VARIOUS TOOLS FOR COMPARATIVE STATICS (ROUGH)

1. The chain rule (or total derivative) for composite functions of several variables

1.1. Chain rule for functions of two variables. When y = f (x_1, x_2) with x_1 = g(t) and x_2 = h(t), then

$$\begin{aligned} \frac{dy}{dt} &= \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} \\ &= \frac{\partial f}{\partial x_1}\frac{dg(t)}{dt} + \frac{\partial f}{\partial x_2}\frac{dh(t)}{dt} \end{aligned} \tag{1}$$

This is usually called the total derivative of y with respect to t.

1.2. Example. Let the function be given by

$$y = f(x_1, x_2) = x_1^2 + x_2^3$$

with

$$\begin{aligned} x_1(t) &= t^2 + 2t + 1 \\ x_2(t) &= 3t \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 2x_1,\ \frac{\partial f}{\partial x_2} = 3x_2^2 \\ \frac{dx_1}{dt} &= 2t + 2,\ \frac{dx_2}{dt} = 3 \\ \Rightarrow \frac{dy}{dt} &= (2x_1)(2t+2) + (3x_2^2)(3) \\ &= (2t^2 + 4t + 2)(2t+2) + (27t^2)(3) \\ &= 4t^3 + 4t^2 + 8t^2 + 8t + 4t + 4 + 81t^2 \\ &= 4t^3 + 93t^2 + 12t + 4 \end{aligned} \tag{2}$$

Date: February 19, 2008.

If we multiply if out we obtain

$$\begin{aligned} y = f(x_1, x_2) &= x_1^2 + x_2^3 \\ &= (t^2 + 2t + 1)^2 + (3t)^3 \\ &= t^4 + 2t^3 + t^2 + 2t^3 + 4t^2 + 2t + t^2 + 2t + 1 + 27t^3 \\ \frac{df}{dt} &= 4t^3 + 6t^2 + 2t + 6t^2 + 8t + 2 + 2t + 2 + 81t^2 \\ &= 4t^3 + 93t^2 + 12t + 4 \end{aligned} \tag{3}$$

In-class exercises
Find the total derivative of each of the following with respect to t.

1. $y = f(x_1, x_2) = x_1^2 + x_2^3$
   $x_1(t) = t^2$
   $x_2(t) = 2t$

2. $y = f(x_1, x_2) = x_1^2 + x_2^2$
   $x_1(t) = t^2 + 2t$
   $x_2(t) = 2t + 1$

3. $y = f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2$
   $x_1(t) = t^2 + 2t + 3$
   $x_2(t) = 2t - t^2$

4. $y = f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_3^2$
   $x_1(t) = t^2 + 2t$
   $x_2(t) = 2t$
   $x_3(t) = t^2 - 5t$

5. $y = f(x_1, x_2) = \frac{x_1^2 + x_2^2}{x_1 + x_2}$
   $x_1(t) = t^2 + 2t$
   $x_2(t) = 2t + 1$

2. Directional Derivatives

2.1. **Idea.** If y = f(x_1, x_2), the partial derivatives, $\frac{\partial f}{\partial x_1}\ \frac{\partial f}{\partial x_2}$ measure the rates of change of f(x_1, x_2), in the directions of the x_1 - axis and the x_2 - axis, respectively. We can also measure the rate of change of the function in other directions. Consider a particular point in the domain of f and denote it (x_1^0, x_2^0) . Any non-zero vector (h, k) is then a direction in which we move away from the point (x_1^0, x_2^0) in a straight line to points of the form

$$(x_1, x_2) = (x_1(t), x_2(t)) = (x_1^0 + th, x_2^0 + tk) \tag{4}$$

Given any initial point (x_1^0, x_2^0) and any direction (h, k), define the directional function g by

$$g(t) = f(x_1^0 + th, x_2^0 + tk) \tag{5}$$

The derivative of this function is

$$\begin{aligned} \frac{dg}{dt} &= \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} \\ &= \frac{\partial f}{\partial x_1}(x_1^0 + th,\, x_2^0 + tk)\,h + \frac{\partial f}{\partial x_2}(x_1^0 + th,\, x_2^0 + tk)\,k \end{aligned} \tag{6}$$

Now let t = 0 so that we are at the point (x_1^0, x_2^0). Then we obtain

$$\frac{dg}{dt}(0) = \frac{\partial f}{\partial x_1}(x_1^0, x_2^0)\,h + \frac{\partial f}{\partial x_2}(x_1^0, x_2^0)\,k \tag{7}$$

If the vector (h, k) has length 1, the derivative of f in the direction (h, k) is called the directional derivative of f in the direction of (h, k) at (x_1^0, x_2^0). Specifically, the directional derivative of f(x_1, x_2) at (x_1^0, x_2^0) in the direction of the unit vector (h, k) is

$$D_{h,k}\, f(x_1^0, x_2^0) = \frac{\partial f}{\partial x_1}(x_1^0, x_2^0)\,h + \frac{\partial f}{\partial x_2}(x_1^0, x_2^0)\,k \tag{8}$$

Note that when the length of (h, k) is one, a move away from (x_1^0, x_2^0) in the direction (h, k) changes the value of f by approximately $D_{h,k}\, f(x_1^0, x_2^0)$. Also notice that the directional derivative is the product of the gradient of f and the vector (h, k).

2.2. **Example.** Consider the function f(x_1, x_2) with the following direction and initial point.

$$\begin{aligned} f(x_1, x_2) &= x_1^2 + 3x_2^2 \\ Direction &= (2, 5) \\ Point &= (1, 1) \end{aligned}$$

First normalize the direction vector. Because the length of the vector is $\sqrt{29}$ we can normalize it as $\left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$. Then find the gradient of f as

$$\begin{aligned} f(x_1, x_2) &= x_1^2 + 3x_2^2 \\ \frac{\partial f}{\partial x_1} &= 2x_1 \\ \frac{\partial f}{\partial x_2} &= 6x_2 \end{aligned}$$

Evaluated at (1,1) we obtain

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 2 \\ \frac{\partial f}{\partial x_2} &= 6 \end{aligned}$$

The directional derivative is then given by

$$(2, 6)\begin{pmatrix} \frac{2}{\sqrt{29}}, \\ \frac{5}{\sqrt{29}} \end{pmatrix} = \frac{34}{\sqrt{29}}$$

3. More general chain rules

3.1. General form of the chain rule. Let y = f(x_1, x_2, ... x_n) and let x_1 = g_1 (t_1, t_2, ..., t_m), x_2 = g_2 (t_1, t_2, ..., t_m), ... , x_n= g_n (t_1, t_2, ..., t_m) where t is an m-vector of other variables upon which the x vector depends. Then the following holds

$$\frac{\partial y}{\partial t_j} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial y}{\partial x_n}\frac{\partial x_n}{\partial t_j},\ j = 1, 2, \cdots, n \tag{9}$$

3.2. Example. Consider the function y = f(x_1, x_2) along with the auxiliary functions x_1(z, w) and x_2 (z, w)

$$\begin{aligned} y = f(x_1, x_2) &= 3x_1 + 2x_1 x_2^2 \\ x_1(z, w) &= 5z + 2zw \\ x_2(z, w) &= zw^2 + 3w \end{aligned}$$

where t_1 = z and t_2 = w from equation 9 . We can find the partial derivative of y with respect to z using equation 9 as follows.

$$\begin{aligned} \frac{\partial y}{\partial z} &= \frac{\partial f(x_1, x_2)}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial f(x_1, x_2)}{\partial x_2}\frac{\partial x_2}{\partial z} \\ &= (3 + 2x_2^2)(5 + 2w) + (4x_1 x_2)(w^2) \\ &= (3 + 2(z^2w^4 + 6zw^3 + 9w^2))(5 + 2w) + 4(5z + 2zw)(zw^2 + 3w)w^2 \\ &= (3 + 2z^2w^4 + 12zw^3 + 18w^2)(5 + 2w) + (20zw^2 + 8zw^3)(zw^2 + 3w) \\ &= 15 + 6w + 10z^2w^4 + 4z^2w^5 + 60zw^3 + 24zw^4 + 90w^2 + 36w^3 \\ &+ 20z^2w^4 + 60zw^3 + 8z^2w^5 + 24zw^4 \\ &= 15 + 6w + 90w^2 + 36w^3 + 120zw^3 + 48zw^4 + 30z^2w^4 + 12z^2w^5 \end{aligned}$$

We can also find the partial of y with respect to w as

$$\begin{aligned} \frac{\partial y}{\partial w} &= \frac{\partial f(x_1, x_2)}{\partial x_1}\frac{\partial x_1}{\partial w} + \frac{\partial f(x_1, x_2)}{\partial x_2}\frac{\partial x_2}{\partial w} \\ &= (3 + 2x_2^2)(2z) + (4x_1 x_2)(2zw + 3) \\ &= (3 + 2(z^2w^4 + 6zw^3 + 9w^2))(2z) + 4(5z + 2zw)(zw^2 + 3w)(2zw + 3) \\ &= (3 + 2z^2w^4 + 12zw^3 + 18w^2)(2z) + (20z + 8zw)(zw^2 + 3w)(2zw + 3) \\ &= (6z + 4z^3w^4 + 24z^2w^3 + 36zw^2) + (20z^2w^2 + 60zw + 8z^2w^3 + 24zw^2)(2zw + 3) \\ &= 6z + 4z^3w^4 + 24z^2w^3 + 36zw^2 + 40z^3w^3 + 120z^2w^2 + 16z^3w^4 + 48z^2w^3 \\ &+ 60z^2w^2 + 180zw + 24z^2w^3 + 72zw^2 \\ &= 6z + 180zw + 108zw^2 + 180z^2w^2 + 96z^2w^3 + 40z^3w^3 + 20z^3w^4 \end{aligned}$$

4. Derivatives of implicit functions

4.1. **A two-variable implicit function theorem.**

4.1.1. *statement of the theorem.* Consider the function defined implicitly by f(x_1, x_2) = c where c is a constant. If $\frac{\partial f}{\partial x_2} \neq 0$, then

$$\frac{\partial x_2}{\partial x_1} = -\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} \tag{10}$$

4.1.2. *examples.*

a: Example 1

$$y^0 = f(x_1, x_2) = x_1^2 - 2x_1 x_2 + x_1 x_2^3$$

To find the partial derivative of x_2 with respect to x_1 we find the two partials of f as follows

$$\begin{aligned}\frac{\partial x_2}{\partial x_1} &= -\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} \\ &= -\frac{2x_1 - 2x_2 + x_2^3}{3x_1 x_2^2 - 2x_1} \\ &= \frac{2x_2 - 2x_1 - x_2^3}{3x_1 x_2^2 - 2x_1}\end{aligned}$$

b: Example 2

$$u^0 = U(x_1, x_2)$$

To find the partial derivative of x_2 with respect to x_1 we find the two partials of U as follows

$$\frac{\partial x_2}{\partial x_1} = -\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}}$$

4.2. **A more general form of the implicit function theorem with p independent variables and 1 implicit equation.**

4.2.1. *Statement of the theorem.* Suppose x_1 is defined implicitly as a differentiable function of the p variables y_1, y_2, ... ,y_p by the equation f (x_1, y_1, y_2, . . . y_p) = 0 or f (x_1, y_1, y_2, . . . y_p) = c where c is a constant. Then

$$\frac{\partial x_1}{\partial y_i} = -\frac{\frac{\partial f}{\partial y_i}}{\frac{\partial f}{\partial x_1}},\ i = 1, 2, \cdots, p \tag{11}$$

4.2.2. *Example.* Consider a utility function given by

$$u = x_1^{\frac{1}{4}} \, x_2^{\frac{1}{3}} \, x_3^{\frac{1}{6}}$$

This can be written implicitly as

$$f\,(u,\, x_1,\, x_2,\, x_3\,) = u - x_1^{\frac{1}{4}} \, x_2^{\frac{1}{3}} \, x_3^{\frac{1}{6}} = 0$$

Now consider some of the partial derivatives arising from this implicit function. First consider the marginal rate of substitution of x_1 for x_2

$$\begin{aligned} \frac{\partial x_1}{\partial x_2} &= \frac{\frac{-\partial f}{\partial x_2}}{\frac{\partial f}{\partial x_1}} \\ &= \frac{-\,(\,-\frac{1}{3}\,)\, x_1^{\frac{1}{4}} \, x_2^{-\frac{2}{3}} \, x_3^{\frac{1}{6}}}{(-\frac{1}{4}\,)\, x_1^{-\frac{3}{4}} \, x_2^{\frac{1}{3}} \, x_3^{\frac{1}{6}}} \\ &= -\,\frac{4}{3}\,\frac{x_1}{x_2} \end{aligned}$$

Now consider the marginal rate of substitution of x_2 for x_3.

$$\begin{aligned} \frac{\partial x_2}{\partial x_3} &= \frac{\frac{-\partial f}{\partial x_3}}{\frac{\partial f}{\partial x_2}} \\ &= \frac{-\,(\,-\frac{1}{6}\,)\, x_1^{\frac{1}{4}} \, x_2^{\frac{1}{3}} \, x_3^{-\frac{5}{6}}}{(-\frac{1}{3}\,)\, x_1^{\frac{1}{4}} \, x_2^{-\frac{2}{3}} \, x_3^{\frac{1}{6}}} \\ &= -\,\frac{1}{2}\,\frac{x_2}{x_3} \end{aligned}$$

Now consider the marginal utility of x_2.

$$\begin{aligned} \frac{\partial u}{\partial x_2} &= \frac{\frac{-\partial f}{\partial x_2}}{\frac{\partial f}{\partial u}} \\ &= \frac{-\,(\,-\frac{1}{3}\,)\, x_1^{\frac{1}{4}} \, x_2^{-\frac{2}{3}} \, x_3^{\frac{1}{6}}}{1} \\ &= \frac{1}{3}\, x_1^{\frac{1}{4}} \, x_2^{-\frac{2}{3}} \, x_3^{\frac{1}{6}} \end{aligned}$$

4.3. **A more general form of the implicit function theorem with p independent variables and m implicit equations.**

4.3.1. *One motivation for implicit function theorem.* Assuming the conditions of the implicit function theorem as discussed below hold, we can eliminate m variables from the constrained optimization problem using the constraint equations. In this way the constrained problem is converted to an unconstrained problem and we can use the results on unconstrained problems to determine a solution..

4.3.2. *description of the system of equations.* Suppose that we have m equations depending on m + p variables (parameters) written in implicit form as follows

$$\begin{aligned} \phi_1\,(x_1,\, x_2,\, \cdots\, ,\, x_m,\;\, y_1,\, y_2,\, \cdots\, ,\, y_p\,) &= 0 \\ \phi_2\,(x_1,\, x_2,\, \cdots\, ,\, x_m,\;\, y_1,\, y_2,\, \cdots\, ,\, y_p\,) &= 0 \\ \cdots &= 0 \\ \vdots\vdots\vdots &= \vdots \\ \phi_m\,(x_1,\, x_2,\, \cdots\, ,\, x_m,\;\, y_1,\, y_2,\, \cdots\, ,\, y_p\,) &= 0 \end{aligned} \tag{12}$$

For example with m = 2 and p = 3, we might have

$$\begin{aligned} \phi_1\,(x_1,\, x_2,\, p,\, w_1,\, w_2) \;&=\; (0.4)\; p\, x_1^{-0.6}\, x_2^{0.2} \;-\; w_1 \;= 0 \\ \phi_2\,(x_1,\, x_2,\, p,\, w_1,\, w_2\,) \;&=\; (0.2)\; p\, x_1^{0.4}\, x_2^{-0.8} \;-\; w_2 \;= 0 \end{aligned} \tag{13}$$

4.3.3. *Jacobian matrix of the system.* The Jacobian matrix of the system in (12) defined as matrix of first partials as follows

$$J \;=\; \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_m} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \phi_m}{\partial x_1} & \frac{\partial \phi_m}{\partial x_2} & \cdots & \frac{\partial \phi_m}{\partial x_m} \end{bmatrix} \tag{14}$$

This matrix will be of rank m if its determinant is not zero.

4.4. **Statement of implicit function theorem.** Suppose ϕ_i are real valued functions defined on a domain D and are continuously differentiable on an open set $D^1 \subset D \subset R^{m+p}$, p ¿ 0 and $\phi_i(x^0, y^0) = 0$ for i=1, 2, ..., m and $(x^0, y^0)\ \epsilon\ D^1$, i.e., the equation system is satisfied at the point (x^0, y^0). Let x be of dimension m and y of dimension p. Assume the Jacobian matrix

$$\left[\frac{\partial Œ_i(x^0,\ y^0)}{\partial x_j} \right]$$

has rank m. Then there exists a neighborhood of (x^0, y^0) ($N_{ffi}(x^0, y^0)$) $\epsilon\ D^1$, an open set $D^2 \subset R^p$ containing y^0, and real valued functions ψ_k k=1,...,m continuously differentiable on D^2 such that the following conditions are satisfied

$$x_k^0 \;=\; g_k\,(\,y^0)\; k \;=\; 1, ..., m\,. \tag{15}$$

This means we can solve the system of m implicit equations for all m of the x variables as functions of the p independent or y variables. In the example system (13), we could solve the two equations for x_1 and x_2 as functions of p, w_1, and w_2, i.e., $x_1 = g_1(p, w_1, w_2)$ and $x_2 = g_2(p, w_1, w_2)$.

For every $y \epsilon D^2$, we have

$$\begin{aligned} \phi_i\,(g_1\,(y),\ g_2\,(y)\ \cdots, g_m\,(y)\,,\ y)\ &= 0\ i = 1\,, \ldots,\ m \\ \Rightarrow\ \phi_i\ (\ g\,(\ y)\,,\ y)\ &= 0\ i = 1\,, \ldots,\ m \end{aligned} \tag{16}$$

where $g(y) = [g_1(y), g_2(y) \ldots g_m(y)]$ and for all $(x, y) \epsilon N_{ffi}(x^0, y^0)$ the Jacobian matrix

$$\frac{\partial \phi_i(x,\ y)}{\partial x_j}\ has\ rank\ m \tag{17}$$

This means that if we plug the x's that we obtain from (16) into the implicit equations (equation 12), the left hand side of the equations will be zero, or the equations will be satisfied.

Furthermore for $y \epsilon D^2$, we can compute the partial derivatives of $\psi_k(y)$ as solutions to the set of equations

$$\sum_{k=1}^{m} \frac{\partial \phi_i(g\,(y),\ y)}{\partial x_k}\,\frac{\partial g_k\,(y)}{\partial y_j}\ =\ -\ \frac{\partial \phi_i(g\,(y),\ y)}{\partial y_j},\ i\ =\ 1,\ 2,\ \cdots,\ m \tag{18}$$

Here the derivative $\frac{\partial \phi_i(\,g\,(y)\,,\,y)}{\partial x_k}$ is just the derivative of ϕ_i with respect to its kth argument. In equation 12 this is x_k, while in equation 16, it is $g_k(y)$. For the example in (13) we can find $\frac{\partial x_1}{\partial p}\ \frac{\partial x_2}{\partial p}$, that is $\left(\frac{\partial g_i}{\partial p}\right)$, from the following two equations derived from equation 13, which is repeated here.

$$\begin{aligned} \phi_1\,(x_1,\ x_2,\ p,\ w_1,\ w_2)\ &=\ (0.4)\ p\,x_1^{-0.6}\,x_2^{0.2}\ -\ w_1\ = 0 \\ \phi_2\,(x_1,\ x_2,\ p,\ w_1,\ w_2\,)\ &=\ (0.2)\ p\,x_1^{0.4}\,x_2^{-\,0.8}\ -\ w_2\ = 0 \\ \left[(-\,0.24)\,p\,x_1^{-1.6}\,x_2^{0.2}\right]\frac{\partial x_1}{\partial p}\ +\ \left[(\,0.08)\,p\,x_1^{-0.6}\,x_2^{-\,0.8}\right]\frac{\partial x_2}{\partial p}\ &=-\ (0.4)\ x_1^{-0.6}\,x_2^{0.2} \\ \left[(\,0.08)\,p\,x_1^{-0.6}\,x_2^{-0.8}\right]\frac{\partial x_1}{\partial p}\ +\ \left[(\,-\,0.16)\,p\,x_1^{0.4}\,x_2^{-\,1.8}\right]\frac{\partial x_2}{\partial p}\ &=-\ (0.2)\ x_1^{0.4}\,x_2^{-\,0.8} \end{aligned} \tag{19}$$

While the system is non-linear, we can (in principle and in practice) solve for the two derivatives in question.

To see the intuition of equation 18, take the total derivative of ϕ_i in (16) with respect to y_j as follows

$$\begin{gathered} \phi_i\,(g_1\,(y)\,,\ g_2\,(y)\,,\ \cdots,\ g_m\,(y)\,,\ y)\ = 0 \\ \frac{\partial \phi_i}{\partial g_1}\,\frac{\partial g_1}{\partial y_j}\ +\ \frac{\partial \phi_i}{\partial g_2}\,\frac{\partial g_2}{\partial y_j}\ +\ \cdots\ +\ \frac{\partial \phi_i}{\partial g_m}\,\frac{\partial g_m}{\partial y_j}\ +\ \frac{\partial \phi_i}{\partial y_j}\ =\ 0 \end{gathered} \tag{20}$$

and then move $\frac{\partial \phi_i}{\partial y_j}$ to the right hand side of the equation. Then perform a similar task for the other equations to obtain m equations in the m partial derivatives, $\frac{\partial x_i}{\partial y_j}\ =\ \frac{\partial g_i}{\partial y_j}$

For the case of only one implicit equation (18) reduces to

$$\sum_{k=1}^{m} \frac{\partial \phi_i(g\,(y),\ y)}{\partial x_k}\,\frac{\partial g_k\,(y)}{\partial y_j}\ =\ -\ \frac{\partial \phi(\psi(y),\ y)}{\partial y_j} \tag{21}$$

With only one implicit equation m = 1 and we obtain

$$\frac{\partial \phi(g(y),\, y)}{\partial x_k}\, \frac{\partial g_k(y)}{\partial y_j} \;=\; -\,\frac{\partial \phi\,(g\,(y),\, y)}{\partial y_j} \tag{22}$$

which can be rewritten as

$$\frac{\partial g_k(y)}{\partial y_j} \;=\; \frac{-\,\frac{\partial \phi(g\,(y),\, y)}{\partial y_j}}{\frac{\partial \phi(g\,(y),\, y)}{\partial x_k}} \tag{23}$$

This is then the same as equation 11.
If there are only two variables, x_1 and x_2 where x_2 is now like y_1, we obtain

$$\begin{aligned} \frac{\partial \phi(g(x_2),\, x_2)}{\partial x_1}\, \frac{\partial g_1(x_2)}{\partial x_2} &= -\,\frac{\partial \phi\,(g\,(x_2),\, x_2)}{\partial x_2} \\ \Rightarrow\; \frac{\partial x_1}{\partial x_2} \;=\; \frac{\partial g_1(x_2)}{\partial x_2} &= \frac{-\,\frac{\partial \phi\,(g\,(x_2),\, x_2)}{\partial x_2}}{\frac{\partial \phi(g\,(x_2),\, x_2)}{\partial x_1}} \end{aligned} \tag{24}$$

which is the same as the two variable case in equation 10 where ϕ takes the place of f.

4.4.1. *examples of implicit function theorem.*

a: example 1

$$\phi_1(x_1,\, x_2,\, y) \;=\; 3x_1 \,+\, 2x_2 \,+\, 4y = 0$$

$$\phi_2\,(x_1,\, x_2,\, y) \;=\; 4x_1 \,+\, x_2 \,+\, y = 0$$

The Jacobian is given by

$$\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

Since this Jacobian is of full rank, we can solve these equations for x_1 and x_2 as functions of y. Move y to the rhs in each equation.

$$3x_1 \,+\, 2x_2 \;=\; -4\,y$$

$$4x_1 \,+\, x_2 \;=\; -y$$

Now solve the second equation for x_2, and plug in the first equation

$$x_2 = -y - 4x_1$$

$$3x_1 + 2(-y - 4x_1) = -4y$$

$$3x_1 - 2y - 8x_1 = -4y$$

$$-5x_1 = -2y$$

$$\Rightarrow x_1 = 2/5\,y = \psi_1(y)$$

Now plug this answer into the equation for y_2 and solve

$$x_2 = -y - 4(2/5y)$$

$$= -y - 8/5y$$

$$= -13/5\,y = \psi_2(y)$$

We can check to see that equation 16 is satisfied

$$\begin{aligned}\phi_1(2/5y, -13/5y, y) &= 3(2/5\,y) + 2(-13/5) + 4y \\ &= 6/5y - 26/5y + 4y = 0\end{aligned}$$

$$\begin{aligned}\phi_2(2/5y, -13/5y, y) &= 4(2/5\,y) + (-13/5) + y \\ &= 8/5\,y - 13/5y + y = 0\end{aligned}$$

Furthermore

$$\frac{\partial x_1}{\partial y} = \frac{\partial g_1(y)}{\partial y} = \frac{2}{5}$$

$$\frac{\partial x_2}{\partial y} = \frac{\partial g_2(y)}{\partial y} = \frac{-13}{5}$$

Rather than solving the system for x_1 and x_2 as functions of y, we can solve for the partial derivatives $\left(\frac{\partial x_1}{\partial y} \; \frac{\partial x_2}{\partial y} \right) \left(\frac{\partial g_1}{\partial y} \; \frac{\partial g_2}{\partial y} \right)$ using the formula in (18) as:

$$\frac{\partial \phi_1}{\partial x_1} \frac{\partial g_1}{\partial y} + \frac{\partial \phi_1}{\partial x_2} \frac{\partial g_2}{\partial y} = \frac{-\partial \phi_1}{\partial y}$$

$$\frac{\partial \phi_2}{\partial x_1} \frac{\partial g_1}{\partial y} + \frac{\partial \phi_2}{\partial x_2} \frac{\partial g_2}{\partial y} = \frac{-\partial \phi_2}{\partial y}$$

$$3 \frac{\partial g_1}{\partial y} + 2 \frac{\partial g_2}{2y} = -4$$

$$4 \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial y} = -1$$

$$=> \frac{\partial g_2}{\partial y} = -1 - 4 \frac{\partial g_1}{\partial y}$$

$$=> 3 \frac{\partial g_1}{\partial y} + 2 (-1 - 4 \frac{\partial g_1}{\partial y}) = -4$$

$$=> 3 \frac{\partial g_1}{\partial y} - 2 - 8 \frac{\partial g_1}{\partial y} = -4$$

$$\Rightarrow -5 \frac{\partial g_1}{\partial y} = -2$$

$$\Rightarrow \frac{\partial g_1}{\partial y} = 2/5$$

$$\frac{\partial g_2}{\partial y} = -1 - 4 (2/5)$$

$$= -1 - 8/5 = -13/5$$

b: example 2

Let the production function for a firm by given by

$$y = 14x_1 + 11x_2 - x_1^2 - x_2^2$$

Profit for the firm is given by

$$\begin{aligned}\pi &= py - w_1 x_1 - w_2 x_2 \\ &= p\,(14x_1 + 11x_2 - x_1^2 - x_2^2) - w_1 x_1 - w_2 x_2\end{aligned}$$

The first order conditions for profit maximization imply that

$$\begin{aligned}\pi &= 14\,p\,x_1 + 11\,p\,x_2 - p\,x_1^2 - p\,x_2^2 - w_1 x_1 - w_2 x_2 \\ \frac{\partial \pi}{\partial x_1} &= \phi_1 = 14\,p - 2\,p\,x_1 - w_1 = 0 \\ \frac{\partial \pi}{\partial x_2} &= \phi_1 = 11\,p - 2\,p\,x_2 - w_2 = 0\end{aligned}$$

We can solve the first equation for x_1 as follows

$$\begin{aligned}\frac{\partial \pi}{\partial x_1} &= 14\,p - 2\,p\,x_1 - w_1 = 0 \\ \Rightarrow\ 2\,p\,x_1 &= 14\,p - w_1 \\ \Rightarrow\ x_1 &= \frac{14\,p - w_1}{2\,p} \\ &= 7 - \frac{w_1}{2\,p}\end{aligned}$$

In a similar manner we can find x_2 from the second equation

$$\begin{aligned}\frac{\partial \pi}{\partial x_2} &= 11\,p - 2\,p\,x_2 - w_2 = 0 \\ \Rightarrow\ 2\,p\,x_2 &= 11\,p - w_2 \\ \Rightarrow\ x_2 &= \frac{11\,p - w_2}{2\,p} \\ &= 5.5 - \frac{w_2}{2\,p}\end{aligned}$$

We can find the derivatives of x_1 and x_2 with respect to p, w_1 and w_2 directly as follows:

$$\begin{aligned} x_1 &= 7 - \tfrac{1}{2} w_1 p^{-1} \\ x_2 &= 5.5 - \tfrac{1}{2} w_2 p^{-1} \\ \frac{\partial x_1}{\partial p} &= \tfrac{1}{2} w_1 p^{-2} \\ \frac{\partial x_1}{\partial w_1} &= -\tfrac{1}{2} p^{-1} \\ \frac{\partial x_1}{\partial w_2} &= 0 \\ \frac{\partial x_2}{\partial p} &= \tfrac{1}{2} w_2 p^{-2} \\ \frac{\partial x_2}{\partial w_2} &= -\tfrac{1}{2} p^{-1} \\ \frac{\partial x_2}{\partial w_2} &= 0 \end{aligned}$$

We can also find these derivatives using the implicit function theorem. The two implicit equations are

$$\begin{aligned} \phi_1(x_1,\ x_2,\ p, w_1, w_2) &= 14\,p - 2\,p\,x_1 - w_1 = 0 \\ \phi_2(x_1,\ x_2\ p, w_1, w_2) &= 11\,p - 2\,p\,x_2 - w_2 = 0 \end{aligned}$$

First we check the Jacobian of the system. It is obtained by differentiating ϕ_1 and ϕ_2 with respect to x_1 and x_2 as follows

$$J = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2\,p & 0 \\ 0 & -2p \end{bmatrix}$$

The determinant is $4p^2$ which is positive. Now we have two we can solve using the implicit function theorem.

The theorem says

$$\begin{aligned} \textstyle\sum_{k=1}^{m} \frac{\partial\phi_i(g(y),y)}{\partial x_k}\,\frac{\partial g_k(y)}{\partial y_j} &= -\,\frac{\partial\phi_i(\psi(y),y)}{\partial y_j}\ ,\ i\ =\ 1,2,\cdots \\ or\ \frac{\partial\phi_1}{\partial x_1}\frac{\partial x_1}{\partial p} + \frac{\partial\phi_1}{\partial x_2}\frac{\partial x_2}{\partial p} &= -\,\frac{\partial\phi_1}{\partial p} \\ \frac{\partial\phi_2}{\partial x_1}\frac{\partial x_1}{\partial p} + \frac{\partial\phi_2}{\partial x_2}\frac{\partial x_2}{\partial p} &= -\,\frac{\partial\phi_2}{\partial p} \end{aligned}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial p} \\ \frac{\partial x_2}{\partial p} \end{bmatrix} = \begin{bmatrix} -\frac{\partial\phi_1}{\partial p} \\ -\frac{\partial\phi_2}{\partial p} \end{bmatrix} \tag{25}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial x_1}{\partial p} \\ \frac{\partial x_2}{\partial p} \end{bmatrix} = \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{\partial\phi_1}{\partial p} \\ -\frac{\partial\phi_2}{\partial p} \end{bmatrix}$$

We can compute the various partial derivatives of ϕ as follows

$$\begin{aligned} \phi_1(x_1,\ x_2,\ p,\ w_1,\ w_2) &= 14\,p\ -\ 2\,p\,x_1\ -\ w_1\ =\ 0 \\ \frac{\partial\phi_1}{\partial x_1} &= -\,2\,p \\ \frac{\partial\phi_1}{\partial x_2} &= 0 \\ \frac{\partial\phi_1}{\partial p} &= 14\ -\ 2\,x_1 \\ \phi_2(x_1,\ x_2\ p,\ w_1,\ w_2) &= 11\,p\ -\ 2\,p\,x_2\ -\ w_2\ =\ 0 \\ \frac{\partial\phi_2}{\partial x_1} &= -\,2\,p \\ \frac{\partial\phi_2}{\partial x_2} &= 0 \\ \frac{\partial\phi_1}{\partial p} &= 11\ -\ 2\,x_2 \end{aligned}$$

Now writing out the system we obtain for the case at hand we obtain

$$
\begin{aligned}
\begin{bmatrix} \frac{\partial x_1}{\partial p} \\ \frac{\partial x_2}{\partial p} \end{bmatrix} &= \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{\partial \phi_1}{\partial p} \\ -\frac{\partial \phi_2}{\partial p} \end{bmatrix} \\
\Rightarrow \begin{bmatrix} \frac{\partial x_1}{\partial p} \\ \frac{\partial x_2}{\partial p} \end{bmatrix} &= \begin{bmatrix} -2p & 0 \\ 0 & -2p \end{bmatrix}^{-1} \begin{bmatrix} 2x_1 - 14 \\ 2x_2 - 11 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2p} & 0 \\ 0 & -\frac{1}{2p} \end{bmatrix} \begin{bmatrix} 2x_1 - 14 \\ 2x_2 - 11 \end{bmatrix} \\
&= \begin{bmatrix} \frac{7 - x_1}{p} \\ \frac{5.5 - x_2}{p} \end{bmatrix}
\end{aligned} \tag{26}
$$

Given that

$$
\begin{aligned}
x_1 &= 7 - \tfrac{1}{2} w_1 p^{-1} \\
x_2 &= 5.5 - \tfrac{1}{2} w_2 p^{-1}
\end{aligned}
$$

we obtain

$$\begin{aligned} \frac{\partial x_1}{\partial p} &= \frac{7 - x_1}{p} \\ &= \frac{7 - (7 - \frac{1}{2} w_1 p^{-1})}{p} \\ &= \frac{1}{2} w_1 p^{-2} \\ \frac{\partial x_1}{\partial p} &= \frac{5.5 - x_2}{p} \\ &= \frac{5.5 - (5.5 - \frac{1}{2} w_2 p^{-1})}{p} \\ &= \frac{1}{2} w_2 p^{-2} \end{aligned}$$

which is the same as before.

c: example 3 (exercise)

Verify the implicit function theorem derivatives $\frac{\partial x_1}{\partial w_1}$, $\frac{\partial x_1}{\partial w_2}$, $\frac{\partial x_2}{\partial w_1}$, $\frac{\partial x_2}{\partial w_2}$ for the function in example 2.

d: example 4

A firm sells its output into a perfectly competitive market and faces a fixed price p. It hires labor in a competitive labor market at a wage w, and rents capital in a competitive capital market at rental rate r. The production is f(L, K). The production function is strictly concave. The firm seeks to maximize its profits which are

$$ß = pf(L, K) - wL - rK$$

The first-order conditions for profit maximization are

$$\begin{aligned} \pi_L &= p f_L(L^*, K^*) - w = 0 \\ \pi_K &= p f_K(L^*, K^*) - r = 0 \end{aligned}$$

This gives two implicit equations for K and L.

The second order conditions are

$$\frac{\partial^2 \pi}{\partial L^2}(L^*,\ K^*) \ < \ 0 \ \frac{\partial^2 \pi}{\partial L^2}\frac{\partial^2 \pi}{\partial K^2} \ - \ \left[\frac{\partial^2 \pi}{\partial L \, \partial K}\right]^2 \ > \ 0 \ at \ (L^*,\ K^*)$$

We can compute these derivatives as

$$\begin{aligned} \frac{\partial^2 \pi}{\partial L^2} &= \frac{\partial\,(p\, f_L(L^*, K^*) - w)}{\partial\, L} = p\, F_{LL} \\ \frac{\partial^2 \pi}{\partial K^2} &= \frac{\partial\,(p\, f_K(L^*, K^*) - r)}{\partial\, L} = p\, F_{KK} \\ \frac{\partial^2 \pi}{\partial\, L\, \partial\, K} &= \frac{\partial\,(p\, f_L(L^*, K^*) - w)}{\partial\, K} = p\, F_{KL} \end{aligned}$$

The first of the second order conditions is satisfied by concavity of f. We then write the second condition as

$$\begin{aligned} D &= \begin{vmatrix} p\, f_{LL} & p\, f_{LK} \\ p\, f_{KL} & p\, f_{KK} \end{vmatrix} > 0 \\ &\Rightarrow p^2 \begin{vmatrix} f_{LL} & f_{LK} \\ f_{KL} & f_{KK} \end{vmatrix} > 0 \\ &\Rightarrow p^2 (f_{LL}\, f_{KK} - f_{KL}\, f_{LK}) > 0 \end{aligned} \tag{27}$$

The expression is positive because f is assumed to be strictly concave.

Now we wish to determine the effects on input demands, L^* and K^*, of changes in the input prices. Using the implicit function theorem in finding the partial derivatives with respect to w, we obtain for the first equation

$$\begin{aligned} \pi_{LL}\, \frac{\partial\, L^*}{\partial\, w} + \pi_{LK}\, \frac{\partial\, K^*}{\partial\, w} + \pi_{Lw} &= 0 \\ \Rightarrow\ p\, f_{LL}\, \frac{\partial\, L^*}{\partial\, w} + p\, f_{KL}\, \frac{\partial\, K^*}{\partial\, w} - 1 &= 0 \end{aligned}$$

For the second equation we obtain

$$\begin{aligned} \pi_{KL}\, \frac{\partial\, L^*}{\partial\, w} + \pi_{KK}\, \frac{\partial\, K^*}{\partial\, w} + \pi_{Kw} &= 0 \\ \Rightarrow\ p\, f_{KL}\, \frac{\partial\, L^*}{\partial\, w} + p\, f_{KK}\, \frac{\partial\, K^*}{\partial\, w} &= 0 \end{aligned}$$

We can write this in matrix form as

$$\begin{bmatrix} pf_{LL} & pf_{LK} \\ pf_{KL} & pf_{KK} \end{bmatrix} \begin{bmatrix} \partial L^*/\partial w \\ \partial K^*/\partial w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Using Cramer's rule, we then have the comparative-statics results

$$\begin{aligned} \frac{\partial L^*}{\partial w} &= \frac{\begin{vmatrix} 1 & pf_{LK} \\ 0 & pf_{KK} \end{vmatrix}}{D} = \frac{p\, f_{KK}}{D} \\ \frac{\partial K^*}{\partial w} &= \frac{\begin{vmatrix} pf_{LL} & 1 \\ pf_{KL} & 0 \end{vmatrix}}{D} = -\frac{p\, f_{KL}}{D} \end{aligned}$$

The sign of the first is negative because $f_{KK} < 0$ and $D > 0$ from either strict concavity of f or the second-order conditions. Thus, the demand curve for labor has a negative slope.

However, in order to sign the effect of a change in the wage rate on the demand for capital, we need to know the sign of f_{KL}, the effect of a change in the labor input on the marginal product of capital. It is plausible to assume that this is positive (though it may not be) and so an increase in the wage rate would also decrease the demand for capital. We can derive the effects of a change in the rental rate of capital in a similar way.

e: example 5

Let the production function be Cobb-Douglas of the form y = $L^{ff}K^{fi}$. Find the comparative-static effects ∂L*/∂w and ∂K*/∂w.

Profits are given by

$$\begin{aligned}\pi &= p\, f\,(L,\, K) \;-\; wL \;-\; rK\\ &= p\, L^{\alpha}\, K^{\beta} \;-\; w\, L \;-\; r\, K\end{aligned}$$

The first-order conditions are for profit maximization are

$$\begin{aligned}\pi_L &= \alpha\, p\, L^{\alpha\,-\,1}\, K^{\beta} \;-\; w \;=\; 0\\ \pi_K &= \beta\, p\, L^{\alpha}\, K^{\beta\,-\,1} \;-\; r \;=\; 0\end{aligned}$$

This gives two implicit equations for K and L.

The second order conditions are

$$\frac{\partial^2\,\pi}{\partial L^2}\,(L^*,\; K^*) \;<\; 0\, \frac{\partial^2\,\pi}{\partial L^2}\, \frac{\partial^2\,\pi}{\partial K^2} \;-\; \left[\frac{\partial^2\,\pi}{\partial L\,\partial K}\right]^2 \;>\; 0\, at\; (L^*,\; K^*)$$

We can compute these derivatives as

$$\begin{aligned}\frac{\partial^2\,\pi}{\partial L^2} &= \alpha\,(\alpha \;-\; 1)\; p\; L^{\alpha\,-\,2}\, K^{\beta}\\ \frac{\partial^2\,\pi}{\partial K^2} &\equiv \beta\,(\beta \;-\; 1)\; p\; L^{\alpha}\, K^{\beta\,-\,2}\\ \frac{\partial^2\,\pi}{\partial\, L\, \partial\, K} &= \alpha\, \beta\, p\; L^{\alpha\,-\,1}\, K^{\beta\,-\,1}\end{aligned}$$

The first of the second order conditions is satisfied as long as $\alpha,\, \beta$ ¡ 1. We can write the second condition as

$$D \;=\; \begin{vmatrix} \alpha\,(\alpha \;-\; 1)\; p\; L^{\alpha\,-\,2}\, K^{\beta} & \alpha\, \beta\, p\; L^{\alpha\,-\,1}\, K^{\beta\,-\,1}\\ \alpha\, \beta\, p\; L^{\alpha\,-\,1}\, K^{\beta\,-\,1} & \beta\,(\beta \;-\; 1)\; p\; L^{\alpha}\, K^{\beta\,-\,2}\end{vmatrix} \;>\; 0 \tag{28}$$

We can simplify D as follows

$$
\begin{aligned}
D &= \begin{vmatrix} \alpha\,(\alpha\,-\,1)\,p\,L^{\alpha\,-\,2}\,K^{\beta} & \alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1} \\ \alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1} & \beta\,(\beta\,-\,1)\,p\,L^{\alpha}\,K^{\beta\,-\,2} \end{vmatrix} \\
&= p^2 \begin{vmatrix} \alpha\,(\alpha\,-\,1)\,L^{\alpha\,-\,2}\,K^{\beta} & \alpha\,\beta\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1} \\ \alpha\,\beta\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1} & \beta\,(\beta\,-\,1)\,L^{\alpha}\,K^{\beta\,-\,2} \end{vmatrix} \\
&= p^2\,\alpha\,\beta\,(\alpha\,-\,1)\,(\beta\,-\,1)\,L^{\alpha\,-\,2}\,K^{\beta}\,L^{\alpha}\,K^{\beta\,-\,2}\,-\,\alpha^2\,\beta^2\,L^{2\alpha\,-\,2}\,K^{2\beta\,-\,2} \\
&= p^2\,\alpha\,\beta\,L^{\alpha\,-\,2}\,K^{\beta}\,L^{\alpha}\,K^{\beta\,-\,2}\,(\,(\alpha\,-\,1)\,(\beta\,-\,1)\,-\,\alpha\,\beta\,) \\
&= p^2\,\alpha\,\beta\,L^{\alpha\,-\,2}\,K^{\beta}\,L^{\alpha}\,K^{\beta\,-\,2}\,(\,\alpha\,\beta\,-\,\beta\,-\,\alpha\,+\,1\,-\,\alpha\,\beta\,) \\
&= p^2\,\alpha\,\beta\,L^{\alpha\,-\,2}\,K^{\beta}\,L^{\alpha}\,K^{\beta\,-\,2}\,(\,1\,-\,\alpha\,-\,\beta\,)
\end{aligned}
\tag{29}
$$

The condition is then that

$$D \,=\, p^2\,\alpha\,\beta\,L^{2\alpha-2}\,K^{2\beta-2}\,(1\,-\,\alpha\,-\,\beta)\,>\,0$$

This will be true if $\alpha + \beta < 1$.

Now we wish to determine the effects on input demands, L^* and K^*, of changes in the input prices. Using the implicit function theorem in finding the partial derivatives with respect to w, we obtain for the first equation

$$
\begin{aligned}
\pi_{LL}\,\frac{\partial\,L^*}{\partial\,w}\,+\,\pi_{LK}\,\frac{\partial\,K^*}{\partial\,w}\,+\,\pi_{Lw} &= 0 \\
\Rightarrow\;\alpha\,(\alpha\,-\,1)\,p\,L^{\alpha\,-\,2}\,K^{\beta}\,\frac{\partial\,L^*}{\partial\,w}\,+\,\alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1}\,\frac{\partial\,K^*}{\partial\,w}\,-\,1 &= 0
\end{aligned}
$$

For the second equation we obtain

$$
\begin{aligned}
\pi_{KL}\,\frac{\partial\,L^*}{\partial\,w}\,+\,\pi_{KK}\,\frac{\partial\,K^*}{\partial\,w}\,+\,\pi_{Kw} &= 0 \\
\Rightarrow\;\alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1}\,\frac{\partial\,L^*}{\partial\,w}\,+\,\beta\,(\beta\,-\,1)\,p\,L^{\alpha}\,K^{\beta\,-\,2}\,\frac{\partial\,K^*}{\partial\,w} &= 0
\end{aligned}
$$

We can write this in matrix form as

$$
\begin{bmatrix} \alpha\,(\alpha\,-\,1)\,p\,L^{\alpha\,-\,2}\,K^{\beta} & \alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1} \\ \alpha\,\beta\,p\,L^{\alpha\,-\,1}\,K^{\beta\,-\,1} & \beta\,(\beta\,-\,1)\,p\,L^{\alpha}\,K^{\beta\,-\,2} \end{bmatrix}
\begin{bmatrix} \partial L^*/\partial w \\ \partial K^*/\partial w \end{bmatrix}
=
\begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

Using Cramer's rule, we then have the comparative-statics results

$$\frac{\partial L^*}{\partial w} = \frac{\begin{vmatrix} 1 & \alpha\,\beta\,p\,L^{\alpha-1}\,K^{\beta-1} \\ 0 & \beta\,(\beta-1)\,p\,L^{\alpha}\,K^{\beta-2} \end{vmatrix}}{D} = \frac{\beta\,(\beta-1)\,p\,L^{\alpha}\,K^{\beta-2}}{D}$$

$$\frac{\partial K^*}{\partial w} = \frac{\begin{vmatrix} \alpha\,(\alpha-1)\,p\,L^{\alpha-2}\,K^{\beta} & 1 \\ \alpha\,\beta\,p\,L^{\alpha-1}\,K^{\beta-1} & 0 \end{vmatrix}}{D} = \frac{-\alpha\,\beta\,p\,L^{\alpha-1}\,K^{\beta-1}}{D}$$

The sign of the first of is negative because $p\,\beta\,(\beta-1)\,L^{\alpha}\,K^{\beta-2} < 0\,(\beta < 1)$, and D > 0 by the second order conditions. The cross partial derivative $\frac{\partial K^*}{\partial w}$ is also less than zero because $p\,\alpha\,\beta\,L^{\alpha-1}K^{\beta-1} > 0$ and D > 0. So, for the case of a Cobb-Douglas production function, an increase in the wage unambiguously reduces the demand for capital.

5. The General Equation for the Tangent to $F(X_1, X_{2,...}) = C$ and Properties of the Gradient

5.1. The general equation for a tangent plane. Consider the surface defined by

$$y = f(x_1, x_2, \cdots, x_n) \tag{30}$$

at the point $(x_1^0, x_2^0, \cdots, x_n^0)$

$$y^0 = f(x_1^0, x_2^0, \cdots, x_n^0) \tag{31}$$

The equation of the plane tangent to the surface at this point is given by

$$\begin{aligned} (y - y^0) &= f_1(x_1 - x_1^0) + f_2(x_2 - x_2^0) + \cdots + f_n(x_n - x_n^0) \\ \Rightarrow f_1(x_1 - x_1^0) &+ f_2(x_2 - x_2^0) + \cdots + f_n(x_n - x_n^0) - (y - y^0) = 0 \\ \Rightarrow f(x_1, x_2, \cdots, x_n) &= f(x_1^0, x_2^0, \cdots, x_n^0) + f_1(x_1 - x_1^0) + f_2(x_2 - x_2^0) + \cdots + f_n(x_n - x_n^0) \end{aligned} \tag{32}$$

where the partial derivatives f_i are evaluated at $(x_1^0, x_2^0, \cdots, x_n^0)$. We can also write this as

$$f(x) - f(x^0) = \nabla f(x^0)(x - x^0) \tag{33}$$

Compare this to the tangent equation with one variable

$$\begin{aligned} y - f(x^0) &= f'(x^0)(x - x^0) \\ \Rightarrow y &= f(x^0) + f'(x^0)(x - x^0) \end{aligned} \tag{34}$$

5.2. Vector functions of a real variable. Let $x_1, x_2, \ldots, x_n$ be functions of a variable t defined in an interval I and write

$$x(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \tag{35}$$

The function $t \rightarrow x(t)$ is a transformation from R to R^n and is called a vector function of a real variable. As x runs through I, x(t) traces out a set of points in n-space called a curve. In particular, if we put

$$x_1(t) = x_1^0 + t a_1, \; x_2(t) = x_2^0 + t a_2, \cdots, x_n(t) = x_n^0 + t a_n \tag{36}$$

the resulting curve is a straight line in n-space. It passes through the point $x^0 = (x_1^0, x_2^0, \cdots, x_n^0)$ at t = 0, and it is in the direction of the vector $a = (a_1, a_2, \cdots, a_n)$.

We can define the derivative of x(t) as

$$\frac{dx}{dt} = \dot{x}(t) = \left(\frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}, \ldots, \frac{dx_n(t)}{dt} \right) \tag{37}$$

If K is a curve in n-space traced out by x(t), then $\dot{x}(t)$ can be interpreted as a vector tangent to K at the point t.

5.3. **Tangent vectors.** Consider the function $r(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ and its derivative $\dot{r}(t) = (\dot{x}_1(t), \dot{x}_2(t), \cdots, \dot{x}_n(t))$. This function traces out a curve in R^n. We can call the curve K. For fixed t,

$$\dot{r}(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h} \tag{38}$$

If $\dot{r}(t) \neq 0$, then for t + h close enough to t, the vector r(t+h) - r(t) will not be zero. As h tends to 0, the quantity r(t+h) - r(t) will come closer and closer to serving as a direction vector for the tangent to the curve at the point P. This can be seen in the following graph

It may be tempting to take this difference as approximation of the direction of the tangent and then take the limit

$$\lim_{h \to 0} [r(t+h) - r(t)] \tag{39}$$

and call the limit the direction vector for the tangent. But the limit is zero and the zero vector has no direction. Instead we use the a vector that for small h has a greater length, that is we use

$$\frac{r(t+h) - r(t)}{h} \tag{40}$$

For any real number h, this vector is parallel to r(t+h) - r(t). Therefore its limit

$$\dot{r}(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h} \tag{41}$$

can be taken as a direction vector for the tangent line.

5.4. **Tangent lines, level curves and gradients (Sydsaeter).** Consider the equation

$$f(x) = f(x_1, x_2, \cdots, x_n) = c \tag{42}$$

which defines a level curve for the function f. Let $x^0 = (x_1^0, x_2^0, \cdots, x_n^0)$ be a point on the surface and let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ represent a differentiable curve K lying on the surface and passing through x^0 at t = t^0. Because K lies on the surface, f [x(t)] = f [x_1(t), x_2(t), ,,, x_n(t)] = c for all t. Now differentiate this equation with respect to t

$$\begin{aligned} \frac{\partial f(x)}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial f(x)}{\partial x_2} \frac{\partial x_2}{\partial t} + \cdots + \frac{\partial f(x)}{\partial x_2} \frac{\partial x_2}{\partial t} &= 0 \\ \Rightarrow \nabla f(x^0) \cdot \dot{x}(t^0) &= 0 \end{aligned} \tag{43}$$

Because the vector $\dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t), \cdots, \dot{x}_n(t))$ has the same direction as the tangent to the curve K at x^0, the gradient of f is orthogonal to the curve K at the point x^0.

6. Linear Approximations and Differentials

6.1. **Differentials.** Consider a function $y = f(x_1, x_2, \ldots, x_n)$. If $dx_1, dx_2, \ldots, dx_n$ are arbitrary real numbers (not necessarily small), we define the differential of $y = f(x_1, x_2, \ldots, x_n)$ as

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n \tag{44}$$

When x_i is changed to $x_i + dx_i$, then the actual change in the value of the function is the **increment**

$$\Delta y = f(x_1 + dx_1,\ x_2 + dx_2, \cdots, x_n + dx_n) - f(x_1, x_2, \cdots, x_n) \tag{45}$$

If dx_i is small in absolute value, the Δy can be approximated by dy

$$\Delta y \approx dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n \tag{46}$$

6.2. **Rules for differentials.**

1: $dc = 0$ (c is a constant)
2: $d(cx^n) = cnx^{n-1}\,dx$
3: $d(af + bg) = a\,df + b\,dg$ (a and b are constants)
4: $d(fg) = g\,df + f\,dg$
5: $d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2}$ $g \neq 0$
6: $d(fgh) = gh\,df + fh\,dg + fg\,dh$
7: If $y = g[\,f(x_1, x_2, \ldots, x_n)\,]$ then $dy = g'[\,f(x_1, x_2, \ldots, x_n)\,]\,df$.

6.3. **Differentials and systems of equations.**

6.3.1. *Idea.* We can find partial derivatives of implicit systems using differentials. We take the total differential of both sides of each equation, set all differentials of variables that are not changing equal to zero, and then divide each equation by the differential of the one exogenous variable that is changing. We then solve the resulting system for the various partial derivatives.

6.3.2. *Example 1.* Consider the system

$$\begin{aligned} \phi_1(x_1,\, x_2,\, p,\, w_1,\, w_2) &= 14\,p \;-\; 2\,p\,x_1 \;-\; w_1 \;=\; 0 \\ \phi_2(x_1,\, x_2\; p,\, w_1,\, w_2) &= 11\,p \;-\; 2\,p\,x_2 \;-\; w_2 \;=\; 0 \end{aligned} \tag{47}$$

The total differential of each equation is

$$\begin{aligned} 14\,dp \;-\; 2\,p\,dx_1 \;-\; 2\,x_1\,dp \;-\; d\,w_1 &= 0 \\ 11\,dp \;-\; 2\,p\,dx_2 \;-\; 2\,x_2\,dp \;-\; d\,w_2 &= 0 \end{aligned}$$

Now set $dw_1 = dw_2 = 0$ and divide each equation by dp

$$\begin{aligned} 14 \;-\; 2\,p\,\tfrac{dx_1}{dp} \;-\; 2\,x_1 &= 0 \\ 11 \;-\; 2\,p\,\tfrac{dx_2}{dp} \;-\; 2\,x_2 &= 0 \end{aligned}$$

Solving we obtain

$$\begin{aligned} &2\,p\,\tfrac{\partial x_1}{\partial p} = 14 \;-\; 2\,x_1 \\ &\Rightarrow\; \tfrac{\partial x_1}{\partial p} = \tfrac{14\,-\,2x_1}{2\,p} \;=\; \tfrac{7\,-\,x_1}{p} \\ &2\,p\,\tfrac{\partial x_1}{\partial p} = 11 \;-\; 2\,x_1 \\ &\Rightarrow\; \tfrac{\partial x_2}{\partial p} = \tfrac{11\,-\,2x_1}{2\,p} \;=\; \tfrac{5.5\,-\,x_2}{p} \end{aligned}$$

6.3.3. *Example 2.* Consider the following macroeconomic model:

$$\begin{aligned} Y &= C + I + G \\ C &= f(Y - T) \\ I &= h(r) \\ r &= m(M) \end{aligned} \tag{48}$$

Where Y is national income, C is consumption, I is investment, G is public expenditure, T is tax revenue, r is the interest rate, and M is money supply. There are seven variables and four equations so we can potentially solve for 4 endogenous variables in terms of 3 exogenous variables. If we assume that f, h, and m are differentiable functions with $0 < f' < 1$, $h' < 0$, and $m' < 0$, then these equations determine Y, C, I, and r as differentiable functions of M, T, and G. We can also find the differentials of Y, C, I, and r in terms of the differentials of M, T, and G.

The total differential of the system is

$$\begin{aligned} dY &= dC + dI + dG \\ dC &= f'(Y - T)\,(dY - dT) \\ dI &= h'(r)\,dr \\ dr &= m'(M)\,dM \end{aligned} \tag{49}$$

We need to solve this system for the differential changes dY, dC, dI, and dr in terms of the differential changes dM, dT, and dG in the exogenous policy variables M, T, and G. From the last two equations in (41), we can find dI and dr as follows

$$\begin{aligned} dr &= m'(M)\,dM \\ dI &= h'(r)\,m'(M)\,dM \end{aligned} \tag{50}$$

Inserting the expression for dI from (50) into the first two equations in (49) gives

$$\begin{aligned} dY - dC &= h'(r)\,m'(M)\,dM + dG \\ f'(Y - T)\,dY - dC &= f'(Y - T)\,dT \end{aligned} \tag{51}$$

This gives two equations to determine the two unknowns dY and dC in terms of dM, dG, and dT. We can write this in matrix form as follows

$$\begin{bmatrix} 1 & -1 \\ f'(Y - T) & -1 \end{bmatrix} \begin{bmatrix} dY \\ dC \end{bmatrix} = \begin{bmatrix} h'(r)\,m'(M)\,dM + dG \\ f'(Y - T)\,dT \end{bmatrix} \tag{52}$$

We can use Cramer's rule to solve this system. The determinant of the coefficient matrix is given by

$$D = \begin{vmatrix} 1 & -1 \\ f'(Y - T) & -1 \end{vmatrix} = (-1) - (-f'(Y - T)) = f'(Y - T) - 1 \tag{53}$$

First solving for dY we obtain

$$dY = \frac{\begin{vmatrix} h\prime(r)\; m\prime(M)\; dM + dG & -1 \\ f\prime(Y - T)\; dT & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ f\prime(Y - T) & -1 \end{vmatrix}} = \frac{\begin{vmatrix} h\prime(r)\; m\prime(M)\; dM + dG & -1 \\ f\prime(Y - T)\; dT & -1 \end{vmatrix}}{f\prime(Y-T)-1}$$

$$\begin{aligned} \Rightarrow dY &= \frac{-h\prime(r)\; m\prime(M)\; dM - dG + f\prime(Y-T)\; dT}{f\prime(Y-T)-1} \\ &= \frac{-h\prime(r)\; m\prime(M)}{f\prime(Y-T)-1}\, dM - \frac{1}{f\prime(Y-T)-1}\, dG + \frac{f\prime(Y-T)}{f\prime(Y-T)-1}\, dT \\ &= \frac{h\prime m\prime}{1-f\prime}\, dM - \frac{f\prime}{1-f\prime}\, dT + \frac{1}{1-f\prime}\, dG \end{aligned} \tag{54}$$

Then solving for dC we obtain

$$dC = \frac{\begin{vmatrix} 1 & h\prime(r)\; m\prime(M)\; dM + dG \\ f\prime(Y - T) & f\prime(Y - T)\; dT \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ f\prime(Y - T) & -1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & h\prime(r)\; m\prime(M)\; dM + dG \\ f\prime(Y - T) & f\prime(Y - T)\; dT \end{vmatrix}}{f\prime(Y-T)-1}$$

$$\begin{aligned} \Rightarrow dC &= \frac{f\prime(Y-T)\; dT - h\prime(r)\; m\prime(M)\; dM\; f\prime(Y-T) - dG\; f\prime(Y-T)}{f\prime(Y-T)-1} \\ &= \frac{-h\prime(r)\; m\prime(M)\; f\prime(Y-T)}{f\prime(Y-T)-1}\, dM - \frac{f\prime(Y-T)}{f\prime(Y-T)-1}\, dG + \frac{f\prime(Y-T)}{f\prime(Y-T)-1}\, dT \\ &= \frac{f\prime h\prime m\prime}{1-f\prime}\, dM - \frac{f\prime}{1-f\prime}\, dT + \frac{f\prime}{1-f\prime}\, dG \end{aligned} \tag{55}$$

We have now found the differentials dY, dC, dI, and dr as linear functions of dM, dT, and dG. If we set dM and dG equal to zero, then

$$\begin{aligned} dY &= -\frac{f\prime}{1-f\prime}\, dT \\ \Rightarrow \frac{\partial Y}{\partial T} &= -\frac{f\prime}{1-f\prime} \end{aligned} \tag{56}$$

Similarly $\frac{\partial r}{\partial T}$= 0 and $\frac{\partial I}{\partial T}$ = 0. Because we assumed that $0 < f' < 1$, $\frac{\partial Y}{\partial T} = \frac{-f'}{(1-f')} < 0$.
If dM, dT, and dG are small in absolute value, then

$$\Delta Y = Y(M_0 + dM,\, T_0 + dT,\, G_0 + dG) - Y(M_0,\, T_0,\, G_0) \approx dY$$

Literature Cited

Sydsaeter, Knut. *Topics in Mathematical Analysis for Economists*. New York: Academic Press, 1981.