$$

By (10), for any $D \in [0, y]$, the partial derivatives of $v^{Hy}(D, t)$ are given by

$$
v_2^{Hy}(D, t) = -y \int_{z=0}^{D/y} \frac{\partial H(z|t)}{\partial t} dz \quad \text{and} \quad (18)
$$

$$
v_1^{Hy}(D, t) = 1 - H \left( \frac{D}{y} \right) = \int_{z=D/y}^{1} \frac{\partial H(z|t)}{\partial z} dz. \quad (19)
$$

For all $D \in [0, y]$ and $t \in [0, 1]$,

$$
\frac{\partial v_2^{Hy}(D, t)}{\partial D} = -\frac{\partial}{\partial t} H \left( \frac{D}{y} \right) \in \left[ k_2 \frac{D}{y} \left( 1 - \frac{D}{y} \right), k_3 \right) \quad (20)
$$

by (18) and L-H;

$$
\left| \frac{\partial v_2^{Hy}(D, t)}{\partial t} \right| \leq k_4 D \quad (21)
$$

by (18) and BSD;

$$
\frac{\partial v_1^{Hy}(D, t)}{\partial D} = -\frac{\partial}{\partial D} H \left( \frac{D}{y} \right) \in \left( -\frac{k_3}{y}, -\frac{k_2}{y} \right) \quad (22)
$$

by (19) and L-H;

$$
k_3 D^2 \left[ 3y - 2D \right] < v_2^{Hy}(D, t) \quad (23)
$$

by (19) and L-H;

$$
k_2 \left( 1 - \frac{D}{y} \right) \left< v_1^{Hy}(D, t) \quad (24)
$$

by (18) and L-H;
by (19) and L-\( H \); for all \( t' \in [0, 1] \),
\[
\left| v_{1}^{H_{y}}(D, t) - v_{1}^{H_{y}}(D, t') \right| = \left| H \left( \frac{D}{y} \right) - H \left( \frac{D}{y} | t' \right) \right| \leq k_{3} \left| t - t' \right| \tag{25}
\]
by (19) and L-\( H \); for all \( D' \in [0, y] \),
\[
\left| v_{1}^{H_{y}}(D, t) - v_{1}^{H_{y}}(D', t) \right| = \left| H \left( \frac{D}{y} | t \right) - H \left( \frac{D'}{y} | t \right) \right| < \frac{k_{3}}{y} | D - D' | \tag{26}
\]
by L-\( H \). By (23) and (24),
\[
0 \leq \frac{k_{2}D^{2}(3y - 2D)}{6yk_{3}(y - D)} < \frac{v_{2}^{H_{y}}(D, t)}{v_{1}^{H_{y}}(D, t)} < \frac{k_{3}y}{k_{2}} \left( \frac{D}{y - D} \right), \tag{27}
\]
and
\[
\frac{6yk_{3}(y - D)}{k_{2}D^{2}(3y - 2D)} > \frac{v_{1}^{H_{y}}(D, t)}{v_{2}^{H_{y}}(D, t)} > \frac{k_{2}}{k_{3}y} \left( \frac{y - D}{D} \right). \tag{28}
\]
Finally, by (20), (22), (23), (24), and (28),
\[
\frac{\partial}{\partial D} \left( \frac{v_{2}^{H_{y}}(D, t)}{v_{1}^{H_{y}}(D, t)} \right) = \frac{\partial}{\partial D} \frac{v_{2}^{H_{y}}(D, t)}{v_{1}^{H_{y}}(D, t)} - \frac{v_{2}^{H_{y}}(D, t) \frac{\partial}{\partial D} \frac{v_{1}^{H_{y}}(D, t)}{v_{1}^{H_{y}}(D, t)}}{\left[ \frac{\partial}{\partial D} \frac{v_{1}^{H_{y}}(D, t)}{v_{1}^{H_{y}}(D, t)} \right]^{2}} \in \left[ \gamma_{1}^{\gamma}(D), \gamma_{2}^{\gamma}(D) \right] \tag{29}
\]
where
\[
\gamma_{1}^{\gamma}(D) = \frac{k_{2}D}{k_{3}y} \left[ 1 + \frac{Dk_{2}(3y - 2D)}{6k_{3}(y - D)^{2}} \right] \tag{30}
\]
and
\[
\gamma_{2}^{\gamma}(D) = \frac{k_{3}y}{k_{2}(y - D)} \left[ 1 + \frac{Dk_{3}}{k_{2}(y - D)} \right]. \tag{31}
\]
Note that \( \gamma_{1}^{\gamma}(D) \) lies in \((0, \infty)\) if \( D \in (0, y) \), is zero if \( D = 0 \), and is \( \infty \) if \( D = y \). Moreover, \( \gamma_{2}^{\gamma}(D) \) is increasing in \( D \), lies in \((0, \infty)\) if \( D \in [0, y) \), and is \( \infty \) if \( D = y \). Finally,
\[
\max_{D \in [a, b]} \gamma_{2}^{\gamma}(D) \leq \frac{k_{3}y}{k_{2}(y - b)} \left[ 1 + \frac{bk_{3}}{k_{2}(y - b)} \right], \tag{32}
\]
which is positive and, if \( b < y \), finite.

For any real number \( \ell \), let \([\ell, \infty]\) and \([\ell, \infty)\) denote the sets \((\ell, \infty) \cup \{\infty\}\) and \([\ell, \infty) \cup \{\infty\}\), respectively. Define the function
\[
\phi(D, t) = \frac{v_{2}^{H_{y}}(D, t)}{v_{1}^{H_{y}}(D, t)} = -\frac{y}{y_{z=0}} \frac{\partial H(z | t)}{\partial t} dz \leq \frac{1 - H \left( \frac{D}{y} | t \right)}{1 - H \left( \frac{D}{y} | t \right)} \tag{33}
\]

(by (10)). For \( u \in [0,1] \) and \( a \in (0,y] \), and \( k \in (0,\infty) \), consider the following initial value problem in model \( \infty \):

**Continuous Initial Value Problem with Parameters \( \Phi, y, u, a, k (CP^{\Phi y}_{uak}) \).**

The differential equation

\[
\frac{dD^{Hy}_{uak}}{dt} = -\min \left\{ \frac{1}{1-\delta} \varphi \left( D^{Hy}_{uak}(t), k \right) \right\}
\]

with \( D^{Hy}_{uak} : [u, 1] \rightarrow \mathbb{R} \), together with the initial value \( D^{Hy}_{uak}(u) = a \).

Clearly, any \( D^{Hy}_{0y0} \) that solves \( CP^{Hy}_{0y0} \) must also be a solution \( D^{Hy} \) to \( CP^{Hy} \) and vice-versa.

**Claim 4** For any \( u \in [0,1], a \in (0,y], \) and \( k \in (0,\infty) \):

1. If either \( a < y \) or \( k < \infty \) (or both) then there exists a unique solution \( D^{Hy}_{uak} \) to \( CP^{Hy}_{uak} \), which is decreasing and differentiable in \( t \) and takes values in \((0,a]\).

2. Let \( a' \in (0,a] \) and \( k' \in [k,\infty] \). Then \( D^{Hy}_{uak}(t) \geq D^{Hy}_{uak'k'}(t) \) for all signals \( t \in [u,1] \).

3. If either \( a < y \) or \( k < \infty \) (or both), then the function \( D^{Hy}_{uak} \) is Lipschitz continuous in the signal \( t \) with Lipschitz constant \( \min \{k,k_{a}\} \) where

\[
k_{a} = \frac{k_{ya}}{(1-\delta)k_{2}(y-a)}.
\]

4. If \( a < y \), then for all \( t \in [u,1], D^{Hy}_{uak}(t) - D^{Hy}_{uak}(t) \in [0,y-a] \).

**Proof of Claim 4.** Part 1. We first show that \( \min \left\{ \frac{1}{1-\delta} \varphi \left( D(t), k \right) \right\} \) is (a) continuous in \( t \in [0,1] \) and (b) Lipschitz continuous in \( D \in [0,a] \). Since \( \min \) is a Lipschitz-continuous function, it suffices to show that \( \varphi \left( D(t), k \right) \) has properties (a) and (b) whenever \( \varphi \left( D(t), k \right) \leq (1-\delta)k \). If \( k = \infty \), then \( D \leq a < y \); if \( k < \infty \) and \( \varphi \left( D(t), k \right) \leq (1-\delta)k \) then by (33), \( D < y \) as \( \varphi \left( y,t \right) = \infty \). Since in both cases \( D < y, 1 - H \left( \frac{D}{y} \right) > 0 \). Moreover, \( H \) is continuously differentiable in \( z \) and \( t \). This establishes existence and uniqueness by the Picard-Lindelöf theorem. The solution \( D^{Hy}_{uak} \) is differentiable since the right hand side of (34) is finite. Finally, \( \varphi \left( D(t), 0 \right) < 0 \) for all \( D \in (0,y] \) and \( \varphi \left( 0,t \right) = 0 \). Hence, \( D^{Hy}_{uak} \) is decreasing in \( t \) until and unless it hits zero, where it remains. Thus, by (27), \( D^{Hy}_{uak}(t) \leq a \) for all \( t \in [u,1] \). It remains to show that \( D^{Hy}_{uak} > 0 \). Suppose that \( D^{Hy}_{uak}(t) \) first reaches its minimum value of \( D \) at \( t = \bar{t} > u \).

**Lemma 5** The minimum face value \( D \) is nonzero.
PROOF OF LEMMA 5: For \( t \in [u, \bar{t}] \), the function \( D_{uak}^{Hy}(t) \) has a strictly decreasing inverse \( t_{uak}^{Hy} \) that satisfies the following inverse problem:

**Inverse Continuous Initial Value Problem with Parameters**

\( \Phi, y, u, a, k \) (ICP\( \Phi_{uak} \)). The differential equation

\[
\frac{d t_{uak}^{Hy}}{dD} = -\max \left\{ (1 - \delta) \frac{v_1^{Hy}(D, t_{uak}^{Hy}(D))}{v_2^{Hy}(D, t_{uak}^{Hy}(D))} \frac{1}{k} \right\}
\]

(36)

with \( t_{uak}^{Hy} : [D, a] \to [u, \bar{t}] \), together with the terminal value \( t_{uak}^{Hy}(a) = u \).

By (28), \( \frac{dt_{uak}^{Hy}}{dD} \leq - (1 - \delta) \frac{k_2}{k_y} \left( \frac{y-D}{D} \right) \). Hence, as \( t_{uak}^{Hy}(a) = u \),

\[
\bar{t} = t_{uak}^{Hy}(D) = t_{uak}^{Hy}(a) - \int_{D=D}^{a} \frac{dt_{uak}^{Hy}}{dD} dD
\]

\[
\geq u + \frac{(1 - \delta) k_2}{k_y} \int_{D}^{y} \left( \frac{y-D}{D} \right) dD = u + \frac{(1 - \delta) k_2}{k_1} F \left( \frac{D}{y} \right),
\]

where \( F(x) \) denotes \( x - \ln x - 1 \), which is finite and differentiable for all finite \( x > 0 \). \( F \) is decreasing in \( x \in (0, 1) \): \( F'(x) = 1 - 1/x < 0 \). Thus, \( F \left( \frac{D}{y} \right) < \frac{k_1}{(1 - \delta) k_2} (\bar{t} - u) \), whence

\[
D > yF^{-1} \left( \frac{k_1}{(1 - \delta) k_2} (\bar{t} - u) \right).
\]

Moreover, \( F(1) = 0 \) and \( \lim_{x \to 0} F(x) = \infty \), so the inverse \( F^{-1} \) is decreasing in \( x \in (0, \infty) \) and satisfies \( F^{-1}(0) = 1 \) and \( \lim_{x \to \infty} F^{-1}(x) = 0 \). Hence, as \( \bar{t} - u > 0, D > 0 \) Q.E.D.-Lemma 5

As for part (b), let \( D, D' \in [0, a], D > D' \), be such that \( \max \{ \Phi(D, t), \Phi(D', t) \} \leq (1 - \delta) k \). If \( k = \infty \), then \( D \leq a < y \). If \( k < \infty \), then by (27),

\[
\frac{k_2 D^2}{6k_3 (y-D)} \leq \frac{k_2 D^2 (y + 2[y-D])}{6yk_3 (y-D)} < 
\phi(D, t) \leq (1 - \delta) k
\]

so \( \beta D^2 + D - y \leq 0 \) where \( \beta = k_2 \left[ 6(1 - \delta) k k_3 \right]^{-1} \in (0, \infty) \). As this function is convex in \( D \), it follows that \( D \leq \bar{D} = \frac{-1 + \sqrt{1 + 4\beta y}}{2\beta} \) which is less than \( y \) as

\[
y > 0 \implies 4\beta^2 y^2 + 4\beta y + 1 = (2\beta y + 1)^2 > 1 + 4\beta y
\]

\[
\implies 2\beta y + 1 > \sqrt{1 + 4\beta y} \implies y > \frac{-1 + \sqrt{1 + 4\beta y}}{2\beta} = \bar{D}.
\]

In both cases, \( D' < D \leq \max \{ \bar{D}, a \} < y \). Thus, \( |\phi(D, t) - \phi(D', t)| \leq c |D - D'| \) by (29) and (32), where \( c = \frac{k_3 y}{k_2 y} \left[ 1 + \frac{\max \{ \bar{D}, a \} k_3}{k_2 y} \right] \) is finite. This proves part (b).
Part 2. Define \( \Lambda(t) = D^{Hy}_{uak}(t) - D^{Hy}_{adk}(t) \), which is differentiable (in \( t \)) and thus continuous. Clearly, \( \Lambda(u) = a - a' \geq 0 \). And if, for some \( t \in [0,1] \), \( \Lambda(t) = 0 \), then \( D^{Hy}_{uak}(t) = D^{Hy}_{adk}(t) \) so by (34),

\[
\Lambda'(t) = \min \left\{ \frac{1}{1-\delta} \phi \left( D^{Hy}_{uak}(t), t \right), k' \right\} - \min \left\{ \frac{1}{1-\delta} \phi \left( D^{Hy}_{uak}(t), t \right), k \right\}
\]

which is nonnegative since \( k' \geq k \). Hence, \( \Lambda \) is never negative.

Part 3 follows from (27) and the fact that \( D^{Hy}_{uak} \leq a \).

Part 4. If \( a < y \), then for all \( t \in [u,1] \), \( D^{Hy}_{uak}(t) - D^{Hy}_{uawo}(t) \in [0,y-a] \). By part 2, \( D^{Hy}_{uak} \geq D^{Hy}_{uawo} \). Let \( \Gamma_a(t) = D^{Hy}_{uak}(t) - D^{Hy}_{uawo}(t) \geq 0 \). By (33),

\[
\Gamma_a'(t) = \max \left\{ \frac{1}{1-\delta} \left[ \phi \left( D^{Hy}_{uawo}(t), t \right) - \phi \left( D^{Hy}_{uak}(t), t \right) \right], -k_a \right\}
\]

Both entries in the max are nonpositive by (27), (33), and the fact that \( D^{Hy}_{uawo}(t) \leq a \). Accordingly, for all \( t \in [u,1] \), \( D^{Hy}_{uak}(t) - D^{Hy}_{uawo}(t) \leq \Gamma_a(u) = y - a \). Q.E.D.

Claim 4

Before addressing the discrete case, we prove some useful bounds:

Claim 6 Let \( w, w', \zeta, e, \zeta' \) be in \( (0,y] \) and satisfy \( w' \geq w \), \( \zeta' \geq \zeta \), \( w > \zeta \), and \( w' > \zeta' \). Then

\[
0 \leq \Delta^{Hy}_{2} \left( \zeta', t \right) - \Delta^{Hy}_{2} \left( \zeta, t \right) \leq k_3 \left( \zeta' - \zeta \right). \tag{37}
\]

Moreover, if \( \min \left\{ w, w', \zeta, \zeta' \right\} > y \Delta' \) then

\[
0 \geq \Delta^{Hy}_{1} \left( \zeta', w', t \right) - \Delta^{Hy}_{1} \left( \zeta, w, t \right) \geq -k_3 \left[ \max \left\{ w' - w, \zeta' - \zeta \right\} \right] \tag{38}
\]

and

\[
\frac{\Delta^{Hy}_{2} \left( \zeta, t \right)}{\Delta^{Hy}_{1} \left( \zeta, w, t \right)} \in \left( 0, \frac{k_3 \zeta}{k_2 \left( 1 - \Delta' \left| \frac{w}{y \Delta'} \right| + \Delta'_t \right)} \right) \subset (0,\infty). \tag{39}
\]

Proof of Claim 6. By (5), for any \( \zeta \in [0,y] \) and \( t \in S \setminus \{ 1 \} \),

\[
y^{Hy} \left( \zeta, t + \Delta \right) - y^{Hy} \left( \zeta, t \right) = \int y^{\hat{z}_{\gamma}} \left[ H \left( \Delta'_t \left| \frac{z}{\Delta'_t} \right| t \right) - H \left( \Delta'_t \left| \frac{z}{\Delta'_t} \right| t + \Delta \right) \right] \, dz. \tag{40}
\]

As the integrand is nonnegative, (40) is nondecreasing in \( \zeta \), so \( \Delta^{Hy}_{2} \left( \zeta', t \right) - \Delta^{Hy}_{2} \left( \zeta, t \right) \geq 0 \). By L-H, for any \( z \in [0,y] \), \( H(z|t) - H(z|t + \Delta) < k_3 \Delta \). Equation (37) then follows from (15) and (40).
By (5),
\[
\Delta^H_{1i} (\xi e', w', t) - \Delta^H_{1i} (\xi e, w, t) = \frac{y^{\text{Hy}i} (\xi e', t) - y^{\text{Hy}i} (w', t)}{\zeta e' - w'} - \frac{y^{\text{Hy}i} (\xi e, t) - y^{\text{Hy}i} (w, t)}{\zeta e - w} = \frac{y}{w - \zeta e} \int_{z = \frac{\xi e}{y}}^{w} H \left( \Delta' \left| \frac{z}{\Delta'_{ij}} \right| t \right) dz - \frac{y}{w - \zeta e} \int_{z = \frac{\xi e'}{y}}^{w'} H \left( \Delta' \left| \frac{z}{\Delta'_{ij}} \right| t \right) dz.
\]
Define the change of variables \( z' = \xi + \left( \frac{w - \zeta e'}{w - \zeta e} \right) (z - \frac{\xi e'}{y}) \). When \( z = \frac{\xi e'}{y}, z' = \frac{\xi e'}{y} \), and when \( z = \frac{w y}{y}, z' = \frac{w e}{w - \zeta e} \). Moreover, \( dz = \frac{w - \xi e'}{w - \zeta e} dz' \) and \( z = \frac{\xi e'}{y} + \left( \frac{w - \xi e'}{w - \zeta e} \right) (z' - \frac{\xi e'}{y}) \) which we denote \( \psi(z') \). So \( \int_{z = \frac{\xi e}{y}}^{w} H \left( \Delta' \left| \frac{z}{\Delta'_{ij}} \right| t \right) dz = \int_{z = \frac{\xi e}{y}}^{w} H \left( \Delta' \left| \frac{\psi(z')}{\Delta'_{ij}} \right| t \right) d z' \). Renaming \( z' \) to \( z \) and simplifying,
\[
\Delta^H_{1i} (\xi e', w', t) - \Delta^H_{1i} (\xi e, w, t) = \frac{y}{w - \zeta e} \int_{z = \frac{\xi e}{y}}^{w} \left[ H \left( \Delta' \left| \frac{\psi(z') - w}{\Delta'_{ij}} \right| t \right) - H \left( \Delta' \left| \frac{\psi(z') - w}{\Delta'_{ij}} \right| t \right) \right] d z.
\]
(41)
We can write
\[
z - \psi(z) = \frac{1}{w - \zeta e} \left( \left( z - \frac{\xi e'}{y} \right) [w - \zeta e] - \left( z - \frac{\xi e}{y} \right) [w - \zeta e'] \right).
\]
(42)
As the right hand side is linear in \( z \), it reaches its maximum and minimum at the endpoints of the interval of integration. At the lower endpoint (at \( z = \frac{\xi e}{y} \)), the right hand side of (42) equals \( \frac{z - \xi e'}{y} \), while at the upper endpoint (at \( z = \frac{w y}{y} \)), it equals \( \frac{w - w e}{y} \). Thus, \( -w \leq z - \psi(z) \leq -w \) where \( w = y^{-1} \min \{ w' - w, \xi' - \xi \} \) and \( \bar{w} = y^{-1} \max \{ w' - w, \xi' - \xi \} \). As \( w \) and \( \bar{w} \) are both nonnegative, \( z \leq \psi(z) \), which by (41) establishes the first inequality in (38). Finally, by (41) and L-H,
\[
\Delta^H_{1i} (\xi e', w', t) - \Delta^H_{1i} (\xi e, w, t) \geq \frac{y}{w - \zeta e} \int_{z = \frac{\xi e}{y}}^{w} \left[ H \left( \Delta' \left| \frac{\psi(z') - w}{\Delta'_{ij}} \right| t \right) - H \left( \Delta' \left| \frac{\psi(z') - w}{\Delta'_{ij}} \right| t \right) \right] d z
\]
\[
\geq \frac{yk_3}{w - \zeta e} \int_{z = \frac{\xi e}{y}}^{w} \left[ \Delta' \left| \frac{\psi(z') - w}{\Delta'_{ij}} \right| - \Delta' \left| \frac{\psi(z') - w}{\Delta'_{ij}} \right| \right] d z
\]
\[
\geq -\frac{yk_3}{w - \zeta e} \int_{z = \frac{w + \Delta'}{y}}^{w} [\bar{w} + \Delta'] d z = -k_3 [\bar{w} + \Delta'].
\]
This establishes the second inequality in (38). Finally, (39) follows from (16) and (17). Q.E.D. Claim 6
Claim 7. Fix $H$, $y$, and $i$. Define
\[ \phi(D, t) = \psi^{I_i}(D, t + \Delta_i) - \delta \psi^{I_i}(D, t). \] (43)

For any $D \in (y\Delta_i, y]$ and any $t \in S_i$ there exists a unique solution $D^* = D^* (D)$, which lies in $(y\Delta_i, D)$, to
\[ \phi(D^*, t) = (1 - \delta) \psi^{I_i}(D, t). \] (44)
Moreover, $D^* (D)$ is increasing in $D$ and $D - D^* (D)$ is nondecreasing in $D$. Finally,
\[ \frac{\partial \phi}{\partial D} \geq k_2 (1 - \delta) \left( 1 - \frac{D}{y} \right). \] (45)

Proof of Claim 7. We first show three properties.

1. $\phi(y\Delta_i, t) < (1 - \delta) \psi^{I_i}(D, t)$. Proof: by (5), for all $t$ in $S_i$, $\psi^{I_i}(y\Delta_i, t) = y\Delta_i$. Hence, $\phi(y\Delta_i, t) = (1 - \delta) y\Delta_i$. Moreover, $\psi^{I_i}(D, t)$ is strictly increasing in $D \in [0, y]$ by part 2 of Claim 2, so $\psi^{I_i}(D, t) > y\Delta_i$. The result then follows from (43) and (44).

2. $\phi(D, t) > (1 - \delta) \psi^{I_i}(D, t)$. Proof: for $D \in (y\Delta_i, y]$, $\psi^{I_i}(D, t)$ is increasing in $t$ by part 1 of Claim 2. The result then follows from (43) and (44).

3. $\phi$ is continuous and increasing in $D \in [0, y]$. Proof: $\psi^{I_i}(D, t)$ is continuous in $D$ by (8), so $\phi(D, t)$ also is continuous in $D$. By (5),
\[ \phi(D, t) = (1 - \delta) D - y \left[ \int_{z=0}^{D/y} H \left( \Delta_i \left| \frac{z}{\Delta_i} \right| t + \Delta_i \right) - \delta H \left( \Delta_i \left| \frac{z}{\Delta_i} \right| t \right) dz \right]. \]

By L-H, $1 - H(z|t) > k_2 (1 - z)$ for all $z \in [0, 1]$ (and thus, substituting $z = 0, k_2 < 1$), so letting $z_0 = \Delta_i \left| \frac{D}{\Delta_i} \right| \leq D/y$,
\[ \frac{\partial \phi}{\partial D} = 1 - \delta - \left[ H(z_0|t + \Delta_i) - \delta H(z_0|t) \right] \geq (1 - \delta) \left[ 1 - H(z_0|t) \right] \geq (1 - \delta) k_2 (1 - z_0) \geq k_2 (1 - \delta) \left( 1 - \frac{D}{y} \right). \]

This establishes the result as well as equation (45).

Facts 1-3 imply that for any $D \in (y\Delta_i, y]$, there exists a unique $D^* = D^* (D)$ satisfying (44), and that it lies in $(y\Delta_i, D)$. Moreover, $D^* (D)$ is increasing in $D$.

Finally, let $\tilde{D}_0 > D$ and let $\tilde{D}^* = D^* \left( \tilde{D}_0 \right)$. To show that $D - D^* (D)$ is nondecreasing in $D$, we must show that $\tilde{D}_0 - \tilde{D}^* \geq D - D^*$ or, equivalently, that $\tilde{D}_0 - D \geq \tilde{D}^* - D^*$. By (43),
\[ \phi(D, t) = \psi^{I_i}(D, t + \Delta_i) - \delta \psi^{I_i}(D, t) = (1 - \delta) \psi^{I_i}(D, t) + \psi^{I_i}(D, t + \Delta_i) - \psi^{I_i}(D, t). \]
Hence, by (15), (37), and (44), and since \( \hat{D}^* > D^* \),

\[
(1 - \delta) \left[ v^{Hyi} \left( \hat{D}_0, t \right) - y^{Hyi} (D, t) \right] = \phi \left( \hat{D}^*, t \right) - \phi \left( D^*, t \right)
\geq (1 - \delta) \left[ v^{Hyi} \left( \hat{D}^*, t \right) - y^{Hyi} (D^*, t) \right],
\]

whence by part 2 of Claim 2,

\[
\frac{\hat{D}^* - D^*}{\hat{D}_0 - D} \leq \frac{\frac{v^{Hyi}(\hat{D}_0, t) - v^{Hyi}(D, t)}{v^{Hyi}(D, t) - v^{Hyi}(D^*, t)}}{\frac{D_0 - D}{D^* - D^*}},
\]

which is in \((0, 1)\) by (14) and (38). Thus, \( \hat{D}^* - D^* \leq \hat{D}_0 - D \) as claimed. Q.E.D.

Claim 7

For any real number \( \ell \), let \((\ell, \infty)\) and \([\ell, \infty)\) denote the sets \((\ell, \infty) \cup \{\infty\}\) and \([\ell, \infty) \cup \{\infty\}\), respectively. For any constants \( u \in S_i \), \( a \in (0, y] \), and \( k \in (0, \infty) \), consider the following initial value problem, where \( S^u_i \) denotes the set of signals \( t \geq u \) in \( S_i \):

**Discrete Initial Value Problem with Parameters** \( \Phi, y, u, a, k \) \( (DP^{Hyi}(uak)) \).

The condition

\[
D^{Hyi}_{uak}(t + \Delta_i) = \max \left\{ D^{Hyi+}_{uak}(t + \Delta_i), D^{Hyi}_{uak}(t) - k\Delta_i \right\}
\]

(46)

with \( D^{Hyi}_{uak} : S^u_i \to \mathbb{R} \), where \( D^{Hyi+}_{uak}(t + \Delta_i) \) is the (by Claim 7) unique solution \( D^* \in \left( y\Delta_i, D^{Hyi}_{uak}(t) \right) \) to

\[
v^{Hyi}(D^*, t + \Delta_i) - \delta v^{Hyi}(D^*, t) = (1 - \delta) v^{Hyi} \left( D^{Hyi}_{uak}(t), t \right),
\]

(47)

together with the initial value \( D^{Hyi}_{uak}(u) = a > y\Delta_i \).

Clearly, any \( D^{Hyi}_{0\infty} \) that solves \( DP^{Hyi}_{0\infty} \) must also be a solution \( D^{Hyi}_{uak} \) to \( DP^{Hyi}_{uak} \) and vice-versa.

**Claim 8** For any \( u \in S_i \), \( a \in (y\Delta_i, y] \), and \( k \in (0, \infty) \):

1. There exists a unique solution \( D^{Hyi}_{uak} \) to \( DP^{Hyi}_{uak} \). This function is decreasing in \( t \in S_i^u \) and takes values in \((y\Delta_i, a]\).

2. Let \( a' \in (y\Delta_i, a] \) and \( k' \in [k, \infty) \). Then \( D^{Hyi}_{uak}(t) \geq D^{Hyi}_{uak}(k') \) for all signals \( t \in S_i^u \).

3. Let \( k_a = \frac{k_y (y + a)}{k_2 (1 - \delta) (y (1 - \Delta_i) - a)} \). For all signals \( t \in S_i^u \), \( 0 > D^{Hyi+}_{uak}(t + \Delta_i) - D^{Hyi}_{uak}(t) \geq -\Delta_i k_a \) and hence \( 0 > D^{Hyi}_{uak}(t + \Delta_i) - D^{Hyi}_{uak}(t) \geq -\Delta_i \min \{ k, k_a \} \).
4. For all \( t \in S_t^u \), \( D_{uak}^{Hyi} (t) - D_{uak}^{Hyi} (t) \in [0, y - a] \).


Part 2. Clearly, \( D_{uak}^{Hyi} (u) - D_{uak}^{Hyi} (u) = a - a' \geq 0 \). And if, for some \( t \in S_t^u \setminus \{1\} \), \( D_{uak}^{Hyi} (t) \geq D_{uak}^{Hyi} (t) \), then \( D_{uak}^{Hyi} (t) - k_\Delta \geq D_{uak}^{Hyi} (t) - k'_\Delta \) and, by Claim 7, \( D_{uak}^{Hyi} (t + \Delta) \geq D_{uak}^{Hyi} (t + \Delta) \), so \( D_{uak}^{Hyi} (t + \Delta) \geq D_{uak}^{Hyi} (t + \Delta) \).

Part 3. Let \( D' = D_{uak}^{Hyi} (t) \), \( D'' = D_{uak}^{Hyi} (t + \Delta) \), and \( D* = D_{uak}^{Hyi} (t + \Delta) \). By Claim 7, \( D^* - D' < 0 \) and \( \min \{ D', D'', D* \} > y_\Delta \). We will show that \( D^* - D' \leq -Dk_a \) which, by (46), implies \( 0 > D'' - D' \geq -\Delta \min \{k, k_a\} \). The result is trivial when \( a = y \) since \( k_y = \infty \).

Suppose \( a < y \). By (15), (17), (43), and (44),

\[
\phi(D', t) - \phi(D^*, t) = \phi(D', t) - (1 - \delta) \phi(D', t) = \phi(D', t + \Delta) - \phi(D', t) \leq k_3 D' \Delta t \leq k_3 a \Delta t.
\]

But by (45), \( \phi(D', t) - \phi(D^*, t) \geq [D' - D^*] k_2 (1 - \delta) \left(1 - \frac{a}{y}\right) \). Combining the two inequalities and using (35) yields the result.

Part 4. Fix \( u \in [0, 1] \) and \( a \in (0, y) \), whence \( k_a \in (0, \infty) \). By part 2, \( D_{uak}^{Hyi} (t) \geq D_{uak}^{Hyi} (t) \) for all signals \( t \in S_t^u \). Let \( \Gamma_a (t) = D_{uak}^{Hyi} (t) - D_{uak}^{Hyi} (t) \geq 0 \). As \( D_{uak}^{Hyi} (t + \Delta) = D_{uak}^{Hyi} (t + \Delta) \),

\[
\Gamma_a (t + \Delta) - \Gamma_a (t) = \max \left\{ \frac{D_{uak}^{Hyi} (t + \Delta) - D_{uak}^{Hyi} (t) - [D_{uak}^{Hyi} (t + \Delta) - D_{uak}^{Hyi} (t)]}{\Delta}, \ -k_a \Delta t - [D_{uak}^{Hyi} (t + \Delta) - D_{uak}^{Hyi} (t)] \right\}
\]

for any \( t < 1 \) in \( S_t^u \) by (46). By Claim 7, \( D' - D^* (D') \) is nondecreasing in \( D' \), so

\[
D_{uak}^{Hyi} (t + \Delta) - D_{uak}^{Hyi} (t) - [D_{uak}^{Hyi} (t + \Delta) - D_{uak}^{Hyi} (t)] \leq 0,
\]

and by part 3, \( D_{uak}^{Hyi} (t + \Delta) - D_{uak}^{Hyi} (t) \geq -Dk_a \). Hence, \( \Gamma_a (t + \Delta) \in \{0, \Gamma_a (t)\} \), so \( t \in S_t^u \), \( D_{uak}^{Hyi} (t) - D_{uak}^{Hyi} (t) \leq \Gamma_a (u) = y - a \). Q.E.D. Claim 8

As noted in section 6, we extend any function defined on \( S_t \) to any \( t \in [0, 1] \) by evaluating it at \( \tau_i \). For any \( t \in [0, 1] \),

\[
(1 - \delta) \Delta_i^{Hyi} \left( D_{uak}^{Hyi} (t + \Delta_i), D_{uak}^{Hyi} (t), \tau_i^t \right) \Delta D_{uak}^{Hyi} (t) + \Delta_2^{Hyi} \left( D_{uak}^{Hyi} (t + \Delta_i), \tau_i^t \right) = 0 \quad (48)
\]
where we define

\[\Delta D_{uaok}^{Hyi}(t) = \frac{D_{uaok}^{Hyi}(t+\Delta_i) - D_{uaok}^{Hyi}(t)}{\Delta_i}.\]  

Equation (48) can be rewritten

\[\Delta D_{uaok}^{Hyi}(t) = -\frac{1}{1-\delta} \Delta_1^{Hyi} \left( D_{uaok}^{Hyi}(t+\Delta_i), \tau_i^t \right) \]

whence \(D_{uaok}^{Hyi}(t+\Delta_i) = D_{uaok}^{Hyi}(t) - \frac{1}{1-\delta} \Delta_1^{Hyi} \left( D_{uaok}^{Hyi}(t+\Delta_i), D_{uaok}^{Hyi}(t), \tau_i^t \right) \Delta_i\) and thus, by (46),

\[D_{uaok}^{Hyi}(t+\Delta_i) = D_{uaok}^{Hyi}(t) - \Delta_i \min \left\{ \frac{1}{1-\delta} \Delta_1^{Hyi} \left( D_{uaok}^{Hyi}(t+\Delta_i), D_{uaok}^{Hyi}(t), \tau_i^t \right), k \right\},\]

and so

\[\Delta D_{uaok}^{Hyi}(t) = \frac{D_{uaok}^{Hyi}(t+\Delta_i) - D_{uaok}^{Hyi}(t)}{\Delta_i} = \min \left\{ \frac{1}{1-\delta} \Delta_1^{Hyi} \left( D_{uaok}^{Hyi}(t+\Delta_i), D_{uaok}^{Hyi}(t), \tau_i^t \right), k \right\}.\]

**Lemma 10** In the above formula for \(\Delta D_{uaok}^{Hyi}(t)\), we may replace \(D_{uaok}^{Hyi}(t+\Delta_i)\) by \(D_{uaok}^{Hyi}(t)\). That is,

\[\Delta D_{uaok}^{Hyi}(t) = \min \left\{ \frac{1}{1-\delta} \Delta_1^{Hyi} \left( D_{uaok}^{Hyi}(t+\Delta_i), D_{uaok}^{Hyi}(t), \tau_i^t \right), k \right\}.\]  

**Proof of Lemma 10.** Let \(w = w' = D_{uaok}^{Hyi}(t)\). Let \(\zeta e = D_{uaok}^{Hyi}(t+\Delta_i)\) and \(\zeta e' = D_{uaok}^{Hyi}(t+\Delta_i)\). By Claim 7 and part 1 of Claim 8, \(\min\{w, \zeta, w', \zeta\} \geq y\Delta_i\). Hence, by Claim 6, \(\Delta_1^{Hyi}(\zeta', w', \tau_i^t) \leq \Delta_1^{Hyi}(\zeta, w, \tau_i^t)\) and \(\Delta_2^{Hyi}(\zeta', \tau_i^t) \leq \Delta_2^{Hyi}(\zeta, \tau_i^t)\). Thus,

\[\frac{\Delta_2^{Hyi}(\zeta', \tau_i^t)}{\Delta_1^{Hyi}(\zeta', w', \tau_i^t)} \leq \frac{\Delta_2^{Hyi}(\zeta, \tau_i^t)}{\Delta_1^{Hyi}(\zeta, w, \tau_i^t)},\]

with equality when \(\zeta e' = \zeta e\). Accordingly, there are two cases. If \(\frac{\Delta_2^{Hyi}(\zeta, \tau_i^t)}{1-\delta \Delta_1^{Hyi}(\zeta, w, \tau_i^t)} \leq k\), then \(\zeta e' = \zeta e\), which implies (50). If \(\frac{\Delta_2^{Hyi}(\zeta, \tau_i^t)}{1-\delta \Delta_1^{Hyi}(\zeta, w, \tau_i^t)} \geq k\), then \(\Delta D_{uaok}^{Hyi}(t) = -k\) but by (51), \(\frac{\Delta_2^{Hyi}(\zeta', \tau_i^t)}{1-\delta \Delta_1^{Hyi}(\zeta', w', \tau_i^t)} \geq k\) as well, so (50) holds. Q.E.D. Lemma 10
For all $t' \in [u, 1]$, since $D_{u \alpha 0 k}^{Hy}(u) = a_0$,
\[
D_{u \alpha 0 k}^{Hy}(t') = a_0 + \Delta_i \sum_{t \in S_i', t < \tau_i'} \Delta D_{u \alpha 0 k}^{Hy}(t)
\]
\[
= a_0 - \Delta_i \sum_{t \in S_i', t < \tau_i'} \min \left\{ \frac{1}{1-\delta} \frac{\Delta_i^{Hy} \left( D_{u \alpha 0 k}^{Hy}(t+\Delta_i), t \right)}{\Delta_i^{Hy} \left( D_{u \alpha 0 k}^{Hy}(t+\Delta_i), D_{u \alpha 0 k}^{Hy}(t), t \right)}, k \right\}
\]
\[
= a_0 - \int_{t=u}^{t'} \min \left\{ \frac{1}{1-\delta} \frac{\Delta_i^{Hy} \left( D_{u \alpha 0 k}^{Hy}(t+\Delta_i), t \right)}{\Delta_i^{Hy} \left( D_{u \alpha 0 k}^{Hy}(t+\Delta_i), D_{u \alpha 0 k}^{Hy}(t), \tau_i \right)}, k \right\} dt, \tag{52}
\]
By (34) and (33), for all $t' \in [u, 1]$,
\[
D_{u \alpha 1 k}^{Hy}(t') = a_1 - \int_{t=u}^{t'} \min \left\{ \frac{1}{1-\delta} \frac{v_D \left( D_{u \alpha 1 k}^{Hy}(t) \right)}{v_D \left( D_{u \alpha 1 k}^{Hy}(t), \tau_i \right)}, k \right\} dt. \tag{53}
\]
Define the analogous quantities to $\Delta_i^{Hy}$ and $\Delta_i^{Hy}$ with $v^{Hy}$ replaced by $v^{Hy}$: for any $D', D'' \in [0, y]$, let
\[
\Delta_i^{Hy} \left( D', D'', t \right) = \frac{v^{Hy}(D', t) - v^{Hy}(D'', t)}{D' - D''}
\]
\[
\in \left( k_2 \left( 1 - \max \left\{ D', D'' \right\} \right), k_3 \left( 1 - \frac{\min \left\{ D', D'' \right\}}{y} \right) \right) \quad \text{and} \tag{54}
\]
\[
\Delta_i^{Hy} \left( D', t, \Delta_i \right) = \frac{v^{Hy}(D', t+\Delta_i) - v^{Hy}(D', t)}{\Delta_i}
\]
\[
\in \left( k_2 (D')^2 \left[ 3y - 2D' \right], k_3 D' \right), \tag{55}
\]
where the bounds follow from (23) and (24) and imply that
\[
\frac{\Delta_i^{Hy} \left( D', t, \Delta_i \right)}{\Delta_i^{Hy} \left( D', D'', t \right)} \in \left( \frac{k_2 (D')^2 \left[ 3y - 2D' \right]}{6yk_3 (y - \min \left\{ D', D'' \right\})}, \frac{yk_3 D'}{k_2 (y - \max \left\{ D', D'' \right\})} \right). \tag{56}
\]
If $a < y$ then by (39), (56), (27), and (35), for all $D, D' \in [y \Delta', a]$ and all $t \in [0, 1]$, the ratios
\[
\frac{\Delta_i^{Hy} \left( D', \tau_i \right)}{\Delta_i^{Hy} \left( D', D'', \tau_i \right)} \quad \text{and} \quad \frac{v_D^{Hy}(D', \tau_i)}{v_D^{Hy}(D', \tau_i)}
\]
are all at most $(1 - \delta) k_a$. Let
\[
\kappa = (1 - \delta) \min \{ k, k_a \} < \infty. \tag{57}
\]
It follows that by (52), (53) and the triangle inequality that for all $t' \in [u, 1]$,
\[
(1 - \delta) \left| D_{u \alpha 0 k}^{Hy}(t') - D_{u \alpha 1 k}^{Hy}(t') \right| = \left| \int_{t=u}^{t'} \min \left\{ \frac{\Delta_i^{Hy} \left( D_{u \alpha 0 k}^{Hy}(t+\Delta_i), \tau_i \right)}{\Delta_i^{Hy} \left( D_{u \alpha 0 k}^{Hy}(t+\Delta_i), D_{u \alpha 0 k}^{Hy}(t), \tau_i \right)}, \kappa \right\} dt \right|
\]
\[
- \left| \int_{t=u}^{t'} \min \left\{ \frac{v_D^{Hy} \left( D_{u \alpha 1 k}^{Hy}(t) \right)}{v_D^{Hy} \left( D_{u \alpha 1 k}^{Hy}(t), \tau_i \right)}, \kappa \right\} dt \right| + a_1 - a_0
\]
\[
\leq |a_1 - a_0| + A_1 + A_2 + A_3 + A_4 + A_5 + A_6
\]
where

\[
A_1 = \left| \int_{t'=t'}^{t_{\Delta_i}} \min \left\{ \frac{\Delta^2_H y_i \left( D_{u \alpha i k} (t + \Delta_i) \cdot \tau^i \right)}{\Delta^2_H y_i \left( D_{u \alpha i k} (t + \Delta_i) \cdot D_{u \alpha i k} (t) \cdot \tau^i \right)} , \kappa \right\} dt \right| ,
\]

\[
A_2 = \int_{t'=t'}^{t_{\Delta_i}} \min \left\{ \frac{\Delta^2_H y_i \left( D_{u \alpha i k} (t + \Delta_i) \cdot \tau^i \right)}{\Delta^2_H y_i \left( D_{u \alpha i k} (t + \Delta_i) \cdot D_{u \alpha i k} (t) \cdot \tau^i \right)} , \kappa \right\} dt ,
\]

\[
A_3 = \int_{t'=t'}^{t_{\Delta_i}} \min \left\{ \frac{\Delta^2_H y_i \left( D_{u \alpha i k} (t + \Delta_i) \cdot \tau^i \right)}{\Delta^2_H y_i \left( D_{u \alpha i k} (t + \Delta_i) \cdot D_{u \alpha i k} (t) \cdot \tau^i \right)} , \kappa \right\} dt ,
\]

\[
A_4 = \int_{t'=t'}^{t_{\Delta_i}} \min \left\{ \frac{v^H y_i \left( D_{u \alpha i k} (t + \Delta_i) \cdot \tau^i \right)}{v^H y_i \left( D_{u \alpha i k} (t) \cdot \tau^i \right)} , \kappa \right\} - \min \left\{ \frac{v^H y_i \left( D_{u \alpha i k} (t) \cdot \tau^i \right)}{v^H y_i \left( D_{u \alpha i k} (t) \cdot \tau^i \right)} , \kappa \right\} dt ,
\]

\[
A_5 = \int_{t'=t'}^{t_{\Delta_i}} \min \left\{ \frac{v^H y_i \left( D_{u \alpha i k} (t) \cdot \tau^i \right)}{v^H y_i \left( D_{u \alpha i k} (t) \cdot \tau^i \right)} , \kappa \right\} - \min \left\{ \frac{v^H y_i \left( D_{u \alpha i k} (t) \cdot \tau^i \right)}{v^H y_i \left( D_{u \alpha i k} (t) \cdot \tau^i \right)} , \kappa \right\} dt ,
\]

and

\[
A_6 = \int_{t'=t'}^{t_{\Delta_i}} \min \left\{ \frac{v^H y_i \left( D_{u \alpha i k} (t) \cdot \tau^i \right)}{v^H y_i \left( D_{u \alpha i k} (t) \cdot \tau^i \right)} , \kappa \right\} - \min \left\{ \frac{v^H y_i \left( D_{u \alpha i k} (t) \cdot \tau^i \right)}{v^H y_i \left( D_{u \alpha i k} (t) \cdot \tau^i \right)} , \kappa \right\} dt .
\]

Clearly, \( A_1 \leq \kappa \Delta_i \). For \( A_2, A_3, A_4, \) and \( A_5 \), we require the following claim.

**Lemma 11** For any \( a, b, c, d \geq 0 \), \( \max \{ |a - b|, |c - d| \} \) is an upper bound on both

\[
\left| \min \{ a, c \} - \min \{ b, d \} \right|
\]

and \( \max \{ a, c \} - \max \{ b, d \} \).

**Proof of Lemma 11.** We prove the result for \( \min \{ a, c \} - \min \{ b, d \} \); the proof for \( \max \{ a, c \} - \max \{ b, d \} \) is identical with the keyword \( \min \) replaced \( \max \) throughout. First, assume \( a \geq b \) and \( c \geq d \). Then \( \min \{ a, c \} - \min \{ b, d \} = \min \{ a, c \} - \min \{ b, d \} \). And \( \max \{ |a - b|, |c - d| \} = \max \{ a - b, c - d \} \). But

\[
\min \{ a, c \} - \min \{ b, d \} \leq \max \{ a - b, c - d \}
\]
since
\[
\min \{a, c\} \leq \min \{\max \{a, b + c - d\}, \max \{c, d + a - b\}\} = \min \{b + \max \{a - b, c - d\}, d + \max \{a - b, c - d\}\} = \min \{b, d\} + \max \{a - b, c - d\}.
\]

The other cases (in which \(b > a\) or \(c > d\) or both) are analogous. Q.E.D.

This Lemma leads to the following useful bound.

**Lemma 12** For any \(a, b, \kappa \in (0, \infty)\) and \(c, d \in [0, \infty)\),
\[
\left| \min \left\{ \frac{a}{c}, \kappa \right\} - \min \left\{ \frac{b}{d}, \kappa \right\} \right| \leq \frac{a - b}{\max \{c, a/\kappa\}} + \frac{b}{\max \{d, b/\kappa\}} \frac{\max \{c - d\}, a - b\} / \kappa\max \{c, a/\kappa\}.
\]

**Proof of Lemma 12.** For any \(a, b \geq 0\) and \(c, d > 0\),
\[
\frac{a - b}{c - d} = \frac{ad - bc}{cd} \leq \frac{ad - bd}{cd} + \frac{bd - bc}{cd} = \frac{a - b}{c} + \frac{b}{d} \frac{d - c}{c}.
\]

Moreover, for any \(a > 0\) and \(c, \kappa \geq 0\),
\[
\min \left\{ \frac{a}{c}, \kappa \right\} = \kappa \min \left\{ \frac{a}{c \kappa}, 1 \right\} = \kappa \min \left\{ \frac{a}{c \kappa}, a \right\} = \frac{a \kappa}{\max \{c, a/\kappa\}} = \frac{a}{\max \{c, a/\kappa\}}.
\]

By (59) and (60) and using Lemma (in that order),
\[
\left| \min \left\{ \frac{a}{c}, \kappa \right\} - \min \left\{ \frac{b}{d}, \kappa \right\} \right| = \left| \frac{a}{\max \{c, a/\kappa\}} - \frac{b}{\max \{d, b/\kappa\}} \right| \leq \frac{a - b}{\max \{c, a/\kappa\}} + \frac{b}{\max \{d, b/\kappa\}} \frac{\max \{d, b/\kappa\} - \max \{c, a/\kappa\}}{\max \{c, a/\kappa\}} \frac{\max \{c - d\}, a - b\} / \kappa\max \{c, a/\kappa\}.
\]

Q.E.D.

Let \(D_t' = D_{H^y_t}(t + \Delta_t) \in (y\Delta_t', a_0'), D_t'' = D_{H^y_t}(t) \in (y\Delta_t', a_0'), a_t = \Delta_{H^y_t}(D_t', \tau_t'), b_t = \Delta_{H^y_t}(D_t', \tau_t', \Delta_t), c_t = \Delta_{H^y_t}(D_t', D_t'', \tau_t'), \) and \(d_t = \Delta_{H^y_t}(D_t', D_t'', \tau_t'). \) By (58),
\[
A_2 \leq \int_{t=0}^{t_f} a_t - b_t \left( \frac{b_t}{\max \{d_t, b_t/\kappa\}} \frac{\max \{c_t - d_t, a_t - b_t\} / \kappa\} \right) dt + \int_{t=0}^{t_f} \left( \frac{b_t}{\max \{d_t, b_t/\kappa\}} \frac{\max \{c_t - d_t, a_t - b_t\} / \kappa\} \right) dt,
\]

By (14), (15), (54), (55), and parts 3 and 4 of Claim 2, for any \(\epsilon > 0\) there is an \(i^* < \infty\) such that if \(i > i^*, \max \{|a_t - b_t|, |c_t - d_t|\} < \epsilon. \) By (55), \(b_t \in \left[ \frac{k_2(D_t)^2}{6\eta}, k_3y \right]. \) By part 1 of
Claim 2, \( a_t > k_2 \frac{(D')^2(3y-2D')}{6y} > k_2 \frac{(D')^2}{6y} \) where \( D' = D'_t - 2y\Delta_i' < y \). Hence, \( \min \{a_t, b_t\} > k_2 \frac{(D' - 2y\Delta_i')^2}{6y} \). By (14), (54), and part 2 of Claim 2, \( \min \{c_t, d_t\} \geq k_2 \left(1 - \frac{D''}{y}\right) \) so since \( D' \geq D'' - k\Delta_i \), both \( \max \{c_t, a_t/\kappa\} \) and \( \max \{d_t, b_t/\kappa\} \) are at least

\[
K_2 \max \left\{ 1 - \frac{D''}{y}, \frac{(D' - 2y\Delta_i')^2}{6y\kappa} \right\} \geq K_2 \max \left\{ 1 - \frac{D''}{y}, \frac{(D'' - k\Delta_i - 2y\Delta_i')^2}{6y\kappa} \right\},
\]

which is bounded below by a strictly positive constant \( \kappa' \) for large enough \( i \) as \( y > 0 \). Collecting these bounds, \( A_2 \leq \varepsilon K_2 |t' - u| \) where \( K_2 = \frac{1}{\kappa} \left[1 + \frac{k_3 y}{\kappa} \max \{1, 1/\kappa\} \right] \in (0, \infty) \).

By (58), redefining \( a_t = \Delta H_y (D'_t, \tau'_t, \Delta_i), b_t = v_t H_y (D'_t, \tau'_t), c_t = \Delta H_y (D'_t, D''_t, \tau'_t), \) and \( d_t = v_t H_y (D''_t, \tau'_t) \),

\[
A_3 \leq \int_{t'=u}^{t'} \frac{a_t - b_t}{\max \{c_t, a_t/\kappa\}} dt + \int_{t'=u}^{t'} \left( \frac{b_t}{\max \{d_t, b_t/\kappa\}} \max \{|c_t - d_t|, |a_t - b_t|/\kappa\} \right) dt.
\]

By (23), \( b_t \leq k_3 y \). By the Mean Value Theorem, there is a \( t \in [\tau'_t, \tau'_t + \Delta_i] \) such that \( v_t H_y (D'_t, t) = a_t \). Thus, by (21),

\[
|a_t - b_t| = \left| v_t H_y (D'_t, t) - b_t \right| = \left| v_t H_y (D'_t, t) - v_t H_y (D'_t, \tau'_t) \right| \leq k_4 y \Delta_i.
\]

Also by the Mean Value Theorem, there is a \( D \in [D'_t, D''_t] \) such that \( v_t H_y (D, \tau'_t) = c_t \). By part 3 of Claim 8 and (57),

\[
|D'' - D'_t| \leq \Delta_i \min \{k, k_a\} = \frac{\kappa}{(1 - \delta)} \Delta_i.
\]

Hence, \( |c_t - d_t| = \left| v_t H_y (D, \tau'_t) - v_t H_y (D''_t, \tau'_t) \right| \leq \frac{k_3}{y} |D'' - D'_t| \leq \frac{k_3 \kappa}{y(1 - \delta)} \Delta_i \). By (23), (24), (54), and (55), \( \min \{a_t, b_t\} \geq \frac{k_2 (D')^2}{6y}, \) and \( \min \{c_t, d_t\} \geq k_2 \left(1 - \frac{D''}{y}\right) \) so as shown in the prior paragraph, both \( \max \{c_t, a_t/\kappa\} \) and \( \max \{d_t, b_t/\kappa\} \) are at least \( \kappa' > 0 \). Collecting these bounds, \( A_3 \leq \Delta_i \kappa_3 |t' - u| \) where \( \kappa_3 = \frac{y}{\kappa} \left[ k_4 + \frac{k_3 \kappa}{y(1 - \delta)} \max \left\{ \frac{k_3 \kappa}{y(1 - \delta)}, \frac{k_3 y}{\kappa} \right\} \right] \in (0, \infty) \).

By (58), redefining \( a_t = v_t H_y (D'_t, \tau'_t), b_t = v_t H_y (D''_t, \tau'_t), \) and \( c_t = d_t = v_t H_y (D''_t, \tau'_t) \),

\[
A_4 \leq \int_{t'=u}^{t'} \frac{a_t - b_t}{\max \{c_t, a_t/\kappa\}} dt + \int_{t'=u}^{t'} \left( \frac{b_t}{\max \{d_t, b_t/\kappa\}} \max \{|a_t - b_t|/\kappa\} \right) dt.
\]

By (23), \( b_t \leq k_3 y \). By (26) and (61),

\[
|a_t - b_t| = \left| v_t H_y (D'_t, \tau'_t) - v_t H_y (D''_t, \tau'_t) \right| \leq \frac{k_3 \kappa}{(1 - \delta) y} \Delta_i.
\]
By (23) and (24), \( \min \{ a_t, b_t \} \geq \frac{k_2 (D'')^2}{6y} \), and \( \min \{ c_t, d_t \} \geq k_2 \left( 1 - \frac{D''}{y} \right) \) so as shown in the prior paragraph, both \( \max \{ c_t, a_t / \kappa \} \) and \( \max \{ d_t, b_t / \kappa \} \) are at least \( \kappa' > 0 \). Collecting these bounds, \( A_4 \leq \Delta_i \kappa_4 \left[ t' - u \right] \), where \( \kappa_4 = \frac{k_3}{\kappa' (1 - \delta)} \left[ \frac{\kappa}{y} + \frac{k_3}{\kappa} \right] \in (0, \infty) \).

By (58), redefining \( a_I = v_H^y (D'_I, \tau'_I), b_I = v_H^y (D''_{ua1k} (\tau'_I), \tau'_I), c_I = v_D^H (D'_I, \tau'_I), \) and \( d_I = v_H^y (D''_{ua1k} (\tau'_I), \tau'_I) \),

\[
A_5 \leq \int_{t-u}^{t'} \frac{a_I - b_I}{\max \{ c_I, a_I / \kappa \}} \, dt + \int_{t-u}^{t'} \left( \frac{b_I}{\max \{ d_I, b_I / \kappa \}} \right) \max \{ c_I - d_I, |a_I - b_I| / \kappa \} \, dt.
\]

By (23), \( b_I \leq k_3 y \). By (21), \( |a_I - b_I| \leq k_4 y \left| D'_I - D''_{ua1k} (\tau'_I) \right| \). By (26),

\[
|c_I - d_I| \leq \frac{k_3}{y} \left| D'_I - D''_{ua1k} (\tau'_I) \right|.
\]

By (23), (24), \( \min \{ a_t, b_t \} \geq \frac{k_2 (D'')^2}{6y} \) and \( \min \{ c_t, d_t \} \geq k_2 \left( 1 - \frac{D''}{y} \right) \) so as shown above, both \( \max \{ c_I, a_I / \kappa \} \) and \( \max \{ d_I, b_I / \kappa \} \) are at least \( \kappa' > 0 \). Collecting these bounds and using \( D_{ua1k}^H (t) = D_{ua1k}^H (\tau'_I) \),

\[
A_5 \leq \kappa_5 \int_{t-u}^{t'} \left| D_{ua1k}^H (\tau'_I) - D''_{ua1k} (\tau'_I) \right| \, dt \leq \kappa_5 \left[ t' - u \right] \max_{t \in [u,t']} \left| D_{ua1k}^H (\tau'_I) - D''_{ua1k} (\tau'_I) \right|
\]

\[
\leq \kappa_5 \left[ t' - u \right] \max_{t \in [u,t']} \left| D_{ua1k}^H (t) - D''_{ua1k} (t) \right|
\]

where \( \kappa_5 = \frac{y}{\kappa} \left[ k_4 + \frac{k_3}{\kappa} \max \left\{ \frac{k_3}{y}, k_4 y \right\} \right] \in (0, \infty) \).

Now redefine \( a_I = v_H^y (D_{ua1k}^H (t), \tau'_I), b_I = v_H^y (D_{ua1k}^H (t), t), c_I = v_D^H (D_{ua1k}^H (t), \tau'_I), \) and \( d_I = v_H^y (D_{ua1k}^H (t), t) \). By (58),

\[
A_6 \leq \int_{t-u}^{t'} \frac{a_I - b_I}{\max \{ c_I, a_I / \kappa \}} \, dt + \int_{t-u}^{t'} \left( \frac{b_I}{\max \{ d_I, b_I / \kappa \}} \right) \max \{ c_I - d_I, |a_I - b_I| / \kappa \} \, dt.
\]

By (23), \( b_I \leq k_3 y \). By (21), \( |a_I - b_I| \leq k_4 y \Delta_i \). By (25), \( |c_I - d_I| \leq k_3 \Delta_i \). By (23) and (24), \( \min \{ a_t, b_t \} \geq \frac{k_2 (D'')^2}{6y} \) and \( \min \{ c_t, d_t \} \geq k_2 \left( 1 - \frac{D''}{y} \right) \) so as shown above, both \( \max \{ c_I, a_I / \kappa \} \) and \( \max \{ d_I, b_I / \kappa \} \) are at least \( \kappa' > 0 \). Collecting these bounds, \( A_6 \leq \Delta_i \kappa_6 \left[ t' - u \right] \) where \( \kappa_6 = \frac{y}{\kappa} \left[ k_4 + \frac{k_3}{\kappa} \max \left\{ \frac{k_3}{y}, k_4 y \right\} \right] \in (0, \infty) \).

Summarizing our findings and since \( \lim_{t \to \infty} \Delta_i = 0 \) and \( t' \in [u, 1] \), for all \( \varepsilon > 0 \) there is an \( i^* < \infty \) such that if \( i > i^* \), then

\[
(1 - \delta) \left| D_{ua1k}^H (t') - D_{ua1k}^H (t') \right| \leq |a_1 - a_0| + \kappa' \varepsilon + \kappa_5 \left[ t' - u \right] \max_{t \in [u,t']} \left| D_{ua1k}^H (t) - D_{ua1k}^H (t) \right|
\]
where \( \kappa'' = \kappa + (\kappa_2 + \kappa_3 + \kappa_4 + \kappa_5) [1 - u] \). So for any \( t'' \in [u, t'] \),

\[
(1 - \delta) \left| D_{ua_{0k}}^{Hy_i} (t'') - D_{ua_{1k}}^{Hy} (t'') \right| \\
\leq |a_1 - a_0| + \kappa'' \varepsilon + \kappa_5 \left[ t'' - u \right] \max_{t \in [u, t']} \left| D_{ua_{0k}}^{Hy_i} (t) - D_{ua_{1k}}^{Hy} (t) \right|
\]

and therefore,

\[
(1 - \delta) \max_{t \in [u, t']} \left| D_{ua_{0k}}^{Hy_i} (t) - D_{ua_{1k}}^{Hy} (t) \right| \\
\leq |a_1 - a_0| + \kappa'' \varepsilon + \kappa_5 \left[ t' - u \right] \max_{t \in [u, t']} \left| D_{ua_{0k}}^{Hy_i} (t) - D_{ua_{1k}}^{Hy} (t) \right|
\]

Now for \( t' \in [u, u + b] \) where \( b = \frac{1 - \delta}{2\kappa_5} > 0 \), \( (1 - \delta) - \kappa_5 \left[ t' - u \right] \geq \frac{1 - \delta}{2} \), so

\[
\max_{t \in [u, t']} \left| D_{ua_{0k}}^{Hy_i} (t) - D_{ua_{1k}}^{Hy} (t) \right| \leq \frac{2}{1 - \delta} (|a_1 - a_0| + \kappa'' \varepsilon),
\]

whence \( \max_{t \in [u, u + b]} \left| D_{ua_{0k}}^{Hy_i} (t) - D_{ua_{1k}}^{Hy} (t) \right| \leq \frac{2}{1 - \delta} (|a_1 - a_0| + \kappa'' \varepsilon) \). In particular,

\[
\left| D_{ua_{0k}}^{Hy_i} (u + b) - D_{ua_{1k}}^{Hy} (u + b) \right| \leq \frac{2}{1 - \delta} (|a_1 - a_0| + \kappa'' \varepsilon).
\]

Let \( a_2 = D_{ua_{0k}}^{Hy_i} (u + b) \) and \( a_3 = D_{ua_{1k}}^{Hy} (u + b) \). Since \( D_{ua_{0k}}^{Hy_i} \) and \( D_{ua_{1k}}^{Hy} \) are decreasing functions, \( \max \{ a_2, a_3 \} < a \) so in the above reasoning we can use the same constant \( a \) and thus the same constant \( k_0 \) and thus the same \( \kappa'' \). Accordingly,

\[
\max_{t \in [u + b, u + 2b]} \left| D_{ua_{0k}}^{Hy_i} (t) - D_{ua_{1k}}^{Hy} (t) \right| \leq \frac{2}{1 - \delta} (|a_3 - a_2| + \kappa'' \varepsilon)
\]

\[
\leq \frac{2}{1 - \delta} \left( \frac{2}{1 - \delta} \left( |a_1 - a_0| + \kappa'' \varepsilon \right) + \kappa'' \varepsilon \right)
\]

\[
= \left( \frac{2}{1 - \delta} \right)^2 |a_1 - a_0| + \left[ \frac{2}{1 - \delta} + \left( \frac{2}{1 - \delta} \right)^2 \right] \kappa'' \varepsilon.
\]

Iterating this reasoning \( n = \left\lceil \frac{1 - u}{b} \right\rceil \) (which does not depend on \( i \)) times, we obtain

\[
\max_{t \in [u, 1]} \left| D_{ua_{0k}}^{Hy_i} (t) - D_{ua_{1k}}^{Hy} (t) \right| \leq \left( \frac{2}{1 - \delta} \right)^n |a_1 - a_0| + \kappa'' \varepsilon \sum_{i=1}^{n} \left[ \left( \frac{2}{1 - \delta} \right)^l \right].
\]

Since the constants multiplying \( |a_1 - a_0| \) and \( \varepsilon \) are independent of \( i, t \in [u, 1], y \in (0, y], \) and \( H \in \mathcal{H} \), the result follows. Q.E.D. Claim 9
CLAIM 13 For all $u \in [0, 1]$, there exist unique solutions $D_{uy_\infty}^{Hy}$ and $D_{uy_\infty}^{Hy_\infty}$ to $CP_{uy_\infty}^{Hy}$ and $DP_{uy_\infty}^{Hy_\infty}$, respectively. They are decreasing in $t$ and, in the case of $D_{uy_\infty}^{Hy}$, continuous in $t \in [u, 1]$.

PROOF OF CLAIM 13. Define, for all $t \in [u, 1]$,

$$D_{u}^{Hy}(t) = \lim_{a \uparrow y} D_{uy_\infty}^{Hy}(t) = \lim_{a \uparrow y} D_{uy_\infty}^{Hy_\infty}(t),$$

(62)

and for all $t \in S_i^u$, $D_{u}^{Hy_\infty}(t) = \lim_{a \uparrow y} D_{uy_\infty}^{Hy_\infty}(t) = \lim_{a \uparrow y} D_{uy_\infty}^{Hy}(t)$. The four limits exist by part 2 of Claims 4 and 8, and are unique. Moreover, by part 2 of Claims 4 and 8, $D_{u}^{Hy}$ and $D_{u}^{Hy_\infty}$ exist and are unique. Moreover, by part 2 of Claims 4 and 8, $D_{u}^{Hy}$ and $D_{u}^{Hy_\infty}$ coincide with any solutions $D_{uy_\infty}^{Hy}$ and $D_{uy_\infty}^{Hy_\infty}$ to $CP_{uy_\infty}^{Hy}$ and $DP_{uy_\infty}^{Hy_\infty}$, respectively, if such solutions exist. We now show that such solutions do exist: that the functions $D_{u}^{Hy}$ and $D_{u}^{Hy_\infty}$ satisfy

$$D_{u}^{Hy}(u) = y \text{ and, for } t \in (u, 1), \quad \frac{dD_{u}^{Hy}(t)}{dt} = -\frac{1}{1-\delta} \varphi(D_{u}^{Hy}(t), t)$$

(63)

and

$$D_{u}^{Hy_\infty}(u) = y \text{ and, for } t \in S_i^u \setminus \{1\}, D_{u}^{Hy_\infty}(t + \Delta_t) = D_{u}^{Hy_\infty}(t + \Delta_t)$$

(64)

where $D_{u}^{Hy_\infty}(t + \Delta_t)$ is the (by Claim 7) unique solution $D^* \in (y\Delta_t, D_{u}^{Hy_\infty}(t))$ to

$$v^{Hy_\infty}(D^*, t + \Delta_t) - \delta v^{Hy_\infty}(D^*, t) = (1-\delta)v^{Hy_\infty}(D_{u}^{Hy_\infty}(t), t).$$

(65)

First, $D_{u}^{Hy}(u) = \lim_{a \uparrow y} D_{uy_\infty}^{Hy}(u) = \lim_{a \uparrow y} a = y$. Second, for $t \in (u, 1)$, we must show that $dD_{u}^{Hy}(t)/dt = -\frac{1}{1-\delta} \varphi(D_{u}^{Hy}(t), t)$. By (29) and (33), $\varphi(\zeta, t)$ is continuous in $t$ so, by (34), $-\frac{1}{1-\delta} \varphi(D_{u}^{Hy}(t), t)$ is the limit of the derivatives $dD_{uy_\infty}^{Hy}(t)/dt$ as $a$ goes to $y$. We now invoke the following well known result.\(^3\)

THEOREM 14 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $t \in [u, 1]$. Suppose that they converge uniformly to some function $f$. Assume also that the sequence of derivatives $\{f'_n\}_{n=1}^{\infty}$ converges, uniformly in $t \in [u, 1]$ to some continuous function. Then $f$ is differentiable and $\lim_{n \to \infty} f'_n(t) = f'(t)$ for all $t \in [u, 1]$.

Let $f_n = D_{u_\infty}^{Hy}$ where $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence that converges to $y$. By part 4 of Claim 4, $\{f_n\}_{n=1}^{\infty}$ converges to $f = D_{u}^{Hy}$, uniformly in $t \in [u, 1]$. Fix $w \in (u, 1)$. We will show that the sequence of derivatives $\{f'_n\}_{n=1}^{\infty}$ converges, uniformly in $t \in [w, 1]$, to $f'$. The

---

result then follows by taking \( w \to u \). By (27) and (33), letting \( b = \frac{1}{1 - \delta} \frac{k_2}{\delta k_3} \) and \( x = f_n(w) \),

\[
x = f_n(u) + \int_{t=u}^{w} f'_n(t) \, dt = a_n - \frac{1}{1 - \delta} \int_{t=u}^{w} \varphi(f_n(t), t) \, dt < a_n - \frac{b}{y} \int_{t=u}^{w} [f_n(t)]^2 \, dt
\]

so \( x \) lies below the higher root of \( bx^2 (w - u) + yx - ya_n = 0 \), which (since \( a_n \leq y \)) is at most \( -y + \sqrt{y^2 + 4b(w - u)x^2} \). If we set this equal to \( y(1 - c) \) and solve for \( c \), we obtain \( \frac{c}{(1 - c)^2} = b(w - u) \). Hence, for all \( t \in [w, 1] \),

\[
\frac{f_n(t)}{y} < \frac{f_n(w)}{y} < 1 - c
\]

(66)

where \( c \) is positive and independent of \( n \). Thus, for all \( t \in [w, 1] \) and all \( n \),

\[
|f'_n(t)| = \frac{1}{1 - \delta} \varphi(f_n(t), t) < \frac{1}{1 - \delta} \frac{k_3 y}{k_2 c}
\]

which is a finite constant that is independent of \( n \). Thus, the functions \( (f_n)^{\infty}_{n=1} \), as well as their limit \( f \), are Lipschitz on \( t \in [w, 1] \) with the same Lipschitz constant. As \( \varphi \) is continuous and \( f \) is Lipschitz continuous, the function

\[
\lim_{n \to \infty} f'_n(t) = - (1 - \delta)^{-1} \lim_{n \to \infty} \varphi(f_n(t), t)
\]

\[
= - (1 - \delta)^{-1} \varphi \left( \lim_{n \to \infty} f_n(t), t \right) = - (1 - \delta)^{-1} \varphi(f(t), t)
\]

is continuous in \( t \in [w, 1] \). Moreover, convergence of \( f'_n(t) \) is uniform since, for all \( n, n' \geq 1 \), by (29), (33), and (66), letting \( k_5 = \frac{k_1}{k_2 c} \left[ 1 + \frac{k_1 (1 - c)}{k_2 c} \right] \in (0, \infty) \),

\[
(1 - \delta) |f'_n(t) - f'_{n'}(t)| = |\varphi(f_n(t), t) - \varphi(f_{n'}(t), t)| \leq k_5 |f_n(t) - f_{n'}(t)|
\]

\[
= k_5 |D_{H^y u_{ad}}(u) - D_{H^y u_{ad}}(u)| \leq y - \min \{a_n, a_{n'}\}
\]

where the last inequality is from part 4 of Claim 4. Hence, \( (f'_n)_{n=1}^{\infty} \) is a uniform (in \( t \in [w, 1] \)) Cauchy sequence and thus converges uniformly. This proves existence and uniqueness.

By part 1 of claim 8, \( D_{H^y u_{ad}} \) is decreasing in \( t \). It remains to show that \( D_{H^y u_{ad}} \) is decreasing and continuous in \( t \in [u, 1] \). By L-H and (33), \( \varphi(D, t) \) is finite and negative for all \( D < y \). Thus, by (63), \( D_{H^y u} \) is decreasing and continuous in \( t \in (u, 1] \). By (63), for continuity at \( t = u \) we must show that \( \lim_{t \to u} D_{H^y u} = y \). By (62), \( D_{H^y u}(t) = \lim_{a \to y} D_{H^y a_{ad}}(t) = \lim_{a \to y} D_{H^y u_{ad}}(t) \). By \( C_{H^y u_{ad}} \), \( D_{H^y u}(u) = a \). By the triangle inequality, for any \( a \in (0, y) \),

\[
|D_{H^y u}(t) - y| = |D_{H^y u}(t) - D_{H^y a_{ad}}(t)| + |D_{H^y a_{ad}}(t) - a| + |a - y|.
\]
By parts 2 and 4 of Claim 4, 
\[ |D^H_y(t) - D^H_{au\infty}(t)| \leq |D^H_{u\infty}(t) - D^H_{au\infty}(t)| \leq |y - a|. \]
Finally, 
\[ |D^H_{u\infty}(t) - a| = |D^H_{u\infty}(t) - D^H_{au\infty}(u)| \leq \frac{ky\sqrt{y - a}}{(1 - \delta)k^2} |t - u| \] by part 3 of Claim 4. Collecting terms and letting \( a = y - \sqrt{t - u} \leq y \), 
\[ |D^H_{u}(t) - y| \leq \left[ 2 + \frac{ky^2}{(1 - \delta)k^2} \right] \sqrt{t - u} \] which goes to zero as \( t \downarrow u \). Q.E.D. Claim 13

By part 2 of Claims 4 and 8, for all \( a \in (0,y) \),
\[ D^H_{u\infty}(t) - D^H_{u\infty}(t) \leq D^H_{u\infty}(t) - D^H_{u\infty}(t) \leq D^H_{u\infty}(t) - D^H_{u\infty}(t) \]
and thus, by the triangle inequality,
\[ |D^H_{u\infty}(t) - D^H_{u\infty}(t)| \leq \max \left\{ \left| D^H_{u\infty}(t) - D^H_{u\infty}(t) \right|, \left| D^H_{u\infty}(t) - D^H_{u\infty}(t) \right|, \left| D^H_{u\infty}(t) - D^H_{u\infty}(t) \right| \right\}. \] (67)

Fix \( \varepsilon > 0 \). By Claim 9, there is an \( i^* \), independent of \( y, H, \) and \( t \), such that for \( i > i^* \),
\[ |D^H_{u\infty}(t) - D^H_{u\infty}(t)| \text{ and } |D^H_{u\infty}(t) - D^H_{u\infty}(t)| \text{ are each less than } \varepsilon/2. \]
By part 4 of Claim 4, for all \( a \in[y - \varepsilon/2,y) \), 
\[ |D^H_{u\infty}(t) - D^H_{u\infty}(t)| \leq \varepsilon/2. \] But (67) holds for all \( a \in (0,y) \), so it holds in particular for \( a \in [y - \varepsilon/2,y) \). Thus, for all \( \varepsilon > 0 \) there is an \( i^* \), independent of \( y, H, \) and \( t \), such that for \( i > i^* \), 
\[ |D^H_{u\infty}(t) - D^H_{u\infty}(t)| \leq \varepsilon. \]
We now set \( u = 0 \) to obtain the desired result. Together with Claim 13, this proves that the unique solution \( D^H_{u\infty} \) to \( D^H_{u\infty} \)
converges to the unique solution \( D^H_{u\infty} \) to \( C^H_{u\infty} \) as \( i \to \infty \), uniformly in \( H \in H', y \in [0,y] \), and \( t \in [0,1] \).

We next prove that \( p^H \) and thus \( \pi^H \) is both continuous and decreasing in the signal \( t \). First, \( p^H(t) = v^H(\mathcal{D}^H_{u\infty}(t), t) \). By (10) and L-H, \( v^H \) is continuous in both arguments. Since \( D^H_{u\infty} \) is also continuous in \( t \), so is \( p^H \). By (12),
\[ \frac{d}{dt} [v^H(D^H(t), t)] = v^H(D^H(t), t) \frac{dD^H}{dt} + v^H(D^H(t), t) = \delta v^H(D^H(t), t) \frac{dD^H}{dt} \]
which is negative by (12), (23), and (24). Hence, \( p^H \) is decreasing in \( t \).

We now turn to the convergence of \( p^H_{yi} \) to \( p^H \) and thus of \( \pi^H_{yi} \) to \( \pi^H \). For any \( t \in [0,1] \), recall that \( \tau^i_t = \Delta_i |t/\Delta_i| \). As \( D^H_{yi}(t) = D^H_{yi}(\tau^i_t) \),
\[ \left| p^H_{yi}(t) - p^H(t) \right| = \left| v^H_{yi}(D^H_{yi}(t), \tau^i_t) - v^H(D^H_{yi}(t), \tau^i_t) \right| \leq A_1' + A_2' + A_3' \]
where
\[ A_1' = \left| v^H_{yi}(D^H_{yi}(t), \tau^i_t) - v^H(D^H_{yi}(t), \tau^i_t) \right|, \]
\[ A_2' = \left| v^H(D^H(t), \tau^i_t) - v^H(D^H(t), \tau^i_t) \right|, \text{ and} \]
\[ A_3' = \left| v^H(D^H(t), \tau^i_t) - v^H(D^H(t), \tau^i_t) \right|. \]
By (10), (5), and since \( H(0|\tau_i) \) is zero,

\[
\frac{A'_1}{y} \leq \int_{z=0}^{D^{Hyi}(t)/y} \left| H(z|\tau_i) - H \left( \Delta' \left| \frac{z}{\Delta'} \right| \tau_i \right) \right| dz \leq k_3 \Delta'_i,
\]

where the second inequality is by L-\( H \). By (10), \( A'_2 \leq 2 |D^{Hyi}(t) - D^{Hy}(t)| \). By (10) and L-\( H \),

\[
A'_3 \leq y \int_{z=0}^{D^{Hy}(t)/y} \left| H(z|t) - H(z|\tau_i) \right| dz \leq yk_3 |t - \tau_i| \leq yk_3 \Delta_i.
\]

Hence, by the prior result, for all \( \epsilon > 0 \) there is an \( i^* < \infty \) such that if \( i > i^* \),

\[
\left| p^{Hyi}(t) - p(t) \right| \leq yk_3 \Delta'_i + 2 |D^{Hyi}(t) - D^{Hy}(t)| + yk_3 \Delta_i < \epsilon,
\]

for all \( t \in [0, 1], y \in (0, \bar{y}] \), and \( H \in \mathcal{H} \), as claimed. Hence, \( p^{Hyi} \) converges uniformly to \( p^{Hy} \).

We now show convergence of \( \Pi^{Hyi} \) to \( \Pi^{Hy} \):

\[
\left| \Pi^{Hyi}(t) - \Pi^{Hy}(t) \right| \leq \left| E'[yz|t] - E^\infty[yz|t] \right| + \left| \pi^{Hyi}(t) - \pi^{Hy}(t) \right|
\]

and for \( z \in [(c - 1) \Delta_i, c \Delta_i], c = \left\lfloor \frac{z}{\Delta_i} \right\rfloor + 1, \) so

\[
\left| E'[yz|t] - E^\infty[yz|t] \right| = \sum_{c=1}^{1/\Delta_i} yc\Delta'_i \left[ H(c\Delta_i|t) - H((c - 1)\Delta_i|t) \right] - \int_{z=0}^{1} yzdH(z|t)
\]

\[
= y \sum_{c=1}^{1/\Delta_i} \int_{z=(c-1)\Delta_i}^{c\Delta'_i} \left( \left\lfloor \frac{z}{\Delta_i} \right\rfloor + 1 \right) \Delta'_i dH(z|t) - \int_{z=0}^{1} yzdH(z|t)
\]

\[
\leq y\Delta'_i \int_{z=0}^{1} \left( \left\lfloor \frac{z}{\Delta_i} \right\rfloor + 1 - \frac{z}{\Delta_i} \right) dH(z|t) \leq y\Delta'_i.
\]

Thus, \( E'[yz|t] \) converges uniformly to \( E^\infty[yz|t] \) and hence \( \Pi^{Hyi} \) converges uniformly to \( \Pi^{Hy} \).

As there is no mention of \( G \) in the statement of the problems DP^{Hyi} and CP^{Hy}, any solutions \( D^{Hyi} \) and \( D^{Hy} \) to these problems for one distribution \( G \) are also solutions for any other distribution \( \hat{G} \) that satisfies our assumptions.\(^4\) Since, moreover, the solutions \( D^{Hyi} \) and \( D^{Hy} \) are unique by parts 1 and 2 of the theorem, they must be independent of \( G \). Hence, convergence of \( D^{Hyi}, p^{Hyi}, \pi^{Hyi}, \) and \( \Pi^{Hyi} \) is uniform in \( G \) as well.

We now show that \( E\pi^{GHyi} \) converges uniformly to \( E\pi^{GHy} \). Since \( G \) has no atoms, \( G(0) = 0 \); hence,

\[
E\pi^{GHyi} = \sum_{c=1}^{1/\Delta_i} \pi^{Hyi}(c\Delta_i) \left[ G(c\Delta_i) - G((c - 1)\Delta_i) \right] = \int_{t=0}^{1} \pi^{Hyi}(\tau_i + \Delta_i) dG(t)
\]

\(^4\) In particular, it must put positive weight on signals that are arbitrarily close to zero.
and thus, \( E\pi^{GHy} - E\pi^{GHy} = A''_1 + A''_2 - A''_3 \) where
\[
A''_1 = \int_0^1 \left[ \pi^{Hy}(\tau_i^t + \Delta_i) - \pi^{Hy}(\tau_i^t + \Delta_i) \right] dG(t),
\]
\[
A''_2 = \int_{t=0}^1 \pi^{Hy}(\tau_i^t + \Delta_i) dG(t), \quad \text{and} \quad A''_3 = \int_{t=0}^1 \pi^{Hy}(t) dG(t). \]

By part 3 of PROPOSITION 16, for all \( \varepsilon > 0 \) there is an \( i^* \) such that for all models \( i > i^* \), parameters \( y \) in \( (0, \bar{y}) \), and conditional distribution functions \( H \) in \( \mathcal{H} \),
\[
|A''_1| \leq \int_{t=0}^1 |\pi^{Hy}(\tau_i^t + \Delta_i) - \pi^{Hy}(\tau_i^t + \Delta_i)| dG(t) < \int_{t=0}^1 \frac{\varepsilon}{2} dG(t) = \frac{\varepsilon}{2}.
\]

Moreover, since \( \pi^{Hy} \) is a nonincreasing function of \( t \), it follows that
\[
\pi^{Hy}(\tau_i^t + \Delta_i) \leq \pi^{Hy}(t) \leq \pi^{Hy}(\tau_i^t),
\]
so \( A''_2 \leq A''_3 \leq A''_4 \) where
\[
A''_4 = \int_{t=0}^1 \pi^{Hy}(\tau_i^t) dG(t) = \int_{t=0}^{\Delta_i} \pi^{Hy}(\tau_i^t) dG(t) + \int_{t=\Delta_i}^1 \pi^{Hy}(\tau_i^t) dG(t).
\]

But \( \tau_{i+\Delta_i} = \tau_i + \Delta_i \). Hence, letting \( t' = t - \Delta_i \) and renaming \( t' \) to \( t \),
\[
\int_{t=\Delta_i}^1 \pi^{Hy}(\tau_i^t) dG(t) = \int_{t=0}^{1-\Delta_i} \pi^{Hy}(\tau_i^t + \Delta_i) dG(t + \Delta_i).
\]

Now let \( i^* \) also be large enough that \( \Delta_i < \frac{\varepsilon}{2(1-\delta)} \frac{3}{n(k_0+k_1)} \). Then by Lipschitz-G and since, by (9), \( \pi^{Hy} \in [0, (1 - \delta) y] \), for all \( i > i^* \),
\[
|A''_2 - A''_3| \leq |A''_2 - A''_4|
= \left| \int_{t=0}^1 \pi^{Hy}(\tau_i^t + \Delta_i) dG(t) - \int_{t=0}^{\Delta_i} \pi^{Hy}(\tau_i^t) dG(t) - \int_{t=0}^{1-\Delta_i} \pi^{Hy}(\tau_i^t + \Delta_i) dG(t + \Delta_i) \right|
\leq \int_{t=0}^{\Delta_i} \pi^{Hy}(\tau_i^t) dG(t) + \int_{t=1-\Delta_i}^1 \pi^{Hy}(\tau_i^t + \Delta_i) dG(t)
+ \int_{t=0}^{1-\Delta_i} \pi^{Hy}(\tau_i^t + \Delta_i) |\psi(t) - \psi(t + \Delta_i)| dt
\leq (1 - \delta) y k_0 \Delta_i + (1 - \delta) y k_0 \Delta_i + (1 - \delta) y k_1 \Delta_i < \frac{\varepsilon}{2}.
\]

For all \( i > i^* \), \( G \) in \( \mathcal{G} \), \( H \) in \( \mathcal{H} \), and \( y \in (0, \bar{y}) \), \( |E\pi^{GHy} - E\pi^{GHy}| < \varepsilon \) as claimed.

We now show that \( E^i[yz] \) converges uniformly to \( E^\infty[yz] \). Combined with the preceding result, this will show that \( E\Pi^{GHy} \) converges uniformly to \( E\Pi^{GHy} \). First,
\[
E^i[yz] = \sum_{c=1}^{1/\Delta_i} E^i[yz|t = c\Delta_i] \left[ G(c\Delta_i) - G((c - 1) \Delta_i) \right] = \int_{t=0}^1 E^i[yz|\tau_i^t + \Delta_i] dG(t)
\]
so by the triangle inequality, $|E^i [yz] - E^\infty [yz]| \leq A''_1 + A''_2$ where

$$A''_1 = \int_{t=0}^{1} |E^i [yz| \tau^i + \Delta_i] - E^\infty [yz| \tau^i + \Delta_i]| dG(t)$$

and $A''_2 = \int_{t=0}^{1} |E^\infty [yz| \tau^i + \Delta_i] - E^\infty [yz| t]| dG(t)$. By (68), $A''_1 \leq y\Delta_i$. Integrating by parts, $E^\infty [z| t] = 1 - \int_{z=0}^{1} H(z| t) dz$. Thus, by L-H,

$$|E^\infty [yz| t = \tau^i + \Delta_i] - E^\infty [yz| t]| \leq y \int_{z=0}^{1} |H(z| \tau^i + \Delta_i) - H(z| t)| dz$$

$$\leq yk_3 \int_{z=0}^{1} |\tau^i + \Delta_i - t| dz \leq yk_3 \Delta_i,$$

so $A''_2 \leq yk_3 \Delta_i$. Hence, $E^i [yz]$ converges uniformly to $E^\infty [yz]$ as claimed.

We now turn to part 4. Using (13), equation (12) for $y = 1$ can be rewritten as

$$\frac{dD^{H1}}{dt} = \frac{1}{1 - \delta} \int_{z=0}^{1} \frac{\partial H(z| t)}{\partial t} dz.$$

Hence, for any solution $D^{Hy}$ to $Cp^{Hy}$, $D^{H1} = y^{-1} D^{Hy}$ is a solution to $Cp^{H1}$. But both solutions $D^{Hy}$ and $D^{H1}$ are unique by part 1 of PROPOSITION 16. Hence, $D^{Hy}$ must equal $yD^{H1}$. Accordingly, the expected payout $p^{Hy}(t) = v^{Hy}(D^{Hy}(t), t)$ must equal $v^{H1}(yD^{H1}(t), t)$. But by (10), for any $D$, $v^{Hy}(D, t)$ equals $yv^{H1} \left( \frac{D}{y}, t \right)$, so $p^{Hy}(t) = yp^{H1}(t)$, $\pi^{Hy}(t) = y\pi^{H1}(t)$, and $E\pi^{GHy} = yE\pi^{GH1}$ as claimed. These properties hold for $\Pi^{Hy}$ and $E\Pi^{GHy}$ as well since $E^\infty [yz| t]$ is homogeneous of degree 1 in $y$.

For model $i$, by (8),

$$v^{Hyi}(D, t) = yv^{H1i} \left( \frac{D}{y}, t \right),$$

so if $D^{H1i}$ solves $Dp^{H1i}$, then $y^{-1}D^{H1i}$ solves $Dp^{Hyi}$. But both solutions $D^{Hyi}$ and $D^{H1i}$ are unique by part 2 of PROPOSITION 16. Hence, $D^{Hyi}$ must equal $yD^{H1i}$. Thus, by (69), the expected payout $p^{Hyi}(t) = v^{Hyi}(D^{Hyi}(t), t)$ must equal $yv^{H1i}(D^{H1i}(t), t) = yp^{H1i}(t)$ and so $\pi^{Hyi}(t)$ equals $y\pi^{H1i}(t)$ as claimed. These properties hold for $\Pi^{Hyi}$ and $E\Pi^{GHyi}$ as well since $E^i [yz| t]$ is homogeneous of degree 1 in $y$. Q.E.D. PROPOSITION 16