

# Adaptive Expectations and Stock Market Crashes with an Infinite Horizon

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# 1 Introduction

This appendix studies a stationary, infinite horizon extension of the model presented in Frankel [6]. This extension sheds light on two important issues: whether selling pressure by rational traders can cause a crash and whether naive traders' beliefs are confirmed in equilibrium.

## 1.1 Can the Rational Traders be Net Sellers?

In the original model, rational traders are net buyers of stock on the day of the crash. They collectively cause a crash by making limit orders that correspond to a lower point on their demand curves: they bid a lower price, but for a larger quantity of stock. In this appendix we show that with minor modelling changes, they can actually lower their entire demand curves for stock and become net sellers of stock on the crash day. That is, rational traders cause the crash by choosing simultaneously to dump their shares.

This occurs because we now permit trading after the crash, unlike in the original model. On the day after the crash, naive traders sell their shares en masse. This causes prices to continue to fall, though by far less than on the crash day. Thus, the market actually reaches its lowest point the day *after* the crash. Anticipating this, rational traders have an incentive to sell on the crash day.

For rational traders to sell on the crash day, naive traders must buy.<sup>1</sup> Why aren't they scared away by the falling prices? The answer is that we now assume that naive traders react to volatility with a lag: the crash raises their estimate of future volatility only on the following day. This means that, contrary to the original model, naive traders' stock demand curve on each day is conventionally downwards sloping. Crashes now occur because a large price change on one day will lower the naive traders' demand curve on the *following* day. Lags in updating are consistent with evidence from Shiller [17, p. 388] that the average investor heard about the crash of October 19, 1987 at 1:56 Eastern Time, 2 hours before the market closed and, based on intraday price charts in

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<sup>1</sup>An alternative that might also let rational traders be net sellers is to assume a third group of traders who pursue a naive contrarian strategy, as in Gennotte and Leland [7]. We do not pursue this here.

Kleidon and Whaley [10], well after the major portion of the decline.

With these changes, naive traders are now net buyers on the crash day. On the day after the crash, when their volatility estimate rises, they become net sellers. Real-world examples of such naive buying in a crash might include limit orders placed before the crash, requests to buy mutual fund shares made the day before but too late to be executed, etc.

## 1.2 Are Naive Traders' Beliefs Confirmed?

The infinite horizon model also lets us check the accuracy of naive traders' beliefs. Naive traders sell shares when prices are at their lowest and gradually buy them back as prices rise. In contrast, rational traders buy when prices are at their low point and gradually reduce their stock positions as prices rise. By selling at the lowest price, naive traders fare poorly relative to rational traders. They behave this way due to two errors.

The first is that naive traders do not take into account that, due to risk aversion, expected returns are higher when volatility is high. Instead, they continue to believe after the crash that prices follow a random walk - neither rising nor falling on average. However, we show that crashes can occur even if naive traders take into account the empirical relation between volatility and expected returns. The reason is that, empirically, this relation is not very strong.

Naive traders' second mistake is that they overestimate postcrash volatility: they overestimate the persistence of the volatility shock observed on the day of the crash. This error is essential for our results: if traders correctly predict the variance of future returns, crashes cannot occur. However, it is plausible that naive traders did commit this error after the 1987 crash. First, we show below that traders who used a GARCH(1,1) model to predict volatility following the 1987 crash would have made this mistake. This is because volatility was much more persistent in the preceding decades than during the crash itself. Second, there is reason to believe that traders did sharply raise their estimate of future risk after the crash of 1987. Jackwerth and Rubinstein [9] find evidence for a large and enduring increase in traders' subjective probability of large price declines in the aftermath of the crash. Whether they were basing this on the

persistence of volatility or simply learning that large corrections are possible, traders would have demanded a higher risk premium to own stocks in response.

## 2 Adaptive Volatility Predictions

In contrast with the original model (Frankel [6]), here we assume that naive traders realize the implications of the current price change for future volatility only in the next period. Hence, a large price change in the current period does not lower naive traders' demand in that period, but only in the next period. Let  $p_t$  be the stock price in period  $t$ . The naive traders' period- $t$  estimate  $V_t$  of the variance of  $p_{t+1}$  is a weighted sum of the predicted and realized variances in the prior period:

$$V_t = \alpha(p_{t-1} - p_{t-2})^2 + \beta V_{t-1} \quad (1)$$

Thus, naive traders' prediction of the variance of the price in period  $t + 1$  depends on the realized volatility in period  $t - 1$  rather than in period  $t$  as in the original model.

## 3 Infinite Horizon Model

Time is indexed by  $t = 0, 1, 2, \dots$ . In each period there are young and old agents, each present in measure one. Agents live two periods.<sup>2</sup> Each generation consists of a measure  $\mu$  of rational traders and  $1 - \mu$  of naive traders. Agents work only when young, earning a fixed amount  $L$  of labor income, and consume only when old. There are two assets: one ("stocks"), in fixed unit supply, pays an i.i.d. dividend of  $\delta_t \sim N(\bar{\delta}, \sigma^2)$  per share in each period. The other ("bonds"), in infinitely elastic supply at the price 1, pays interest of  $r$  each period.

For tractability, we assume no margin constraints. This means that crashes require a large proportion of naive traders. This is consistent with evidence that the 1987 crash was driven by a few large sellers. The Brady Commission [15, p. 36] found that the top 15 sellers in the stock market accounted for "about 20 percent of total sales" on

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<sup>2</sup>An equivalent assumption is that agents are infinitely lived but myopic.

October 19, while the top 10 sellers in the futures market constituted “about 50 percent of non-market maker total volume.”<sup>3</sup>

In each period  $t$ , a common signal or “sunspot”,  $\theta_t$ , is first observed by all young rational traders. Following this, young traders of both types submit their demand functions. Old agents then receive a dividend  $\delta_t$  and transfer their shares at the market-clearing price  $p_t$ , which is determined by the condition that the demand of young traders equal the fixed unit supply of old agents.

If a young agent buys  $x_t$  shares of stock in period  $t$ , costing her  $p_t x_t$ , her wealth in period  $t + 1$  is

$$W_{t+1} = x_t (p_{t+1} + \delta_{t+1}) + (L - p_t x_t)(1 + r)$$

Young agents maximize expected utility  $EU(W_{t+1}) = E[-e^{-\lambda W_{t+1}}]$ , where  $\lambda$  is the coefficient of absolute risk aversion.

By Lemma 1 in Frankel [6], if  $p_{t+1}$  is normally distributed, a rational agent’s demand for stocks,  $x_t^R$ , equals the expected difference between stock and bond returns,  $E(p_{t+1} - p_t) + \bar{\delta} - r p_t$ , divided by  $\lambda$  times the variance of the total stock return  $p_{t+1} + \delta_{t+1}$ :

$$x_t^R = \frac{E(p_{t+1} - p_t) + \bar{\delta} - r p_t}{\lambda (\text{Var}(p_{t+1}) + \sigma^2)} \quad (2)$$

where the expectation and variance are conditioned on all information available at time  $t$  (including  $p_t$ ). In the simulations, we use this approximation to rational traders’ demands. As explained in section 5, this has a negligible effect on our results, since nearly all the risk for the rational agents comes from the normally distributed dividend shocks.

The naive traders know that  $\delta_t \sim N(\bar{\delta}, \sigma^2)$  and believe that the ex-dividend return  $r_{t+1} = \frac{p_{t+1} - p_t}{p_t}$  is normally distributed<sup>4</sup> with mean  $R_t$  and variance  $V_t$ .<sup>5</sup> The naive

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<sup>3</sup>These quotes of the Brady Commission report are from Shiller [17, p. 372].

<sup>4</sup>Some functional form for the distribution of the capital gains return must be assumed for the naive traders since they estimate only its mean and variance. We assume a normal distribution because it gives rise to the tractable form for the demand function given in (4).

<sup>5</sup>This represents a slight departure from the base model, where investors predict the volatility of

traders' estimate  $V_t$  of the next period's volatility  $r_{t+1}^2$  is a weighted sum of the past realized and expected volatilities:

$$V_t = \alpha r_{t-1}^2 + \beta V_{t-1} \quad (3)$$

Substituting these into (2), we obtain the demand  $x_t^N$  of a naive trader:<sup>6</sup>

$$x_t^N = \frac{R_t p_t + \bar{\delta} - r p_t}{\lambda (V_t p_t^2 + \sigma^2)} \quad (4)$$

With the exception of section 7.1, we will assume that naive traders have random walk beliefs ( $R_t \equiv 0$ ). The condition for market clearing in each period  $t$  is

$$\mu x_t^R + (1 - \mu) x_t^N = 1 \quad (5)$$

where  $x_t^R$  and  $x_t^N$  are given above.

To rule out bubbles, we assume that an agent who wants to invest  $x$  in stocks must have at least  $m x$  in labor income. This implies that the stock price can never exceed  $L/m$ , where  $L$  is the aggregate amount of labor income. We assume that  $m$  is small enough that it never constrains agents' stock demand in equilibrium.

We define an equilibrium as follows. Consider a pricing function  $f(t, \theta_t, h_t) \rightarrow p_t$ , where  $\theta_t$  is the signal observed by young rational agents in period  $t$  and  $h_t$  is the history of prices and signals through time  $t-1$ ,  $(p_0, \theta_0, \dots, p_{t-1}, \theta_{t-1})$ . This pricing function is an *equilibrium* if  $p_t = f(t, \theta_t, h_t)$  satisfies (5), where  $E[p_{t+1}] = E[f(t+1, \theta_{t+1}, h_{t+1}) | \theta_t, h_t, p_t]$  and where  $\text{Var}(p_{t+1})$  is defined analogously.

## 4 Theoretical Results

The following two theoretical results assume that naive traders have random walk beliefs. The results do not rely on the normal approximation to rational agents' demands. We

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$p_{t+1} - p_t$ . This is necessary to avoid level effects in our calibration of  $\alpha$  and  $\beta$ , which is based on GARCH estimates from returns over several decades.

<sup>6</sup>We use this formulation to make naive traders' beliefs match the GARCH specification, which predicts the volatility of capital gain *returns* rather than the volatility of the capital gain  $p_{t+1} - p_t$  itself. The results are qualitatively the same with either formulation.

first show that there is only one constant-price equilibrium:

**Proposition 1** *There is only one constant-price equilibrium. In this equilibrium, the price always equals  $\bar{p} = \frac{1}{r} [\bar{\delta} - \lambda\sigma^2]$ .*

**Proof.** *If the price is constant,  $V_t \equiv 0$  and  $\text{Var}_t(p_{t+1}) \equiv 0$ , so the normal approximation of rational trader demand is exact. By substituting (2) and (4) into (5) and setting  $V_t = R_t = E[p_{t+1} - p_t] = \text{Var}(p_{t+1}) = 0$ , one finds that  $\bar{p}$  must equal  $\frac{1}{r} [\bar{\delta} - \lambda\sigma^2]$ .*

We call  $\bar{p}$  the *variance-free price*. Proposition 2 shows that under random walk beliefs ( $R_t \equiv 0$ ), even if the price is not a constant, it can never exceed the variance-free price. This places a tight limit on the size of any frenzy: it cannot exceed the difference between the variance-free price and the precrash price (which is set to 0.5% in the simulations).

**Proposition 2** *In any equilibrium, the stock price can never exceed  $\bar{p} = \frac{1}{r} [\bar{\delta} - \lambda\sigma^2]$ .*

**Proof.** *Appendix A.*

The intuition is that since there can be no bubbles, there must be a maximum price  $p^{\max}$  that can ever be attained in equilibrium. When the price is  $p^{\max}$ , rational traders must expect the next period's return to be zero or negative. Naive traders, by assumption, expect it to be zero. Agents' expected returns in the constant price equilibrium are at least as optimistic as this; furthermore, agents in the constant-price equilibrium expect zero volatility. So if the price is  $p^{\max}$ , stocks cannot offer a more attractive return distribution to either type of agent than in the constant-price equilibrium. But then no agent will ever be willing to pay more than the price in that equilibrium, which is  $\bar{p}$ ; hence,  $p^{\max} \leq \bar{p}$ .

## 5 Simulations

In the simulations, the length of a period is one day. This is also the horizon of an investor. However, this is not essential for crashes to occur. Other results (available from the author) show that crashes can occur if the period length is as long as 4 months.

This is because volatility innovations are empirically very persistent: their half-life is estimated to be on the order of 6 months to one year (see, e.g., Campbell and Hentschel [2] and Chou [3]).

One aim of the simulations is to determine the relative importance of frenzies vs. crashes. We thus focus on equilibria in which there is initially a constant probability  $q$  of a crash, in which prices fall by a proportion  $c \in [0, 1]$ , and the *same* constant probability  $q$  of a frenzy, in which prices rise by a proportion  $c' \in [0, 1]$ .<sup>7</sup> Crashes are more important than frenzies insofar as  $c > c'$ .<sup>8</sup>

We assume that after a crash or frenzy occurs, the signal  $\theta$  in every subsequent period is ignored. Thus, a crash is a one-time event. This has the advantage of giving a unique price path after a crash occurs. Since we study equilibria in which the per-period crash probability is very low (about one in 10,000), this assumption has a negligible effect on prices.

The parameters are calibrated as follows. Let  $T$  equal the length of one period as a fraction of a year. We set  $T = 1/250$ , where 250 is the approximate number of trading days in a typical year. We normalized the expected annual dividend to 1, so the expected dividend in one day,  $\bar{\delta}$ , equals  $T$ . We take the annual interest rate to be 5%, so  $r = e^{.05T} - 1$ . Results are virtually unchanged if we use the 3-month t-bill rate for October 1987, which was 6.72%, or the geometric average 3-month t-bill rates for the period 1/31-10/87, which was 4.09%.

In our simulations the parameter  $\sigma^2$  proxies for all of the noncrash risk that accompanies investment in the stock market. This includes not only dividend risk but also

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<sup>7</sup>This can be captured in the model by the existence of two sets  $S_C$  and  $S_F$  of possible values of the signal  $\theta_t$ , each occurring with probability  $q$ , such that there is a crash if  $\theta_t \in S_C$  and a frenzy if  $\theta_t \in S_F$ .

<sup>8</sup>Another potential approach might be to assume that frenzies and crashes are of the same size but arrive at different frequencies. However, this approach does not work, since crashes and frenzies must generally be of specific sizes to be consistent with equilibrium. For example, if naive traders believe that prices follow a random walk ( $R_t \equiv 0$ ), a 20% crash can be sustained for some proportion  $\mu$  of rational traders. But by Proposition 2, the price can never exceed  $\bar{p}$ , so frenzies cannot exceed the difference between the variance-free price  $\bar{p}$  and the precrash price (which is set to 1% in the simulations).

risk due to noise trading, earnings news, release of government reports, etc.<sup>9</sup> We thus calibrate  $\sigma^2$  so that it captures the empirical volatility in the stock market. We use the daily closing S&P Composite Index from January 3, 1928 to October 16, 1987, the last trading day before the 1987 crash. The daily noncrash volatility in the index over this period is  $V^{\text{nc}} = 0.00011638$ .<sup>10</sup> In computing this we first remove any 1-day returns below -5% to obtain the noncrash risk present in the market. The computation of  $\sigma^2$  from  $V^{\text{nc}}$  is straightforward (Appendix C).

Since traders have constant absolute risk aversion, the coefficient of relative risk aversion is not a useful metric. We thus calibrate  $\lambda$ , the coefficient of absolute risk aversion, so that the precrash equity premium equals its historic average of 6% (Kocherlakota [11]). We verify in section 6 that crashes can occur for much lower levels of  $\lambda$  as well. This is important since the historic equity premium may entail implausibly high levels of risk aversion (Mehra and Prescott [13]).

We use the approximation to rational agents' demand that assumes returns are normally distributed (equation (2)). As in Frankel [6], this is true in the limit as the chance of a crash shrinks to zero. In our simulations we assume that the chance of a crash is only one in 10,000, so the effect of this approximation is negligible.

The naive traders predict volatility according to equation (1). To estimate the parameters  $\alpha$  and  $\beta$ , we fit Bollerslev's GARCH model [1] to daily closing S&P Composite Index levels from 1/3/28 to 10/16/87.<sup>11</sup> The specification we fit is as follows:

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<sup>9</sup>In addition, our dividends are i.i.d. while actual dividends are positively correlated over time, so even if we were concerned only with modelling dividend risk,  $\sigma^2$  would have to be higher than the empirical dividend volatility in order to capture the price risk resulting from dividend shocks.

<sup>10</sup>The day- $t$  return is defined conventionally, as  $\frac{p_t - p_{t-1}}{p_{t-1}}$ .

<sup>11</sup>In GARCH, future volatility depends on negative and positive returns symmetrically. Empirical evidence suggests that volatility rises more following market declines. The Exponential GARCH model of Nelson [14] captures this by weighting negative returns more in the volatility estimate. We do not study Exponential GARCH beliefs since they would only increase the potential for crashes and the tendency for skewed returns.

$$\begin{aligned}
r_{t+1} &= a + \varepsilon_{t+1} \\
\varepsilon_{t+1} &\sim N(0, V_t) \\
V_t &= \alpha_0 + \alpha \varepsilon_{t-1}^2 + \beta V_{t-1}
\end{aligned}
\tag{6}$$

where  $r_{t+1} = \frac{p_{t+1} - p_t}{p_t}$  and  $V_t$  is the conditional variance of  $r_{t+1}$ .

Variable	Parameter	t Value
$a$	0.000482	9.24
$\alpha_0$	$1.0154 \times 10^{-6}$	18.77
$\alpha$	0.0903	34.53
$\beta$	0.9034	343.67

Table 1: Parameter estimates from the GARCH model.

The parameter estimates are given in Table 1. In the simulations, we set  $a$  to zero since the theoretical model is stationary. We also set  $\alpha_0$  to zero since noncrash price risk in the model is already proxied by  $\sigma^2$ .

Since the proportion  $\mu$  of rational traders is unobservable, our approach is to specify an empirically relevant crash size of 20% and then let  $\mu$  be that proportion for which such a crash can be sustained.<sup>12</sup> The estimated  $\mu$  is always less than 5%. This is consistent with the evidence, cited above, that the 1987 crash was driven by a few large sellers.

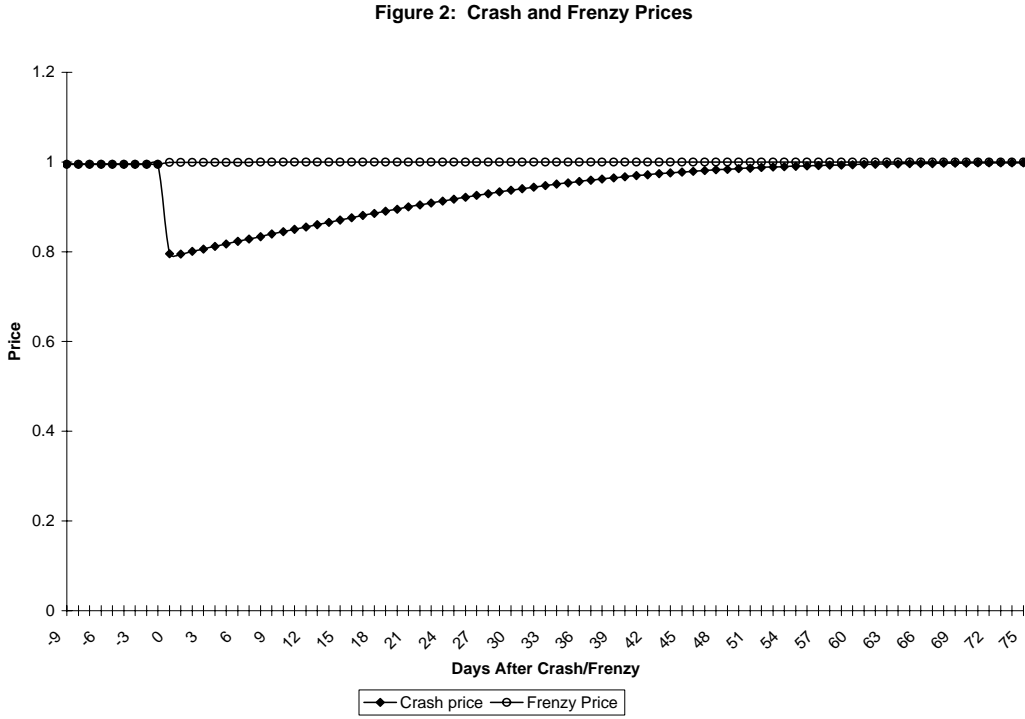
The equilibrium is simulated as follows. First we fix the crash size  $c$  at 20% and the proportion (taken to be 0.5%) by which the precrash price is below the variance-free price. We then find the proportion  $\mu$  for which after a 20% crash, the stock price converges to a finite limit. Based on our calculations, there is a unique such  $\mu$ . Armed with this  $\mu$ , we compute the frenzy size  $c'$  that can be supported in equilibrium (i.e., for which prices converge to a finite limit). Finally, we compute the per-period crash/frenzy probability  $q$  for which, given all the other parameters, the precrash price is really 0.5% below the variance-free price. There is a unique such  $q$ . The precrash

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<sup>12</sup>For a 20% crash, this proportion appears to be unique (based on numerical calculations).

price discount is chosen to be 0.5% so that  $q$  comes out at around 1/10000, implying an average waiting time until a crash of about 40 years. This order was chosen because it gives a straightforward algorithm; since the result is an equilibrium, the order is not important. Appendix B provides the full details of how the equilibrium was computed.

Figure 2 shows the simulation of a 20% crash (marked with solid diamonds) and the corresponding frenzy (marked with open circles).



The parameters from this simulation appear in Table 2.

The average wait until a crash or frenzy is 41 years. The proportion  $\mu$  of rational traders that makes a 20% crash possible is 3.83%. Returns are skewed: while prices fall by 20% in a crash, they rise by only 0.42% in a frenzy. This follows from random-walk beliefs: by Proposition 2, prices cannot rise above the variance-free price  $\bar{p}$  in a frenzy, so the frenzy cannot exceed 0.5% of  $\bar{p}$ .

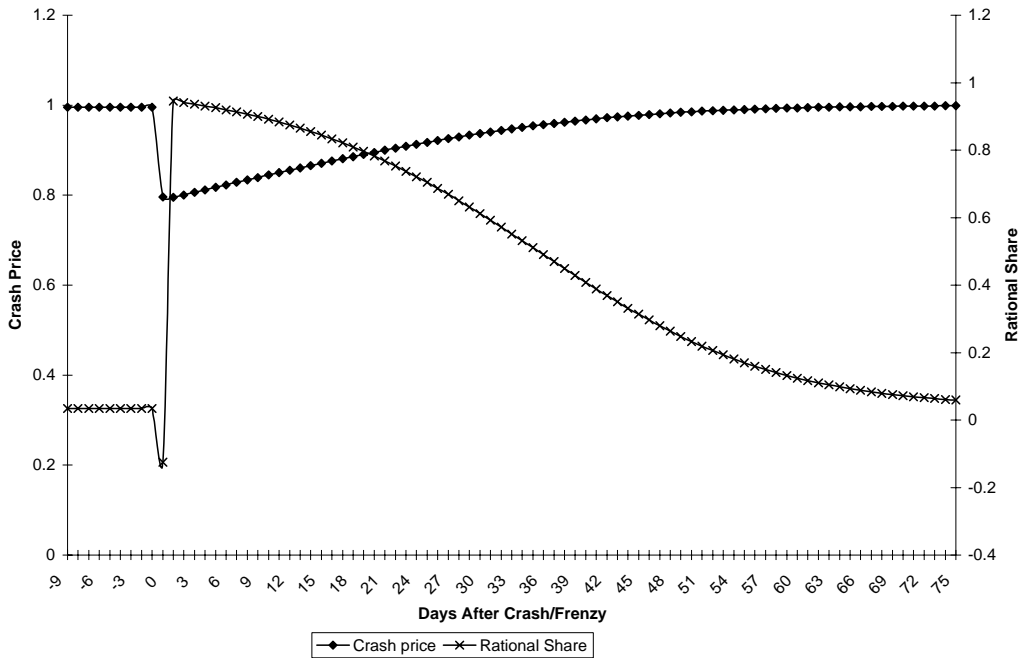
Figure 3 shows the share of stock owned by rational traders during the crash. (The share owned by the naive traders is one minus this number.) For reference, the path of

Parameter	Value
$\lambda$ (CARA)	0.22696
$\mu$ (proportion rational)	3.83%
$c$ (crash proportion)	20%
$c'$ (frenzy proportion)	0.42%
Average wait to crash/frenzy	41 years

Table 2: Simulation parameters and estimates.

prices is also shown.<sup>13</sup>

Figure 3: Prices and Rational Agent Stock Ownership in a Crash



On the day of the crash, rational traders sell about 16% of the market short. This causes a 20% decrease in price. The following day they reverse course, buying nearly

<sup>13</sup>The share of stock owned by rational traders is marked with crosses and corresponds to the vertical axis on the right side. The price path is marked with solid diamonds and corresponds to the left side axis.

100% of the naive traders' shares. (Note that the rational agents' share is measured on the right hand side axis.) In contrast, the total stock and stock index trading volume on October 19, 1987 amounted to only about 1% of total stock market value (Gennotte and Leland [7, p. 999]). However, Gennotte and Leland [7] estimate that about 98% of stock is held by investors who "do not actively alter their their total stock exposure based on information about future economic trends." These investors would sit pat during a crash.<sup>14</sup> Thus, the stock sold by naive traders probably corresponds to no more than 2% of the U.S. stock market. Since some who sold in response to the crash probably did so the day after, our story is consistent with the empirical trading volume of 1% on October 19.

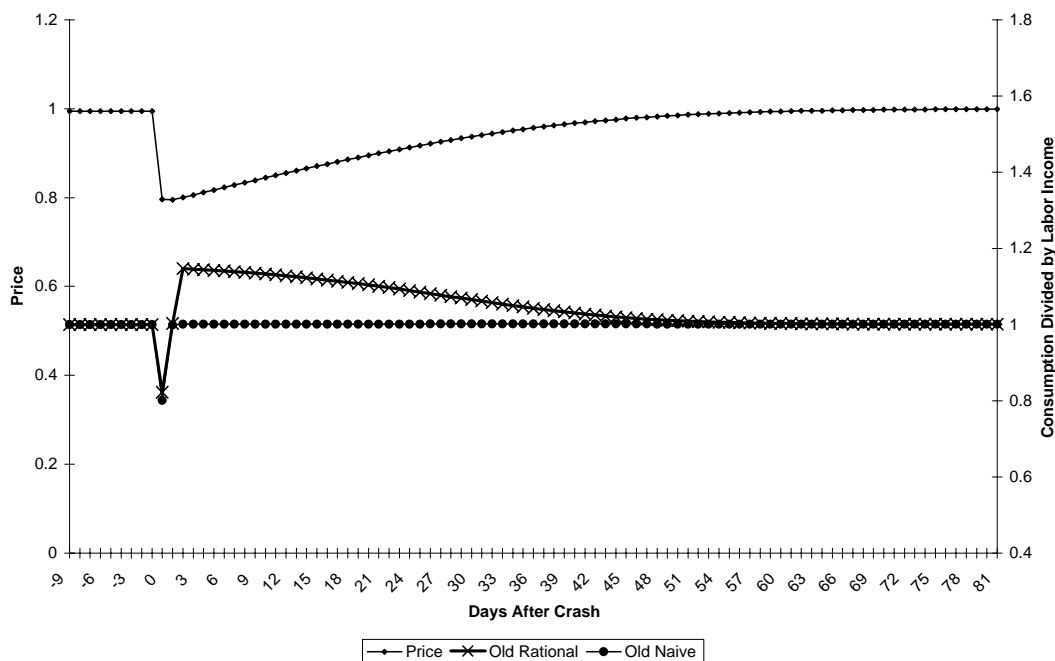
Figure 4 shows the consumption of old agents, according to type.<sup>15</sup>

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<sup>14</sup>The interpretation of Gennotte and Leland [7] is different: in their model, these traders buy when prices fall. However, it seems equally valid to suppose that these investors would not trade at all. This interpretation is consistent with the evidence that trading volume in the 1987 crash was only 1% of market equity.

<sup>15</sup>This chart assumes that stocks pay the average dividend,  $\bar{\delta}$ , in each period. Labor income is normalized to 1. The current consumption of old rational traders is marked by crosses, while the consumption of old naive traders is marked with solid circles. These curves virtually coincide until the second day after the crash. The consumption levels correspond to the right side axis. For reference, the stock price is also shown, above; it corresponds to the left side axis.

Figure 4: Current Consumption of the Old in a Crash



The reason old rational agents don't do better on the crash day is that the crash signal is unexpected, so they cannot lighten up on stocks in anticipation.<sup>16</sup> The day after the crash, the naive traders exit en masse, causing prices to fall again slightly. On this day, old rational traders (who sold short on the crash day) do about as well as naive traders. After this, however, rational traders do much better than naive traders. Unlike naive traders, they know the high volatility of the crash is a short-lived phenomenon. They also know that as the naive traders calm down and reenter the market, prices will rise. Knowing these facts, they hold large long positions and receive large capital gains as the price gradually returns to its fundamental value.

One aspect of Figure 4 that may seem puzzling is the high consumption of old rational

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<sup>16</sup>This might not be true if each rational agent had a truly private signal - e.g., the results of her own technical analysis - that was correlated with other agents' current and future signals (Jackson and Peck [8]). In this case, some agents might come to believe before the crash day that a crash is likely. Indeed, traders' fluctuating assessments of the likelihood of a major correction may help explain the large price swings that occurred in the week that preceded the 1987 crash.

traders after the crash. If total consumption is fixed (at labor income plus dividends) and old rational agents consume more, then why don't old naive agents consume less? In fact, total consumption is *not* fixed in this model. There is an infinitely elastic supply of bonds, which serve as a store of wealth. Old agents consume the portion of the young generation's labor income that the old receive in return for their stocks, plus that part of their *own* labor income that they invested in bonds in the prior period. Thus, there is no reason for total consumption to be the same in different periods.

Indeed, the old naive traders' consumption two days after the crash (1.000537) is actually slightly *greater* than their precrash consumption (1.000444) and remains higher until 117 days after the crash (when it is again 1.000444).<sup>17</sup> Before the crash the stock price is constant: there are no capital gains. After the crash, the stock price rises; the resulting capital gains slightly raise old naive traders' consumption. As the price stabilizes, these capital gains shrink, gradually reducing old naive traders' consumption to the precrash level.

## 6 Sensitivity Analysis

The level of risk aversion  $\lambda$  is chosen to replicate the historic equity premium of 6% (Kocherlakota [11]). However, the historic equity premium is hard to reconcile with other estimates of risk aversion (Mehra and Prescott [13]), so in this section we check what happens if agents are less risk averse. We also examine which crash sizes can be sustained for all possible proportions  $\mu$  of rational agents

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<sup>17</sup>Their consumption finally converges to a level (1.000442) that is very slightly below their precrash consumption. (When we say "their consumption" we actually mean the consumption of the naive traders who are old *in a given period*.) This is because after the crash, the price converges to the variance-free price, which is slightly greater than the precrash price - and thus the dividend yield is lower.

Figure 5: Equilibrium Crash Sizes by Proportion of Rational Agents for Four Different Degrees of Risk Aversion

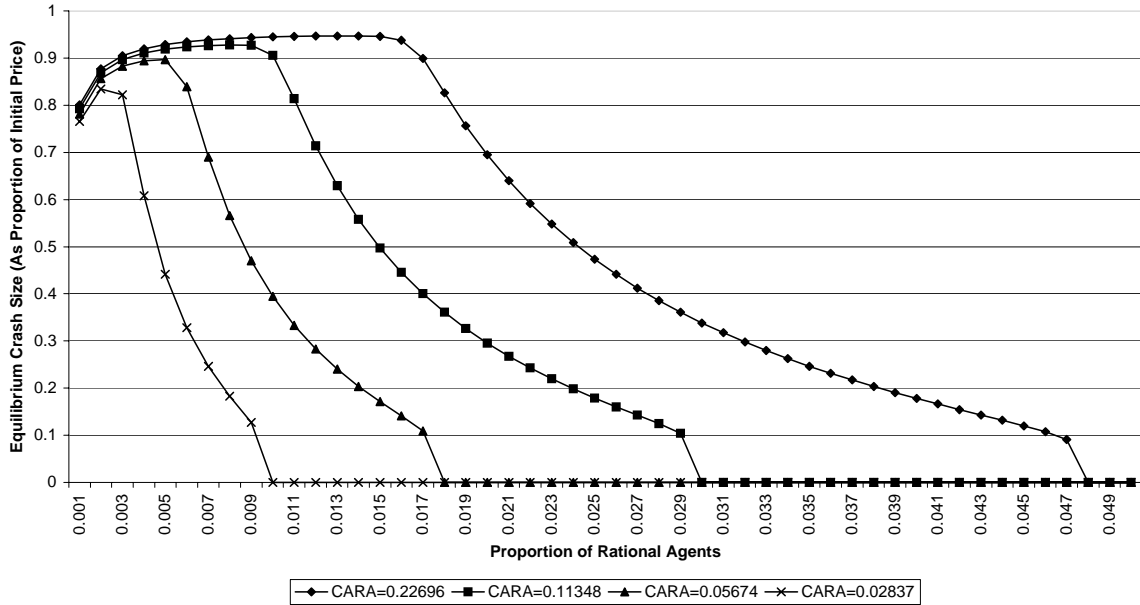


Figure 5 shows the crash sizes that occur in equilibrium for different proportions  $\mu$  and for four different values of  $\lambda$ . ( $\lambda$  is denoted “CARA” in the Figure.) The vertical axis gives the crash size as a proportion of the precrash price. The proportion of rational traders,  $\mu$ , appears on the horizontal axis. There are four curves, each corresponding to a different coefficient of absolute risk aversion. The highest curve uses the original coefficient  $\lambda$  from Table 2. The other three curves correspond to coefficients of absolute risk aversion that are a half, a quarter, and an eighth of this value. At these levels of risk aversion, crashes cannot be sustained if the proportion of rational traders exceeds 5%. Thus, values of  $\mu$  above 0.05 are not depicted.

As shown in Figure 5, the crash size is increasing in the level of risk aversion and decreasing in the proportion of rational agents. This is intuitive. The day after the crash, young naive traders are unwilling to buy stock at almost any price. Their demand elasticity is virtually zero since their demand is proportional to the reciprocal of their variance estimate, which is now very high. Thus, the price is determined almost entirely by the demand of young rational traders. As their proportion shrinks or they become

more risk averse, prices have to fall further the day after the crash to clear the market. The anticipation of this leads rational traders to bid the price down further on the day of the crash, so the crash is larger.

## 7 Naive Traders' Two Mistakes

The crash occurs because selling by rational traders leads naive traders to sell in the following period. Without this postcrash selling, there would be no crash: the rational traders would expect the price to rise right after the crash, which would eliminate their incentive to sell. This postcrash selling occurs because the naive traders make two mistakes.

### 7.1 Mistake #1: Random Walk Beliefs

Popular books (e.g., Malkiel [12]) and best-selling finance texts (e.g., Sharpe [16]) teach that since markets are efficient, stock prices must follow a random walk. However, this is not quite correct: the efficient markets paradigm implies that when stocks are riskier, their excess return should be higher. If naive traders understood this, then they would have less incentive to sell in periods of high volatility - such as after a crash. In this section we assume that the naive traders' expected return  $R_t$ , rather than being identically zero, is an increasing function of their volatility prediction  $V_t$ . Crashes can still occur for empirically reasonable parameter values.

Our approach is to estimate the dependence of expected returns on volatility *empirically*, as we did for the equation governing volatility. Engle, Lilien, and Robins [4] show that if stocks are in fixed supply and agents have constant absolute risk aversion, the relation between  $V_t$  and  $R_t$  has the form

$$R_t = b_0 + b_1 \sqrt{V_t} \tag{7}$$

where  $b_0$  and  $b_1$  are constants.<sup>18</sup> We estimate the following model empirically using the

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<sup>18</sup>They show that it has the form  $R_t = b_0 + b_1 \sqrt{c + V_t}$  but when this model is estimated, the point estimate for  $c$  is 0, so we use the simpler model instead.

daily closing S&P Composite Index from January 3, 1928 to October 16, 1987:

$$\begin{aligned}
 r_{t+1} &= R_t + \varepsilon_{t+1} \\
 R_t &= b_0 + b_1\sqrt{V_t} \\
 \varepsilon_{t+1} &\sim N(0, V_t) \\
 V_t &= \alpha_0 + \alpha\varepsilon_{t-1}^2 + \beta V_{t-1}
 \end{aligned}$$

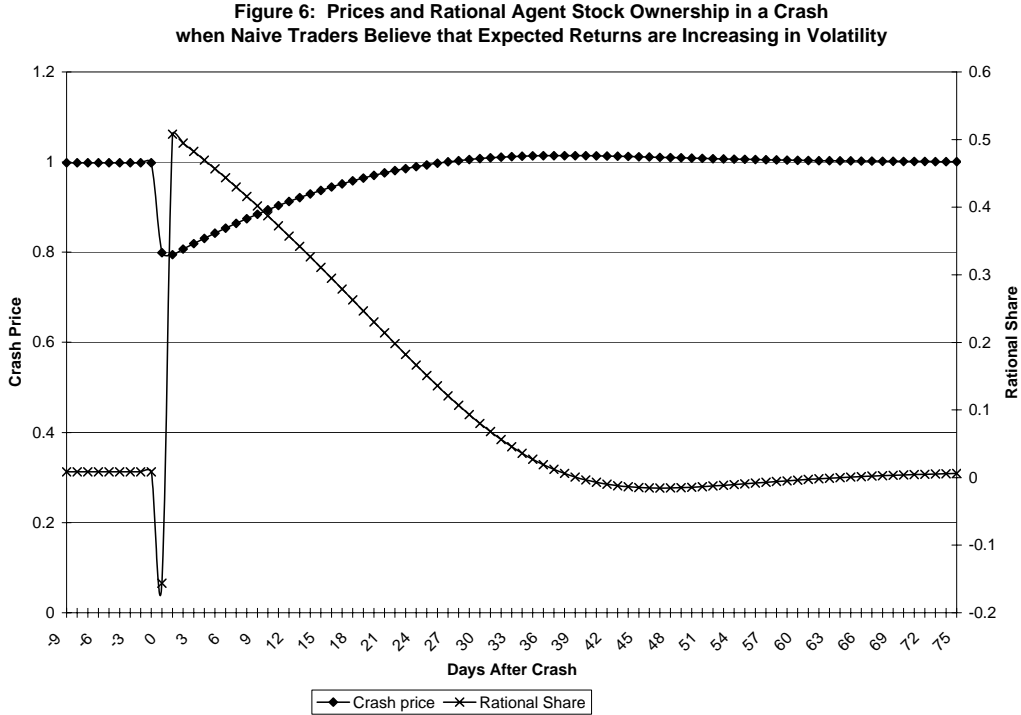
The parameter estimates are shown in Table 3.

<b>GARCH Parameter Estimates</b>		
<b>Variable</b>	<b>Parameter</b>	<b>t Value</b>
$b_0$	0.0000937	0.61
$b_1$	0.0545	2.60
$\alpha_0$	$1.023 \times 10^{-6}$	18.69
$\alpha$	0.0904	34.33
$\beta$	0.9031	339.66

Table 3: Parameter estimates from GARCH model, using S&P Composite Index returns from 1/3/28-10/16/87.

The simulation requires a few minor modifications. In actual market crashes there are two sources of volatility: the crash itself and noncrash risk that comes from earnings surprises, liquidity shocks, and so on. Since the relation between  $V$  and  $R$  is nonlinear, we must add an estimate  $V^{\text{nc}}$  of this noncrash risk to  $V$ . We use the estimate  $V^{\text{nc}} = 0.00011638$  from Section 5, which captures the average noncrash risk in the market. In addition, we subtract  $b_0 + b_1\sqrt{V^{\text{nc}}}$  from the return estimate so that the naive traders expect zero capital gains on average if the crash-related volatility  $V$  is zero. This reflects the fact that dividends are i.i.d. and guarantees that, as before, the price will converge to  $\bar{p}$  as  $V$  gradually shrinks to zero after the crash. Thus, naive traders predict the mean of the ex-dividend return as  $R_t = b_1 \left( \sqrt{V_t + V^{\text{nc}}} - \sqrt{V^{\text{nc}}} \right)$ . We also set  $\alpha_0$  to zero as before.

Crashes can still occur. Figure 6 shows a simulation of a 20% crash. The path of prices is shown, together with the share of stocks owned by rational traders (as in Figure 3).

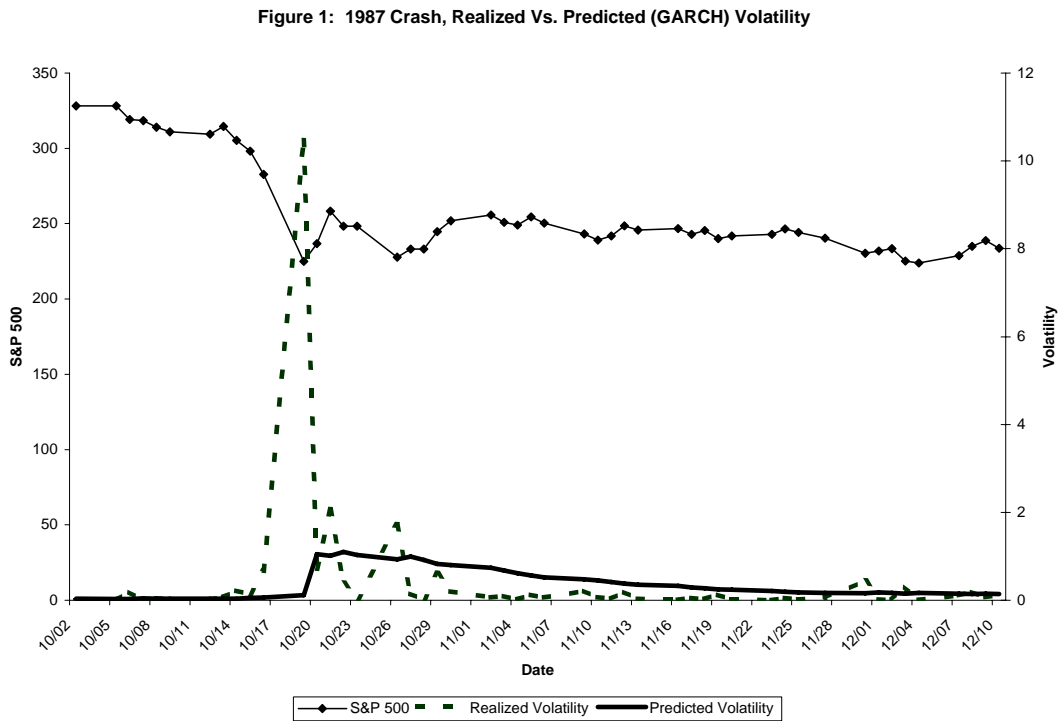


Since  $R_t$  is increasing in  $V_t$ , naive traders expect positive returns after the crash. This does counteract their decision to sell. However, their overestimate of postcrash volatility is large enough by itself to drive them out of the market following a 20% crash. Consequently, a crash remains a self-fulfilling prophecy for the rational traders.

One difference from the base model is that after prices recover, they actually rise very slightly above the variance-free price (here normalized to 1). As long as their prediction of volatility is positive, naive agents expect positive capital gains. For the parameters we calibrate, this makes them willing to pay more than the variance free price when predicted volatility is small.

## 7.2 Mistake #2: Overestimating Postcrash Volatility

Another reason that naive traders sell after the crash is that their reliance on adaptive expectations leads them to overestimate postcrash volatility. This prediction of the model is borne out in the data from the 1987 crash. Figure 1 compares realized volatility to the volatility predicted based on the GARCH model.<sup>19</sup> For most of the postcrash period, GARCH vastly overestimates the volatility. This phenomenon, known as the “threshold effect,” was first documented in 1993 by Engle and Ng [5].



This mistake - overestimating the variance - is essential *within the class of equilibria we simulate*. If the naive traders know that the postcrash variance will be zero and have

<sup>19</sup>All volatilities are annualized by multiplying by 250, the number of trading days in a typical year. Predicted volatility is based on GARCH estimation on the S&P Composite Index for the period 1/3/28 through 10/16/87 (the last trading day before the crash on 10/19/87). The S&P Index is the solid line marked with diamonds. The plain solid line shows predicted volatility while the dashed line gives actual volatility.

random walk beliefs, then there can be no crash, as the following proposition shows:

**Proposition 3** *Consider the class of simulated equilibria, in which there is at most one crash. If naive traders know the true variance but have random-walk beliefs, there is no crash: the price must equal  $\bar{p}$  on the day the “crash signal” is observed and on all subsequent days.*

**Proof.** *Appendix A*

The intuition is that on the day of the crash, the price will never again be as low. Thus, on this day, both types of traders know that the further price variance is zero and both believe that the next period’s capital gain will be nonnegative. But the price  $\bar{p}$  is computed on the assumption that traders expect no variance and zero capital gains. Since beliefs on the crash day are at least as optimistic as this, the price on the crash day cannot be lower than  $\bar{p}$ .

What happens if we do not restrict the form of the equilibrium? This remains an open question. For example, the original model (with random walk beliefs and GARCH expectations of variance) undoubtedly has other equilibria in which the variance of the capital gains return is not zero after the crash - e.g., because of a risk of subsequent crashes. We do not know whether this sort of equilibria would also exist if naive traders knew the true variance but had random walk beliefs.

## 8 How Long a Recovery?

In our main simulation (section 5, below), prices quickly return to their precrash level: 62 trading days after the crash, the entire 20% price decline has been erased. After the 1987 crash, it took 256 trading days for the S&P 500 to return to its precrash level. One reason may be that our naive traders believe that returns are normally distributed. In contrast, one conclusion investors may have drawn from the 1987 crash is that returns are highly skewed and 20% crashes can occur. This is supported by evidence that option prices reflected crash fears only *after* the 1987 crash (Jackwerth and Rubinstein [9]). Thus, the crash led traders not only to expect more volatility in the following

60 days, but also to fear more crashes in the years to come. This may have led to a long-lasting decrease in their willingness to buy stock, thus depressing stock prices for much longer than our model predicts. Adding this dimension to traders' beliefs would be a realistic addition to our model that we do not pursue in the current paper.

## A Proofs

**Proof of Proposition 2.** Since returns may not be normal, we must use the exact exponential utility function for rational trader demand to prove this result. A trader's demand  $D$  in period  $t$ , given the price  $p_t$ , solves

$$\max_D \int_p \int_{\delta} -e^{-\lambda[(1+r)W + D(p + \delta - (1+r)p_t)]} d\Phi(\delta) dF(p) \quad (8)$$

where  $p$  represents the next period's price, which has some c.d.f.  $F$ , and  $\delta$  the next period's dividend, which has the normal c.d.f.  $\Phi$ . Since  $\Phi'(\delta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(\delta - \bar{\delta})^2/2\sigma^2}$ , we drop leading constants and complete the square to show that (8) is equivalent to:

$$\begin{aligned} & \max_D \int_p -e^{-\lambda D(p - (1+r)p_t)} \left( \frac{1}{\sqrt{2\pi}\sigma} \int_{\delta} e^{-\lambda D\delta - (\delta - \bar{\delta})^2/2\sigma^2} d\delta \right) dF(p) \\ = & \max_D \int_p -e^{-\lambda D(p + \bar{\delta} - (1+r)p_t) + (D\lambda\sigma)^2/2} dF(p) \end{aligned}$$

The derivative of this integral with respect to  $D$  is:

$$e^{-\lambda D(\bar{\delta} - (1+r)p_t) + (D\lambda\sigma)^2/2} \int_p \left( \lambda(p + \bar{\delta} - (1+r)p_t) - D\lambda^2\sigma^2 \right) e^{-\lambda Dp} dF(p) \quad (9)$$

Since the leading exponential term is positive, the sign of this expression equals the sign of the integral.

Let  $p^{\max}$  be the supremum of the set of prices reached with positive probability. Since the stock price can never exceed  $L/m$ ,  $p^{\max} < \infty$ . Thus, for any  $\varepsilon > 0$  there is a time  $t$ , a history  $h_t$ , and a signal  $\theta_t$  such that  $p_t$  is within  $\varepsilon$  of  $p^{\max}$ . There are now two cases.

1. If  $x_t^R < 0$ , then by (5),  $x_t^N > 1$ . But  $x_t^N \leq \frac{\bar{\delta} - rp_t}{\lambda\sigma^2}$  by (4)<sup>20</sup> so  $p_t \leq \bar{p}$ , whence  $p^{\max} \leq \bar{p} + \varepsilon$ .

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<sup>20</sup>Naive traders believe that the capital gains return is normally distributed, so the normal approximation always holds exactly for them.

2. If  $x_t^R \geq 0$ , there are two subcases. (a) If  $p_t \leq \bar{p}$  then  $p^{\max} \leq \bar{p} + \varepsilon$  as in case 1.  
 (b) If  $p_t > \bar{p}$  then  $x_t^N \leq \frac{\bar{\delta} - rp_t}{\lambda\sigma^2} < \frac{\bar{\delta} - r\bar{p}}{\lambda\sigma^2} = 1$  so

$$\begin{aligned} 1 &= \mu x_t^R + (1 - \mu)x_t^N < \mu x_t^R + (1 - \mu) \\ \implies x_t^R &> 1 \end{aligned}$$

Thus, letting  $D = x_t^R$  in (9),

$$\lambda(p + \bar{\delta} - (1 + r)p_t) - D\lambda^2\sigma^2 < \lambda(p + \bar{\delta} - (1 + r)p_t) - \lambda^2\sigma^2$$

But  $p - p_t \leq p^{\max} - p_t \leq \varepsilon$ , so

$$\lambda(p + \bar{\delta} - (1 + r)p_t) - \lambda^2\sigma^2 \leq \lambda(\varepsilon + \bar{\delta} - rp_t) - \lambda^2\sigma^2$$

However, the integrand in (9) must be nonnegative for some values of  $p$  or rational traders will set  $x_t^R = -\infty$ . Thus,

$$\begin{aligned} 0 &< \lambda(\varepsilon + \bar{\delta} - rp_t) - \lambda^2\sigma^2 \\ \implies p_t &< \bar{p} + \frac{\varepsilon}{r} \end{aligned}$$

so that  $p^{\max} \leq \bar{p} + \frac{\varepsilon}{r} + \varepsilon$ .

In all cases,  $p^{\max} < \bar{p} + k\varepsilon$  for a fixed constant  $k$ . Since this is true for all  $\varepsilon$ ,  $p^{\max} \leq \bar{p}$ .  
 Q.E.D.

**Proof of Proposition 3:** Since old traders can dispose of their shares freely, the price of stock is never negative. Accordingly, let  $p^{\min} \in [0, \bar{p}]$  be the infimum of prices ever reached in this class of equilibria on or after the period in which the crash or frenzy signal is observed. For any  $\varepsilon > 0$  there is a price  $p_t$ , observed in some equilibrium, that is within  $\varepsilon$  of  $p^{\min}$ . Since all traders know the variance of the ex-dividend return is zero, the normal approximation is exact. By (5), since  $p_{t+1} - p_t \geq -\varepsilon$ ,

$$\begin{aligned} 1 &\geq \mu \frac{-\varepsilon + \bar{\delta} - rp_t}{\lambda\sigma^2} + (1 - \mu) \frac{\bar{\delta} - rp_t}{\lambda\sigma^2} \\ \implies p_t &\geq \bar{p} - \frac{\mu\varepsilon}{r} \end{aligned}$$

Thus,  $p^{\min} \geq p_t - \varepsilon \geq \bar{p} - \frac{\mu\varepsilon}{r} - \varepsilon$ . Since this is true for any  $\varepsilon > 0$ ,  $p^{\min} \geq \bar{p}$ . By Proposition 2, the price must equal  $\bar{p}$  both in and after the period in which the crash or frenzy signal is observed. Q.E.D.

## B Simulated Equilibria

In the simulations, the price is assumed constant in the precrash period. It is determined in the following way. We use the normal approximation to rational traders' demands.<sup>21</sup> Since the precrash price is constant, naive traders' initial predictions of the mean and variance of returns are both zero ( $V_t = R_t = 0$  in any precrash period  $t$ ).<sup>22</sup> For the rational traders,  $E[p_{t+1} - p_t]$  equals  $(qc' - qc)p^{PC} \doteq Ap^{PC}$  and  $\text{Var}(p_{t+1}) = (qc'^2 + qc^2 - (qc' - qc)^2)(p^{PC})^2 \doteq B(p^{PC})^2$ . The precrash price  $p^{PC}$  is the solution to (5) with these values substituted:

$$1 = \mu \frac{Ap^{PC} + \bar{\delta} - rp^{PC}}{\lambda(B[p^{PC}]^2 + \sigma^2)} + (1 - \mu) \frac{\bar{\delta} - rp^{PC}}{\lambda\sigma^2} \quad (10)$$

Suppose that a crash or frenzy occurs at some time  $t' \leq t$ . Since the continuation path of prices is unique,  $\text{Var}(p_{t+1}) = 0$  and  $E(p_{t+1} - p_t) = p_{t+1} - p_t$ . Substituting these into (5) and solving for  $p_{t+1}$ , we obtain

$$p_{t+1} = p_t(1 + r) - \bar{\delta} + \frac{\lambda\sigma^2}{\mu} \left[ 1 - (1 - \mu) \frac{R_t p_t + \bar{\delta} - r p_t}{\lambda(V_t p_t^2 + \sigma^2)} \right] \quad (11)$$

We use this to solve forwards for the price path and impose the requirement that the price converges to  $\bar{p}$ .

The algorithm for computing the simulation is as follows. We first compute the variance-free price  $\bar{p}$  as described in section 5. The precrash price  $p^{PC}$  is set to be 0.5% less than  $\bar{p}$ . We then find crash and frenzy sizes  $c, c'$  that lead to convergent price paths if the price is initially  $p^{PC}$ . Finally, given these crash and frenzy sizes, we find a per-period crash/frenzy probability  $q$  such that  $p^{PC}$  satisfies (10).

<sup>21</sup>This approximation is exact on and after the day of the crash/frenzy since the variance of the next day's price is zero and all risk is due to the normally distributed dividend. It is inexact only before the crash; however, this creates only a small distortion since crashes are very unlikely, so overall risk is dominated by the normally distributed dividend.

<sup>22</sup>This is necessary since if naive traders observe a constant price, they see zero empirical volatility and a zero return. Thus, their adaptive estimates  $V_t$  and  $R_t$  must converge to zero over time unless they are already zero. Since changes in  $V_t$  and  $R_t$  will cause changes in the market-clearing price, they must be identically zero in the precrash, constant-price period.

## C Calibration

We compute  $\sigma^2$  from  $V^{\text{nc}} = 0.00011638$  (section 5) as follows.  $V^{\text{nc}}$  is the variance per dollar invested. The per-share variance is thus  $\bar{p}^2 V^{\text{nc}}$ , where the price is  $\bar{p}$  in a noncrash (constant-price) world. If  $\sigma^2$  is to capture this risk, we must have

$$\sigma^2 = \bar{p}^2 V^{\text{nc}} \quad (12)$$

Moreover, if  $\sigma^2$  represents the only risk to the investor (i.e., if the price is constant), the stock price must equal

$$\bar{p} = \frac{1}{r} (\bar{\delta} - \lambda \sigma^2) \quad (13)$$

Equations (12) and (13) are solved to yield

$$\bar{p} = \frac{1}{2\lambda V^{\text{nc}}} \left( -r + \sqrt{(r^2 + 4\lambda V^{\text{nc}} \bar{\delta})} \right) \quad (14)$$

$\sigma^2$  is computed by substituting (14) into (12). The equations for  $p$  and  $\sigma^2$  depend on  $\lambda$ . We choose  $\lambda$  so that the equity premium equals 6% per year. Since we assume an annual risk-free interest rate of 5%, this is the  $\lambda$  that gives a  $\bar{p}$  for which

$$\bar{\delta}/\bar{p} = e^{.11T} - 1 \quad (15)$$

## References

- [1] Bollerslev, Tim. 1986. "Generalized Autoregressive Conditional Heteroskedasticity." *Journal of Econometrics* 31:307-328.
- [2] Campbell, John Y., and Ludger Hentschel. 1992. "No News is Good News: An Asymmetric Model of Changing Volatility in Stock Returns." *Journal of Financial Economics* 31:281–331.
- [3] Chou, Ray Yeutien. 1988. "Volatility Persistence and Stock Valuations: Some Empirical Evidence Using GARCH." *Journal of Applied Econometrics* 3:279-294.
- [4] Engle, Robert F., David Lilien, and Russel P. Robins. 1987. "Estimating Time Varying Risk Premia in the Term Structure: The ARCH-M Model." *Econometrica* 55:391-407.
- [5] Engle, Robert F., and V.K. Ng. 1993. "Measuring and Testing the Impact of News on Volatility." *Journal of Finance* 48:1749-1778.
- [6] Frankel, David M. 2006. "Adaptive Expectations and Stock Market Crashes." Mimeo, available at [www.econ.iastate.edu/faculty/frankel](http://www.econ.iastate.edu/faculty/frankel).
- [7] Genotte, Gerard, and Hayne Leland. 1990. "Market Liquidity, Hedging, and Crashes." *American Economic Review* 80:999-1021.
- [8] Jackson, Matthew O., and James Peck. 1991. "Speculation and Price Fluctuations with Private, Extrinsic Signals." *Journal of Economic Theory* 55:274-95.
- [9] Jackwerth, Jens C., and Mark Rubinstein. 1996. "Recovering Probability Distributions from Option Prices." *Journal of Finance* 51:1611-1631.
- [10] Kleidon, Allan W., and Robert E. Whaley. July 1992. "One Market? Stocks, Futures, and Options During October 1987." *Journal of Finance Papers and Proceedings*, pp. 851-877.

- [11] Kocherlakota, Narayana R. 1996. "The Equity Premium: It's Still a Puzzle." *Journal of Economic Literature* XXXIV:42-71.
- [12] Malkiel, Burton. 1985. *A Random Walk Down Wall Street, 4th ed.* New York: Norton.
- [13] Mehra, Rajnish, and Edward C. Prescott. 1985. "The Equity Premium: A Puzzle." *Journal of Monetary Economics* 15:145-61.
- [14] Nelson, Daniel B. 1991. "Conditional Heteroskedasticity in Asset Returns: A New Approach." *Econometrica* 59:347-370.
- [15] Presidential Task Force on Market Mechanisms (Brady Commission). January 1988. *Report.* Washington: U.S. Government Printing Office.
- [16] Sharpe, William F. 1981. *Investments.* New Jersey: Prentice-Hall.
- [17] Shiller, Robert J. 1989. *Market Volatility.* Cambridge, Mass.: MIT Press.