

FUNCTIONS AND EQUATIONS

1. SETS AND SUBSETS

1.1. **Definition of a set.** A set is any collection of objects which are called its elements. If x is an element of the set S , we say that x belongs to S and write

$$x \in S.$$

If y does not belong to S , we write

$$y \notin S.$$

The simplest way to represent a set is by listing its members. We use the notation

$$A = \{1, 2, 4\}$$

to denote the set whose elements are the numbers 1, 2 and 4. We can also describe a set by describing its elements instead of listing them. For example we use the notation

$$C = \{x : x^2 + 2x - 3 = 0\}$$

to denote the set of all solutions to the equation $x^2 + 2x - 3 = 0$. The set is $\{1, -3\}$.

1.2. Subsets.

1.2.1. *Definition of a subset.* If all the elements of a set X are also elements of a set Y , then X is a subset of Y and we write

$$X \subseteq Y$$

where \subseteq is the set-inclusion relation. Consider the following five sets.

$$A = \{1, 2, 4\}$$

$$B = \{1, 2, 4, 5\}$$

$$C = \{1, 2, 4, 5\}$$

$$D = \{1, 2, 4, 5, 8, 10\}$$

$$E = \{1, 2, 3, 4, 5, 8, 10\}$$

The following relations are all true.

$$A \subseteq B, \quad A \subseteq C, \quad A \subseteq D, \quad A \subseteq E$$

$$B \subseteq C, \quad B \subseteq D, \quad B \subseteq E$$

$$C \subseteq B, \quad C \subseteq D, \quad C \subseteq E$$

$$D \subseteq E$$

1.2.2. *Definition of a proper subset.* If all the elements of a set X are also elements of a set Y , but not all elements of Y are in X , then X is a **proper** subset of Y and we write

$$X \subset Y$$

where \subset is the proper set-inclusion relation. The following relations are all true.

$$A \subset B, \quad A \subset C, \quad A \subset D, \quad A \subset E$$

$$B \subset D, \quad B \subset E$$

$$C \subset D, \quad C \subset E$$

$$D \subset E$$

1.2.3. *Definition of equality of sets.* Two sets are equal if they contain exactly the same elements, and we write

$$X = Y.$$

For example $B = C$.

1.2.4. *Examples of sets.*

a: all corn farmers in Iowa

b: all firms producing steel east of the Mississippi river in the United States

c: the set of all consumption bundles that a given consumer can afford

d: the set of all combinations of outputs that can be produced by a given set of inputs

1.3. Set operations.

1.3.1. *Intersections.* The intersection, W , of two sets X and Y is the set of elements that are in both X and Y . We write

$$W = X \cap Y = \{x : x \in X \text{ and } x \in Y\}.$$

The definition is symmetric: that is, $A \cap B = B \cap A$. Also, $A \cap B \subseteq A$ and $A \cap B = A \iff A \subseteq B$. As an example consider the following two sets and their intersection.

$$A = \{1, 2, 3\}$$

$$B = \{0, 1, 2, 3, 4\}$$

$$A \cap B = \{1, 2, 3\}$$

We also have $(A \cap B) \cap C = A \cap (B \cap C)$ and write this as $A \cap B \cap C$.

For example consider the sets X , Y , and Z .

$$X = \{1, 2, 3\}$$

$$Y = \{0, 1, 2, 3, 4\}$$

$$Z = \{1, 3, 4, 5\}$$

$$X \cap Y = \{1, 2, 3\}$$

$$Y \cap Z = \{1, 3\}$$

$$(X \cap Y) \cap Z = \{1, 3\}$$

$$X \cap (Y \cap Z) = \{1, 3\}$$

1.3.2. *Empty or null sets.* The empty set or the null set is the set with no elements. The empty set is written \emptyset . For example, if the sets A and B contain no common elements we can write

$$\begin{aligned} A &= \{1, 2, 3\} \\ B &= \{4, 5, 6\} \\ A \cap B &= \emptyset \end{aligned}$$

and these two sets are said to be disjoint.

1.3.3. *Unions.* The union of two sets A and B is the set of all elements in one or the other of the sets. We write

$$V = A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

We have

$$\begin{aligned} A \cup B &= B \cup A \\ A \cup (B \cup C) &= (A \cup B) \cup C = A \cup B \cup C \\ A &\subseteq A \cup B \text{ and} \\ A = A \cup B &\iff B \subseteq A \end{aligned}$$

As an example of the last statement consider the following two sets and their union.

$$\begin{aligned} A &= \{1, 2, 3, 4\} \\ B &= \{2, 4\} \\ A \cup B &= \{1, 2, 3, 4\} \end{aligned}$$

1.3.4. *Distributive laws.*

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \end{aligned}$$

Consider the following three sets.

$$\begin{aligned} A &= \{1, 2, 4\} \\ B &= \{1, 2, 3, 5\} \\ C &= \{2, 4, 6\} \end{aligned}$$

Then we have

$$A \cup B = \{1, 2, 3, 4, 5\}$$

$$A \cup C = \{1, 2, 4, 6\}$$

$$B \cup C = \{1, 2, 3, 4, 5, 6\}$$

$$A \cup B \cup C = \{1, 2, 3, 4, 5, 6\}$$

$$A \cap B = \{1, 2\}$$

$$A \cap C = \{2, 4\}$$

$$B \cap C = \{2\}$$

$$A \cap B \cap C = \{2\}$$

We also have

$$A \cap (B \cup C) = \{1, 2, 4\}$$

$$(A \cap B) \cup (A \cap C) = \{1, 2, 4\}$$

$$A \cup (B \cap C) = \{1, 2, 4\}$$

$$(A \cup B) \cap (A \cup C) = \{1, 2, 4\}$$

1.3.5. *Complements.* The complement of a set X is the set of elements of the universal set U that are not elements of X , and is written X^C . Thus

$$X^C = \{x : x \in U \text{ and } x \notin X\}.$$

We have

$$\emptyset^C = U$$

$$U^C = \emptyset$$

$$(A^C)^C = A$$

$$A \cup A^C = U$$

$$A \cap A^C = \emptyset$$

$$A \subseteq B \iff B^C \subseteq A^C$$

1.3.6. *De Morgan's laws.*

$$(A \cup B)^C = A^C \cap B^C$$

$$(A \cap B)^C = A^C \cup B^C$$

Consider the following three sets.

$$U = \{1, 2, 3, 4, 5\}$$

$$A = \{1, 2, 4\}$$

$$B = \{1, 2, 3, 5\}$$

Then

$$A \cap B = \{1, 2\}$$

$$A \cup B = \{1, 2, 3, 4, 5\} = U$$

$$A^C = \{3, 5\}$$

$$B^C = \{4\}$$

$$(A \cap B)^C = \{3, 4, 5\}$$

$$(A \cup B)^C = \emptyset$$

1.3.7. *Set difference or relative complement.* If A and B are two subsets of X , then define the set difference $B \setminus A$, or the relative complement of A in B , as the set of elements in B that are not in A . Thus

$$B \setminus A = \{x : x \in B \text{ and } x \notin A\}$$

We have $B \setminus A = B \cap A^C$.

Consider the following three sets.

$$X = \{1, 2, 3, 4, 5\}$$

$$A = \{1, 2, 4\}$$

$$B = \{1, 3, 5\}$$

Then

$$A \setminus B = \{2, 4\}$$

$$B \setminus A = \{3, 5\}$$

2. NUMBERS

2.1. **Natural numbers.** The natural numbers are the elements of the set

$$\mathbb{Z}_+ = \{1, 2, 3, \dots\}.$$

A group of three dots, \dots , called an ellipsis, indicates that the numbers continue in the indicated pattern. If the ellipsis contains nothing to the right, it is assumed that the pattern continues forever.

The natural numbers are closed under addition meaning that for numbers a and b that are elements of \mathbb{Z}_+ , then $a + b \in \mathbb{Z}_+$. Similarly, $a \times b \in \mathbb{Z}_+$ and we say that the natural numbers are closed under multiplication.

2.2. **Prime numbers.** A prime number is a natural number greater than one that is divisible only by itself and 1, i.e., $\{2, 3, 5, 7, 11, 13, 17, 19, 23, \dots\}$. The first 25 prime numbers are as follows.

TABLE 1. Prime Numbers

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

2.3. **Composite numbers.** A composite number is a natural number greater than one that is not prime.

$$\{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, \dots\}$$

2.4. **Integers.** The integers are the elements of the set

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

The integers are closed under addition, subtraction and multiplication. The positive integers are the same as the natural numbers.

2.5. **Rational numbers.** The rational numbers are the elements of the set

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z} - \{0\} \right\}.$$

The expression $\frac{a}{b}$ is called a fraction in the sense that a represents a portion of b . We call a the numerator of the fraction and b the denominator of the fraction. An integer n is also a rational number because $n = \frac{n}{1}$. The rational numbers are closed under addition, subtraction, multiplication and division.

2.6. **Irrational numbers.** Numbers that cannot be written as ratios or quotients of integers are called irrational numbers. Examples include $\sqrt{2}$, $\sqrt{5}$, $\sqrt{12}$, π . To see why $\sqrt{2}$ is not a rational number see Appendix A. Irrational numbers have decimal expansions that neither terminate nor become periodic.

2.7. **Real numbers.**

2.7.1. *Definition of the real numbers.* The union of the sets of rational and irrational numbers is the set of real numbers. We can think of the set of real numbers, \mathbb{R} , as extending along a line from zero to infinity in each direction having no breaks or gaps.

2.7.2. *Writing real numbers as decimal fractions.* We can write any real number as an infinite decimal fraction. For example we can write 354.651 as $3 \times 10^2 + 5 \times 10^1 + 4 \times 10^0 + 6 \times 10^{-1} + 5 \times 10^{-2} + 1 \times 10^{-3}$. In general a real number can be written as $x = \pm m.\beta_1\beta_2\beta_3\dots$, where m is an integer and β_n ($n = 1, 2, \dots$) is an infinite series of digits, each in the range 0 to 9.

Rational numbers can be written as finite or recurring decimal fractions, that is, after a certain place in the decimal representation, it either stops or continues to repeat a finite sequence of digits. For example,

$$\frac{3}{4} = 0.75$$

$$\frac{2}{3} = 0.6\bar{6}$$

$$\frac{4}{11} = 0.36\bar{36}$$

$$\frac{4}{13} = 0.307692\overline{307692}$$

Irrational numbers are non-repeating decimal fractions. For example,

$$\pi = 3.1415926535898\dots$$

$$\sqrt{2} = 1.4142135623731\dots$$

$$e = 2.7182818284590\dots$$

$$10^{1/3} = 2.1544346900319$$

Numbers of the form $n^{1/m}$ are irrational unless n is the m^{th} power of an integer. For example, $625^{1/4} = 5$, i.e., $625 = 5^4$. Or, for example, $27^{1/3} = 3$.

2.7.3. *Addition.* One can view addition as a process of *counting on* or *counting up*. Start with one of the numbers say m and view it as set with m objects. Then start counting the elements of the set containing n objects starting with $m+1$ for the first element of the second set. Consider the following example where $n = 3$ and $m = 5$. Label the first set A_1 and the second set A_2 .

$$A_1 = \{1, 2, 3\}$$

$$A_2 = \{1, 2, 3, 4, 5\}$$

Counting the objects in both sets we obtain 8 so that 3 added to 5 gives 8. We often use the word *plus* for addition and use the sign $+$. Then $3 + 5 = 8$.

2.7.4. *Multiplication.* Multiplication is the process of adding numbers together *multiple* times. For example, if we add 3 and 3 ($3 + 3$) we have two groups of 3, or 3 two times. If we add four and four and four ($4+4+4$) we have three groups of 4, or 4 three times. We sometimes use the word *times* to represent multiplication. We use the symbol \times or the symbol $*$ or the symbol \cdot to represent multiplication. So 3 times 2 is written 3×2 or $3 * 2$ or $3 \cdot 2$. When the context is clear, we can represent multiplication by writing two numbers next to each other with a space between them as in $3 \ 2 = 6$. In such cases we often enclose the numbers in parentheses as in $(3)(2) = 6$. It is useful to memorize or at least be able to construct a table listing the first 20 numbers multiplied by each other.

Notice that across the second row or down the second column is counting by twos and that across the fifth row or down the fifth column is counting by fives.

Also notice the following.

TABLE 2. Multiplication Table

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60
4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80
5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	96	100
6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120
7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140
8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160
9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180
10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220
12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260
14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280
15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300
16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320
17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340
18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360
19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380
20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400

- All multiples of 2 are even numbers, that is, they end in 0,2,4,6, or 8.
- The sum of the digits in all multiples of 3 is a lesser multiple of 3.
- All multiples of 5 end in 0 or 5.
- All multiples of 6 are even numbers and the sum of the digits is a lesser multiple of 3.
- A number is a multiple of 8 if the number formed by the last three digits is divisible by 8. For example, 7,120 is divisible by 8 because 120 is divisible by 8, i.e. $8 \times 15 = 120$.
- The sum of the digits in all multiples of 9 is 9 or a multiple of 9.
- All multiples of 10 end in 0.
- Numerals whose alternating sum of digits is divisible by 11 represent numbers divisible by 11. Consider, for example, 121. The alternating sum is $1 - 2 + 1 = 0$, and 0 is divisible by 11, so 121 is divisible by 11. Or consider 2728 where $2 - 7 + 2 - 8 = -11$, which is divisible by 11.
- To determine if a number is divisible by 7, take the last digit off the number, double it and subtract the doubled number from the remaining number. If the result is evenly divisible by 7 (e.g. 14, 7, 0, -7, etc.), then the number is divisible by seven. This may need to be repeated several times. For example consider 196. Drop the 6 and then double 6 to get 12. $19 - 12 = 7$ which is divisible by 7.
- The items in the diagonal of the table representing each number multiplied by itself are called squares. Note that the differences differ by 2, i.e., 5,7,9,11,13,15,17,19, 21, 23, 25, 27, 29, 31, 33, 35, 37, and 39.
- The table is symmetric.

2.7.5. *Field properties of addition and multiplication of real numbers.* For any real numbers a, b, c:

2.7.6. *Definition of subtraction.* For any real numbers a and b

$$a - b = a + -b.$$

TABLE 3. Field Properties of Real Numbers

#		Addition – A	Multiplication – M
0	Closure	$a + b \in \mathbb{R}$	$ab \in \mathbb{R}$.
1	Commutative	$a + b = b + a$	$ab = ba$
2	Associative	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
3	Identity	There is a number 0 with $a + 0 = 0 + a = a$.	There is a number 1 with $(a)(1) = (1)(a) = a$.
4	Inverse	There is a number -a with $a + -a = -a + a = 0$.	If $a \neq 0$, there is a number $\frac{1}{a}$ with $(a)\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)(a) = 1$. $\frac{1}{a}$ is also written as a^{-1} and $aa^{-1}=1$.
D	Distributive	$a(b + c) = ab + ac$	

2.7.7. *Definition of division.* For the real numbers a and b with $b \neq 0$,

$$a \div b = a/b = \frac{a}{b} = (a)\left(\frac{1}{b}\right) = (a)(1/b).$$

2.7.8. *Some uniqueness theorems for real numbers derived from the field properties and the definitions.*

TABLE 4. Uniqueness theorems for real numbers

Theorem #	Description	Condition	Result
Theorem 1a	Uniqueness of 0	If z and a are elements of \mathbb{R} such that $z + a = a$ then	$z = 0$
Theorem 1b	Uniqueness of 1	If u and $b \neq 0$ are elements of \mathbb{R} such that $ub = b$ then	$u = 1$
Theorem 2a	Uniqueness of -a	If a and b are elements of \mathbb{R} such that $a + b = 0$ then	$b = -a$
Theorem 2b	Uniqueness of $\frac{1}{a}$	If $a \neq 0$ and b are elements of \mathbb{R} such that $ab = 1$ then	$b = \frac{1}{a}$

3. OPERATIONS WITH REAL NUMBERS

3.1. **Summary of results.** Using the field properties in section 2.7.5, one can state the following properties of real numbers.

TABLE 5. Operations with Real Numbers

#	Property	Example	Notes
1	$a - b = a + (-b)$	$6 - 2 = 6 + (-2) = 4$	Subtraction
2	$a - (-b) = a + b$	$4 - (-3) = 4 + 3 = 7$	
3	$-a = (-1)a$	$-4 = (-1)4$	
4	$a(b+c) = ab + ac$	$3(2+5) = (3)(2) + (3)(5) = 21$	Distributive
5	$a(b-c) = ab - ac$	$3(1-4) = (3)(1) - (3)(4) = -9$	
6	$-(a+b) = (-1)(a+b) = -a-b$	$-(6+3) = -6-3 = -9$	Distributive
7	$-(a-b) = -a+b$	$-(4-3) = -4+3 = -1$	
8	$-(-a) = a$	$-(-2) = 2$	
9	$a(0) = 0$	$3(0) = 0$	
10	$(-a)(b) = -(ab) = a(-b)$	$(-3)(4) = -(3 \cdot 4) = (3)(-4) = -12$	
11	$(-a)(-b) = ab$	$(-3)(-5) = 3 \times 5 = 15$	
12	$\frac{a}{1} = a$	$\frac{2}{1} = 2$	
13	$\frac{a}{b} = a \left(\frac{1}{b} \right)$	$\frac{2}{5} = 2 \left(\frac{1}{5} \right)$	
14	$\frac{a}{-b} = -\left(\frac{a}{b} \right) = \frac{-a}{b}$	$\frac{2}{-9} = -\left(\frac{2}{9} \right) = \frac{-2}{9}$	
15	$\frac{-a}{-b} = \frac{a}{b}$	$\frac{-2}{-7} = \frac{2}{7}$	
16	$\frac{0}{a} = 0$ when $a \neq 0$	$\frac{0}{7} = 0$	
17	$\frac{a}{a} = 1$ when $a \neq 0$	$\frac{2}{2} = 1, \frac{-5}{-5} = 1$	
18	$a \left(\frac{b}{a} \right) = b$	$2 \left(\frac{7}{2} \right) = 7$	
19	$a \cdot \frac{1}{a} = 1$ when $a \neq 0$	$2 \cdot \frac{1}{2} = 1$	
20	$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{2}{3} \cdot \frac{4}{5} = \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$	
21	$\frac{ab}{c} = \left(\frac{a}{c} \right) b = a \left(\frac{b}{c} \right)$	$\frac{2 \cdot 7}{3} = \frac{2}{3} \cdot 7 = 2 \cdot \frac{7}{3}$	
22	$\frac{a}{bc} = \left(\frac{a}{b} \right) \left(\frac{1}{c} \right) = \left(\frac{1}{b} \right) \left(\frac{a}{c} \right)$	$\frac{2}{3 \cdot 7} = \frac{2}{3} \cdot \frac{1}{7} = \frac{1}{3} \cdot \frac{2}{7}$	
23	$\frac{a}{b} = \left(\frac{a}{b} \right) \left(\frac{c}{c} \right) = \left(\frac{ac}{bc} \right)$ when $c \neq 0$	$\frac{2}{7} = \left(\frac{2}{7} \right) \left(\frac{5}{5} \right) = \frac{2 \cdot 5}{7 \cdot 5}$	
24	$\frac{a}{b(-c)} = \frac{a}{(-b)(c)} = \frac{-a}{bc} = \frac{-a}{(-b)(-c)} = -\frac{a}{bc}$	$\frac{2}{3(-5)} = \frac{2}{(-3)(5)} = \frac{2}{15} = \frac{-2}{(-3)(-5)} = -\frac{2}{3(5)} = -\frac{2}{15}$	
25	$\frac{a(-b)}{c} = \frac{(-a)b}{c} = \frac{ab}{-c} = \frac{(-a)(-b)}{-c} = -\frac{ab}{c}$	$\frac{2(-3)}{5} = \frac{(-2)(3)}{5} = \frac{2(3)}{-5} = \frac{(-2)(-3)}{-5} = \frac{2(3)}{-5} = -\frac{6}{5}$	
26	$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	$\frac{2}{9} + \frac{3}{9} = \frac{2+3}{9} = \frac{5}{9}$	
27	$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$	$\frac{2}{9} - \frac{3}{9} = \frac{2-3}{9} = \frac{-1}{9}$	
28	$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$	$\frac{4}{5} + \frac{2}{3} = \frac{4 \cdot 3 + 5 \cdot 2}{5 \cdot 3} = \frac{22}{15}$	
29	$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$	$\frac{4}{5} - \frac{2}{3} = \frac{4 \cdot 3 - 5 \cdot 2}{5 \cdot 3} = \frac{2}{15}$	
30	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$	$\frac{2}{3} \div \frac{7}{5} = \frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$	
31	$\frac{a}{b} = a \div \frac{b}{c} = a \cdot \frac{c}{b} = \frac{ac}{b}$	$\frac{2}{3} = 2 \div \frac{3}{5} = 2 \cdot \frac{5}{3} = \frac{2 \cdot 5}{3} = \frac{10}{3}$	
32	$\frac{a}{b} \div c = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$	$\frac{2}{3} \div 5 = \frac{2}{3} \cdot \frac{1}{5} = \frac{2}{3 \cdot 5} = \frac{2}{15}$	

3.2. Working with fractions.

3.2.1. *Properties of fractions.*

1. If $a, b, c,$ and d are real numbers and b and d are not zero then *Equality of Fractions* implies that

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc$$

2. If $a, b,$ and x are real numbers and b and x are not zero then the *Fundamental Property of Fractions* implies that

$$\frac{ax}{bx} = \frac{a}{b}$$

3.2.2. *Multiplying fractions.* To multiply two fractions, multiply the numerators and denominators separately, i.e.,

$$\frac{x}{y} \times \frac{a}{b} = \frac{x}{y} \cdot \frac{a}{b} = \frac{xa}{yb} = \frac{xa}{yb}$$

For example,

$$\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{2 \cdot 3}{2 \cdot 2 \cdot 3} = \frac{1}{2}$$

3.2.3. *Dividing fractions.* To divide two fractions, use the inverse multiplication rule, i.e.,

$$\frac{\frac{x}{y}}{\frac{a}{b}} = \frac{x}{y} \cdot \frac{b}{a} = \frac{xb}{ya}$$

For example,

$$\frac{\frac{3}{4}}{\frac{1}{2}} = \left(\frac{3}{4}\right) \left(\frac{2}{1}\right) = \frac{6}{4} = \frac{3}{2}$$

3.2.4. *Simplifying fractions.* The *Fundamental Property of Fractions* allows us to write any fraction in an infinite number of ways by multiplying the numerator and denominator by the same number and then repeating this process for a different number. For example

$$\frac{1}{2} = \frac{2}{4} = \frac{20}{40} = \frac{50}{100} = \frac{2.5}{5} = \frac{.25}{.50}$$

$$\frac{7}{13} = \frac{14}{26} = \frac{35}{65} = \frac{175}{325} = \frac{2.3\bar{3}}{4.3\bar{3}} = \frac{1}{1.857142857142}$$

A fraction which has the smallest possible integers in both the numerator and denominator is said to be in reduced form. The process of eliminating these integer *multipliers* in the numerator and denominator of a fraction is called simplifying the fraction or writing the fraction in reduced form. For example, the fraction $\frac{20}{40}$ can be written in a different form by eliminating the *multiplier* 5 as follows

$$\frac{20}{40} = \frac{5 \times 4}{5 \times 8} = \frac{5}{5} \times \frac{4}{8} = 1 \times \frac{4}{8} = \frac{4}{8}$$

The *multiplier* 5 is often called a factor. The fraction $\frac{4}{8}$ can then be written in a different form by eliminating the *multiplier* 4 as follows

$$\frac{4}{8} = \frac{4 \times 1}{4 \times 2} = \frac{4}{4} \times \frac{1}{2} = 1 \times \frac{1}{2} = \frac{1}{2}$$

Notice that the product of the two factors we eliminated from $\frac{20}{40}$ is $5 \times 4 = 20$. This implies we could find the reduced form of $\frac{20}{40}$ in one step as follows

$$\frac{20}{40} = \frac{20 \times 2}{20 \times 2} = \frac{20}{20} \times \frac{1}{2} = 1 \times \frac{1}{2} = \frac{1}{2}$$

We can *reduce* a fraction by finding common multipliers or factors in the numerator and denominator of a fraction. A *fail safe* way to do this is to factor a number into the primes of which it is made. We can factor 12 as $2 \times 2 \times 3$. And 2 and 3 are both prime.

Definition 1 (Prime Factorization). The *prime factorization* of a number is a group of prime numbers that multiplied together is equal to that number.

For example $357 = 3 \times 7 \times 17$ and $1729 = 7 \times 13 \times 19$. The prime factors in a number may well repeat as in $192 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3$.

To find the prime factorization of a number we can use the following algorithm.

Definition 2 (Algorithm for Prime Factorization).

1. If the number is divisible by 2, then write down 2 as a factor and proceed with the prime factorization of the quotient.
2. If the number is not divisible by 2, then see if the number is divisible by 3. If the number is divisible by 3, then write down 3 as a factor and proceed with the prime factorization of the quotient.
3. If the number is not divisible by 2 or 3, then see if the number is divisible by 5. If the number is divisible by 5, then write down 5 as a factor and proceed with the prime factorization of the quotient.
4. If the number is not divisible by 2 or 3 or 5, then see if the number is divisible by 7. If the number is divisible by 7, then write down 7 as a factor and proceed with the prime factorization of the quotient.
5. Continue in this fashion until the quotient is a prime number or the prime factor you are trying is the smallest prime less than the square root of the smallest perfect square larger than the number being factored.

Consider the prime factorization of 693.

1. 693 is not divisible by 2, so proceed with 3.
2. The sum of the digits in 693 is 18 so 693 is divisible by 3. Forming the quotient we obtain $\frac{693}{3} = 231$.
3. 231 is not divisible by 2, so proceed with 3.
4. The sum of the digits in 693 is 6 so 231 is divisible by 3. Forming the quotient we obtain $\frac{231}{3} = 77$.
5. 77 is not divisible by 2, so proceed with 3.
6. 77 is not divisible by 3, so proceed with 5.
7. 77 is not divisible by 5, so proceed with 7.
8. 77 is divisible by 7. The quotient is 11 so we are finished because 11 is a prime number.

The prime factorization of 693 is then $3 \times 3 \times 7 \times 11$.

Finding the prime factorizations of the numerator and denominator of a fraction and then canceling like terms will always reduce a fraction to its simplest form, i.e. this will always identify the factors that are common in the numerator and denominator of a fraction. In many cases, however, it may be easier to look for larger common factors than those in the prime factorization. Consider the following example

$$\frac{420}{378} = \frac{2 \times 2 \times 3 \times 5 \times 7}{2 \times 3 \times 3 \times 3 \times 7} = \frac{\cancel{2} \times 2 \times \cancel{3} \times 5 \times \cancel{7}}{\cancel{2} \times \cancel{3} \times 3 \times 3 \times \cancel{7}} = \frac{2 \times 5}{3 \times 3} = \frac{10}{9}$$

We might do this faster by noticing that the numerator and denominator are divisible by 2 because they are even and divisible by 3 because the sum of the digits is divisible by 3 (6 and 18). This would then give

$$\frac{420}{378} = \frac{6 \times 70}{6 \times 63} = \frac{\cancel{6} \times 70}{\cancel{6} \times 63} = \frac{70}{63}$$

Remembering our seven times table we can simplify as

$$\frac{420}{378} = \frac{70}{63} = \frac{7 \times 10}{7 \times 9} = \frac{10}{9}$$

When using this approach the following facts or rules are useful.

1. All even numbers have 2 as a factor.
2. If the last two digits of a number divide by 4, the whole number divides by 4. For example the last two digits of 124 divide by 4 and $124/4 = 31$.
3. If the last three digits of a number divide by 8, the whole number divides by 8. For example the last three digits of 3128 divide by 8 and $3128/8 = 391$. Similarly for 16, 32, ...
4. If a number ends in 5 or 0, it is divisible by 5.
5. If the last two digits of a number are divisible by 25, then the whole number is divisible by 25. For example, the last two digits of 675 are divisible by 25 so 675 is divisible by 25, i.e., $675/25 = 27$. Similarly for 625, 3125, ...
6. All numbers ending in zero are divisible by 10. Numbers ending in two zeroes are divisible by 100 and so on.
7. If the sum of the digits in a number is divisible by 3, the whole number is divisible by 3.
8. If the sum of the digits in a number is divisible by 9, the whole number is divisible by 9.
9. If the sum of the digits in a number is divisible by 3 and the number is even, the whole number is divisible by 6.
10. For 11, add alternate digits, if the sums are identical, differ by 11, or by multiple of 11, then the number is divisible by 11. Consider 121 where $1 + 1 = 2$ and $121/11=11$. Or 28,347 where $2 + 3 + 7 = 8 + 4$ so that $28347/11 = 2577$. Or consider 869 where $8 + 9 = 17$ and $17 - 6 = 11$. Therefore $869/11=79$.

A useful fact to remember in factoring a number is that if it is not a prime number, it will have at least one factor less than the square root of the number which is the smallest perfect square greater than the number in question. Consider, for example, 139. None of the rules we listed seem to help in factoring 139. Because 144 is the smallest perfect square greater than 139, we know that the only primes numbers we need to try are 2, 3, 5, 7, and 11. We can easily eliminate 2, 3, and 5. We can eliminate 11 because $10 \neq 3$ and $10 - 3$ is not divisible by 11. Or we can just remember our eleven times table. So we can try 7.

$$\begin{array}{r} 19 \\ 7 \overline{)139} \\ \underline{70} \\ 69 \\ \underline{63} \\ 6 \end{array}$$

We find that $\frac{139}{7} = 19\frac{6}{7}$. This does not divide evenly so we know that 139 is a prime number.

3.2.5. *Adding and subtracting fractions.* To add or subtract two fractions, the denominators must be the same (common denominator). If the denominators are the same, then the addition is completed by adding the numerators, i.e.,

$$\frac{x}{y} + \frac{a}{y} = \frac{x+a}{y}$$

For example

$$\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

If the denominators in the fractions we want to add are not the same, we must rewrite them so that they have the same denominator. For example

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

or

$$\frac{3}{24} + \frac{5}{16} = \frac{6}{48} + \frac{15}{48} = \frac{21}{48}$$

But the $\frac{21}{48}$ is not in reduced form, i.e.,

$$\frac{21}{48} = \frac{\cancel{3} \times 7}{\cancel{3} \times 16} = \frac{7}{16}$$

If we first reduce $\frac{3}{24}$ to $\frac{1}{8}$, we obtain

$$\frac{1}{8} + \frac{5}{16} = \frac{2}{16} + \frac{5}{16} = \frac{7}{16}$$

So making sure all fractions in a product are in reduced form will help us obtain a sum that is in reduced form. But how do we find common denominators. In the examples so far, a reasonable choice has seemed obvious. Is there a more systematic way? One way that will always work is to simply multiply the denominators together to obtain the common denominator and then multiply the numerators by the denominators of the other fractions. For example

$$\begin{aligned} \frac{1}{8} + \frac{5}{16} &= \frac{16}{16 \times 8} + \frac{8 \times 5}{16 \times 8} \\ &= \frac{16}{128} + \frac{40}{128} = \frac{56}{128} \\ &= \frac{14}{32} = \frac{7}{16} \end{aligned}$$

Of course it would have been easier to convert $\frac{1}{8}$ to $\frac{2}{16}$ at the beginning and obtain

$$\frac{1}{8} + \frac{5}{16} = \frac{2}{16} + \frac{5}{16} = \frac{7}{16}$$

Multiplying all the denominators together will often lead to a very large number for the denominator. An alternative approach is to find the least common denominator among the denominators of the fractions being added or subtracted.

Definition 3 (Least Common Denominator(LCD)). The least common denominator for a group of fractions is the smallest number that can be divided by all the denominators.

For example, 6 is the least common denominator for 2 and 3, and 4 is the least common denominator of 2 and 4, while 48 is the least common denominator of 12 and 16. Notice that 48 is much less than $12 \times 16 = 192$.

The most straightforward way to find the least common denominator for a group of fractions is to first find the prime factorization of each denominator. We then create the least common denominator by using each factor the greatest number of times that it appears in any one denominator. For example, consider the least common denominator for 12 and 16. First find the prime factorization of each.

$$12 = 2 * 2 * 3$$

$$16 = 2 * 2 * 2 * 2$$

The least common denominator is then

$$2 * 2 * 2 * 2 * 3 = 48$$

Consider a slightly more complicated example where we want to add $\frac{1}{2}$, $\frac{5}{12}$, $\frac{7}{16}$, $\frac{13}{20}$ and $\frac{6}{35}$. First find the prime factorization of each number.

$$12 = 2 * 2 * 3$$

$$16 = 2 * 2 * 2 * 2$$

$$20 = 2 * 2 * 5$$

$$35 = 5 * 7$$

The least common denominator is then $(2*2*2*2)*3*5*7=1680$. Notice that $(2*2*2*2)$, $(2*2*3)$, $(2*2*5)$, and $(5*7)$ are all contained in this expression. Now multiply each fraction by the appropriate factor so that the denominators are all 1680.

$$\frac{1}{2} = \frac{1}{2} * \frac{2}{2} * \frac{2}{2} * \frac{2}{2} * \frac{3}{3} * \frac{5}{5} * \frac{7}{7} = \frac{840}{1680}$$

$$\frac{5}{12} = \frac{5}{12} * \frac{2}{2} * \frac{2}{2} * \frac{5}{5} * \frac{7}{7} = \frac{700}{1680}$$

$$\frac{7}{16} = \frac{7}{16} * \frac{3}{3} * \frac{5}{5} * \frac{7}{7} = \frac{735}{1680}$$

$$\frac{13}{20} = \frac{13}{20} * \frac{2}{2} * \frac{2}{2} * \frac{3}{3} * \frac{7}{7} = \frac{1092}{1680}$$

$$\frac{6}{35} = \frac{6}{35} * \frac{2}{2} * \frac{2}{2} * \frac{2}{2} * \frac{2}{2} * \frac{3}{3} = \frac{288}{1680}$$

Now add and reduce

$$\begin{aligned} \frac{1}{2} + \frac{5}{12} + \frac{7}{16} + \frac{13}{20} + \frac{6}{35} &= \frac{840}{1680} + \frac{700}{1680} + \frac{735}{1680} + \frac{1092}{1680} + \frac{288}{1680} \\ &= \frac{840 + 700 + 735 + 1092 + 288}{1680} \\ &= \frac{3655}{1680} = \frac{5 \times 731}{5 \times 336} \\ &= \frac{731}{336} \\ &= 2 \frac{59}{336} \end{aligned}$$

3.2.6. Separating Fractions. Given the rule for adding and subtracting fractions we can always write a sum over a denominator with a single term as a sum of fractions with the numerator of each coming from the respective terms to be added in the numerator of the original fraction and the denominator being the common denominator. Consider the following example.

$$\begin{aligned}\frac{80 + 23 + 7 + 16}{40} &= \frac{80}{40} + \frac{23}{40} + \frac{7}{40} + \frac{16}{40} \\ &= \frac{126}{40}\end{aligned}$$

There is no simple way to reduce fractions with more than one term in the denominator as in $\frac{15}{3+x}$. On the other hand, $\frac{3+x}{15} = \frac{3}{15} + \frac{x}{15} = \frac{1}{5} + \frac{x}{15}$.

3.3. Rules of exponents.

3.3.1. *Product rule for exponents.* If m and n are natural numbers then

$$x^m x^n = x^{m+n}$$

Note that the product rule applies to exponential expressions with the same base. A product of two powers with different bases such as $x^3 y^4$, cannot be simplified.

Consider the following examples.

$$\begin{aligned}5^3 5^2 &= (5 * 5 * 5) * (5 * 5) = 5 * 5 * 5 * 5 * 5 = 5^5 \\ x^p x^{2p} &= x^p * x^p * x^p = x^{3p}\end{aligned}$$

3.3.2. *Power rules for exponents.* If m and n are natural numbers then

$$\begin{aligned}(x^m)^n &= x^{mn} \\ (xy)^n &= x^n y^n \\ \left(\frac{x}{y}\right)^n &= \frac{x^n}{y^n}, \quad y \neq 0\end{aligned}$$

For example,

$$\begin{aligned}(4^2)^3 &= 16^3 = 16 * 16 * 16 = 256 * 16 = 4096 = 4^6 = 4 * 4 * 4 * 4 * 4 * 4 = 64 * 64 = 4096 \\ (4 * 5)^3 &= 20^3 = 20 * 20 * 20 = 400 * 20 = 8000 = 4^3 * 5^3 = 64 * 125 = 8000\end{aligned}$$

3.3.3. *Zero rule for exponents.*

$$x^0 = 1, \quad x \neq 0$$

3.3.4. *Negative exponents.* If n is an integer and $x \neq 0$, then

$$\begin{aligned}x^{-n} &= \frac{1}{x^n} \\ \frac{1}{x^{-n}} &= x^n\end{aligned}$$

For example $3^{-2} = \frac{1}{9}$.

3.3.5. *Quotient rule for exponents.* If m and n are integers and $x \neq 0$, then

$$\frac{x^m}{x^n} = x^{m-n}$$

For example,

$$\frac{3^4}{3^2} = \frac{3 * 3 * 3 * 3}{3 * 3} = \frac{81}{9} = 9 = 3^{4-2} = 3^2 = 9$$

Notice that when the numerator and denominator contain the same number, but to different powers, we can cancel terms just like we did in simplifying fractions.

Or consider the following example.

$$\frac{3^4 * x^2 * y^3}{3 * x * y^2} = 3^3 * x * y = 27xy$$

3.3.6. *A fraction to a negative power.* If n is a natural number, then

$$\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n, \quad x \neq 0 \text{ and } y \neq 0$$

For example $\left(\frac{2}{3}\right)^{-2} = \frac{1}{\left(\frac{2}{3}\right)^2} = \frac{1}{\frac{4}{9}} = \frac{9}{4} = \frac{3^2}{2^2} = \left(\frac{3}{2}\right)^2$.

3.3.7. *Fractional exponents.*

(i) $x^{\frac{p}{q}}$ where p is an integer, q is a natural number, and $x > 0$.

$$\begin{aligned} x^{\frac{p}{q}} &= \left(x^{1/q}\right)^p \\ &= \left(x^p\right)^{1/q} \\ &= \sqrt[q]{x^p} \end{aligned}$$

Consider the following example.

$$\begin{aligned} 16^{\frac{3}{4}} &= \left(16^{1/4}\right)^3 = 2^3 = 8 \\ &= \left(16^3\right)^{1/4} = 4096^{1/4} = 8 \\ &= \sqrt[4]{16^3} = \sqrt[4]{4096} = 8 \end{aligned}$$

(ii) $x^{\frac{p}{q}}$ where p is an integer, q is an odd number, and $x < 0$.

For this rule to work, the fraction $\frac{p}{q}$ must be reduced to its lowest terms. Then

$$\begin{aligned} x^{\frac{p}{q}} &= \left(x^p\right)^{1/q} \\ &= \sqrt[q]{x^p} \end{aligned}$$

Consider the following example.

$$\begin{aligned} (-8)^{\frac{2}{3}} &= \left((-8)^2\right)^{1/3} = 64^{1/3} = 4 \\ &= \left((-8)^{1/3}\right)^2 = (-2)^2 = 4 \end{aligned}$$

3.3.8. *Some more examples.*

$$1. \frac{x^3}{yx^2} = \frac{x}{y}$$

$$2. (2x)^{-1} = \frac{1}{2x}$$

$$3. 2x^{-1} = \frac{2}{x}$$

$$4. \frac{(3x)^3 2^4}{9x4x^2} = 12$$

$$5. \frac{3^3 * 2^2 * 5^3}{4^2 * 3^{-1}} = \frac{3^4 * 5^3}{2^2}$$

$$6. \frac{4^2 * 6^2}{3^3 * 2^3} = \frac{8}{3}$$

$$7. \frac{x_1^3 x_2^{1/2}}{x_1^2} = \frac{x_1 x_2^{1/2}}{1} = x_1 x_2^{1/2}$$

$$8. \frac{t^p t^q - 1}{t^r t^s - 1} = t^{p+q-r-s}$$

$$9. \frac{1}{4} \frac{Ax_1^{1/4} x_2^{1/2}}{x_1} = \frac{1}{4} Ax_1^{-3/4} x_2^{1/2}$$

$$10. \alpha_2 \frac{x_1^{\alpha_2} x_2^{\alpha_2}}{x_2} = \alpha_2 x_1^{\alpha_2} x_2^{\alpha_2 - 1}$$

3.4. **Order of operations.** If an expression does not have parentheses or grouping symbols, follow these steps,

1. Find the values of any exponential expressions
2. Perform all multiplications and/or divisions, working from left to right.
3. Perform all additions and/or subtractions, working from left to right.

4. EXPRESSIONS AND EQUATIONS

4.1. **Variables.** A variable is a symbol that can be replaced by any one of a set of numbers or other objects. For example, we might pick a variable denoted by r . Because r is a variable, it can then be replaced by various members of a given set such as the integers. For example, if r is a variable representing the age of an arbitrary person in years, where fractions of years are rounded down to the next integer, r could be replaced by any of the numbers $\{0, 1, 2, 3, \dots, 121, 122, \dots\}$ where the upper bound on the set is a bit uncertain. (If one likes Biblical dating, 969 might be a reasonable number.)

4.2. Combining variables and numbers to form expressions.

4.2.1. *Definition of an expression.* When numbers and variables are combined, the result is called an algebraic expression, or simply an expression.

4.2.2. *Some examples.*

a: Five times the same number is written as $5x$

b: 12 less than twice a number is written as $2x - 12$

c: The product of output (y) and price (p) less the cost of producing that output $c(y)$ is written $py - c(y)$.

4.2.3. *Evaluating an expression.* Substituting for the variables and calculating a result is called evaluating an expression. For example, consider the expression

$$\frac{1}{2}bh.$$

If b is 5 and h is 6, then the expression is evaluated as $\frac{1}{2}(5)(6) = 15$.

4.2.4. *Factoring expressions.* Just as we factored numbers, we can factor algebraic expressions. For example $63 = 3 \times 3 \times 7$. And $5x^2 + 15x = 5x(x + 3)$. Here are some more examples.

1. $12x^2 - 14x - 6 = (6x + 2)(2x - 3)$

2. $x^2 - x + \frac{1}{4} = (x - \frac{1}{2})^2$

3. $4u^2 + 8u + 4 = (2u + 2)^2 = 4(u + 1)^2$

4. $2x^2y - 6x + 6xy - 18 = (xy - 3)(2x + 6)$

4.2.5. *The verbs of algebra.* The most common verbs used in algebra are

$=$	equal
$>$	greater than
$<$	less than
\geq	greater than or equal to
\leq	less than or equal to
\neq	not equal to
\approx	approximately equal to

4.2.6. *Combining expressions and verbs to form sentences.*

- (i) Definition of a sentence. We can combine expressions and verbs to form sentences.
- (ii) Sentences that are assumed to be true are either definitions of terms or postulates.
- (iii) Sentences that are either true or false are called statements or propositions. A true proposition might be "All individuals who can breathe on their own are alive." A false proposition might be "All individuals who can breathe on their own are alert and able to speak."
- (iv) An open proposition is one that is true for some values of the variables but not others. For example, the proposition $x^2 - 1 = 0$ is true for some values of x but not all values of x .
- (v) Statements that are shown to be true using definitions and postulates are called theorems.

4.2.7. *Equations.* An equation is a sentence stating that two expressions are equal. For example we might have the equation

$$\frac{1}{2}b + c = \frac{1}{2}c + d.$$

An equation is an open proposition in that it may be true for some values of the variables but not others.

4.2.8. *Formulas.* A formula is a sentence stating that a single variable is equal to an expression with one or more variables on the other side. Thus

$$c = 2d - b$$

is both an equation and a formula. But $a + b = b + a$ is an equation but not a formula. If we evaluate the expression in a formula and substitute the result for the variable on the other side, the equation will be true for those values of the variables.

4.2.9. *Examples of equations.* Here are some examples of typical equations one might encounter.

1: $2x - 10 = -3x$

2: $2x - 16 = x + y$

3: $x + y = 5x - 3$

4: $\frac{x}{y} = 4$

5: $x + 3y = 10x + 21$

6: $\frac{1}{2}y - 6x = 4x - 3$

7: $\frac{x}{3} + 2y = \frac{14x}{6} + 2$

8: $y = 5x - 4y + 25$

9: $-14x + 3y = \frac{y}{5} - 28$

10: $6x + 5y = 21x - 2y$

11: $x + y = 5$

12: $\frac{2}{3} + \frac{3x}{4} = \frac{y}{2}$

4.2.10. *Some theorems useful in analyzing equations derived from the field properties and definitions.* Some theorems that are useful in analyzing and simplifying equations are contained in table 6.

4.3. **Factoring quadratic expressions.** Some of the most common expressions we will encounter are quadratic expressions of the form $ax^2 + bx + c$. For example $3x^2 + 7x - 6$ is a quadratic expression. Just as we can factor numbers into the product of two other numbers, we can factor quadratic polynomials into the product of two linear expressions. For example $x^2 + 6x + 8 = (x + 1)(x + 2)$, $6x^2 + 11x - 10 = (2x + 5)(3x - 2)$ and $3x^2 + 7x - 6 = (3x - 2)(x + 3)$ The easiest cases are when $a = 1$ or we have $x^2 + bx + c$.

4.3.1. *Factoring quadratic expressions when $a = 1$.* When a in the quadratic expression is equal to one we have $x^2 + bx + c$ and there will be two factors of the form

$$(a) (x + m)(x + n) \text{ where we let } m \text{ and } n \text{ be of either sign}$$

or

$$(b) (x \pm m)(x \pm n) \text{ where } m \text{ and } n \text{ are both positive}$$

It is usually easiest to think of a solution using (a).

1. We want to find m and n such that
 - i. $m + n = b$
 - ii. $mn = c$
2. The signs on m and n can be both positive, both negative, or one positive and one negative.
 - i. If the sign on c in the quadratic expression is positive, then the signs on m and n must both be the same.
 - ii. If the sign on c in the quadratic expression is negative, then m and n must have opposite signs.

The idea then is to find two numbers m and n whose product is equal to c and whose sum or difference is equal to b . One might list the solution approach in steps as follows.

1. Find all pairs of (m, n) such that $mn = c$.
2. Determine if one of these pairs satisfies $m + n = b$

TABLE 6. Theorems Useful for Analyzing Equations

Theorem #	Description	Condition	Result
Theorem 3a	Unique solutions to $a + x = b$	Let a, b be arbitrary elements of R , then	$a + x = b$ has the unique solution $x = (-a) + b$
Theorem 3b	Unique solutions to $ax = b$	Let a, b be arbitrary elements of R , then	if $a \neq 0$, $ax = b$ has the unique solution $x = \frac{1}{a}b$
Theorem 4a	Multiplication property of 0	For any $a \in R$	$(a)(0) = 0$
Theorem 4b	Multiplication property of -1	For any $a \in R$	$(a)(-1) = -a$
Theorem 4c	Opposite of an opposite	For any $a \in R$	$-(-a) = a$
Theorem 4d	Product of (-1) and (-1)	For any $a \in R$	$(-1)(-1) = 1$
Theorem 4e	Opposite of a sum	For all a and $b \in R$	$-(a+b) = -a+-b$
Theorem 5	Cancellation for addition	Let x, y, z , be arbitrary elements of R . If $x + y = x + z$ then	$y = z$
Theorem 6a	Product of two factors = 0	Let a and b be elements of R . We have $ab = 0$ if and only if	$a = 0$ or $b = 0$ or both
Theorem 6b	Product of two factors = 0	Let a and b be elements of R . We have $a \neq 0$ and $b \neq 0$ iff	$ab \neq 0$
Theorem 7a	Reciprocal of a reciprocal	Let a be an element of R . If $a \neq 0$, then $1/a \neq 0$ and	$\frac{1}{\left(\frac{1}{a}\right)} = a$
Theorem 7b	Cancellation for multiplication	Let a, b, c , be elements of R . If $ab = ac$ and $a \neq 0$, then	$b = c$
Theorem 7c	Reciprocal of a product	Let a and b be elements of R . If $a \neq 0$ and $b \neq 0$,	$\frac{1}{ab} = \left(\frac{1}{b}\right)$
Theorem 8	Multiplication of fractions	Let a, b, c, d , be elements of R . If $b \neq 0$ and $d \neq 0$, then	$bd \neq 0$ and $\frac{a}{b} = \left(\frac{b}{c}\right) = \frac{ac}{bd}$
Theorem 9a	Cancellation for fractions	Let a, b, c , be elements of R . If $b \neq 0$ and $c \neq 0$, then	$\left(\frac{a}{b}\right) = \frac{ac}{bc}$
Theorem 9b	Addition of fractions	Let a, b, c , be elements of R . If $c \neq 0$, then	$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
Theorem 9c	Signs for fractions	Let a and b be elements of R . If $b \neq 0$, then	$\left(\frac{-a}{b}\right) = \left(\frac{a}{-b}\right) = -\left(\frac{a}{b}\right)$
Theorem 9d	Fractions divided by fractions	Let a, b, c, d , be elements of R . If $b \neq 0, c \neq 0$, and $d \neq 0$, then	$\left(\frac{a/b}{c/d}\right) = \frac{ad}{bc} = \left(\frac{a}{b}\right) \left(\frac{d}{c}\right)$
Theorem 9e	Cross multiplication for fractions	Let a, b, c, d , be elements of R . If $b \neq 0$ and $d \neq 0$, then	$\left(\frac{a}{b}\right) + \left(\frac{c}{d}\right) = \frac{ad+bc}{bd}$

3. Verify the solution by multiplying out the proposed factors.

An example of factoring

Consider the following quadratic expression.

$$x^2 + 6x + 8$$

1. Find all pairs of (m,n) such that $mn = 8$ There are a number of possibilities $(8,1)$, $(-8,-1)$, $(4,2)$, and $(-4,-2)$.
2. Determine if one of these pairs satisfies $m + n = 6$
 - i) $8+1 = 9 \neq 6$
 - ii) $-8+(-1) = -9 \neq 6$
 - iii) $4+2 = 6 \checkmark$
 - iv) $-4 + (-2) = -6 \neq 6$
3. Verify the solution by multiplying out the proposed factors. $(x + 4)(x + 2) = x^2 + 6x + 8$

Another example of factoring

Consider the following quadratic expression.

$$x^2 - x - 12$$

1. Find all pairs of (m,n) such that $mn = -12$ There are a number of possibilities $(12,-1)$, $(-12,1)$, $(6,-2)$, $(-6,2)$, $(4,-3)$, and $(-4,3)$.
2. Determine if one of these pairs satisfies $m + n = -1$
 - i) $12+(-1) = 11 \neq -1$
 - ii) $-12+1 = -11 \neq -1$
 - iii) $6+(-2) = 4 \neq -1$
 - iv) $-6+2 = -4 \neq -1$
 - v) $4 + (-3) = 1 \neq -1$
 - vi) $-4+3 = -1 \checkmark$
3. Verify the solution by multiplying out the proposed factors. $(x - 4)(x + 3) = x^2 - x - 12$

The following facts are useful to remember.

- i) If c is positive, then the factors you're looking for are either both positive or else both negative;
 - If b is positive, then the factors are positive.
 - If b is negative, then the factors are negative.
- ii) If c is negative, then the factors you're looking for are of alternating signs;
 - If b is positive, then the larger factor is positive.
 - If b is negative, then the larger factor is negative.

4.3.2. *Factoring quadratic expressions when $a \neq 1$.* When a in the quadratic expression is not equal to one we have $ax^2 + bx + c$ and there will be two factors of the form $(px + m)(qx + n)$. Before proceeding to factor we should determine if there is a common factor in each term of $ax^2 + bx + c$. If there is, we should then find the greatest common factor (GCF). Once we have this GCF, we should factor it out of the expression.

Consider for example the quadratic expression $6x^2 + 36x + 48$. Clearly each coefficient in the expression is divisible by 6 so that we can write the expression as follows.

$$\begin{aligned} 6x^2 + 36x + 48 &= 6(x^2 + 6x + 8) \\ &= 6(x + 4)(x + 2) \end{aligned}$$

Once we have factored out the GCF, we can proceed as before. We want to find two factors $(px + m)(qx + n)$ such that $(px + m)(qx + n) = ax^2 + bx + c$. If we multiply out this expression we obtain

$$\begin{aligned} (px + m)(qx + n) &= ax^2 + bx + c \\ \Rightarrow pqx^2 + (pn + qm)x + mn &= ax^2 + bx + c \\ \Rightarrow pq &= a \\ \Rightarrow (pn + qm) &= b \\ \Rightarrow mn &= c \end{aligned}$$

The coefficients p, q, m and n can be of any sign. But we know for example that if a is negative, p and q must be of opposite sign and that if c is positive, then m and n must be of the same sign.

One might list the solution approach in steps as follows.

1. Find all pairs of (p, q) such that $pq = a$. As p and q are not interchangeable in the expression $(px + m)(qx + n)$, i.e. $pqx^2 + (pn + qm) + mn \neq (qx + m)(px + n) = pqx^2 + (qn + pm)x + mn$, we must list both (p, q) and (q, p) . For example if $a = 2$ then we repeat pairs such as $(2, 1)$, $(1, 2)$ and $(-2, -1)$, $(-1, -2)$.
2. Find all pairs of (m, n) such that $mn = c$.
 - i) As $(px + m)(qx + n) = (qx + n)(px + m)$, if we interchange p and q , we do not need to interchange m and n as interchanging both creates an equivalent expression.
 - ii) Also, as $(px + m)(qx + n) = (-px + (-m))(-qx + (-n))$, we do not need to include both (r, s) and $(-r, -s)$.
3. Substitute all combinations of (p, q) and (m, n) into the expression $(px + m)(qx + n)$, multiply the expressions together and determine if one of these is equal to $ax^2 + bx + c$.

An example of factoring

Consider the following quadratic expression.

$$3x^2 + 7x - 6$$

- Find all pairs of (p,q) such that $pq = 3$. We have the following four possibilities $(3,1), (1,3), (-3,-1), (-1,-3)$.
- Find all pairs of (m,n) such that $mn = -6$.
 - The first pair is $(-6,1)$ and we need not consider $(6,-1)$.
 - The second pair is $(-3,2)$ and we need not consider $(-2,3)$.
- Now multiply out the various combinations.

(p,q)	(m,n)	$(px+m)(qx+n)$	Product
$(3,1)$	$(-6,1)$	$(3x-6)(x+1)$	$3x^2 - 3x - 6$
$(3,1)$	$(-3,2)$	$(3x-3)(x+2)$	$3x^2 + 3x - 6$
$(1,3)$	$(-6,1)$	$(x-6)(3x+1)$	$3x^2 - 17x - 6$
$(1,3)$	$(-3,2)$	$(x-3)(3x+2)$	$3x^2 - 7x - 6$
$(-3,-1)$	$(-6,1)$	$(-3x-6)(-x+1)$	$3x^2 + 3x - 6$
$(-3,-1)$	$(-3,2)$	$(-3x-3)(-x+2)$	$3x^2 - 3x - 6$
$(-1,-3)$	$(-6,1)$	$(-x-6)(-3x+1)$	$3x^2 + 17x - 6$
$(-1,-3)$	$(-3,2)$	$(-x-3)(-3x+2)$	$3x^2 + 7x - 6 \checkmark$

- We can check as follows $(-x-3)(-3x+2) = (x+3)(3x-2) = 3x^2 + 7x - 6$.

This can become tedious quite quickly and so one should eliminate combinations that make no sense as soon as possible. But, this is hard to do. There are also a number of other algorithms that are sometimes suggested including the box method and the grouping method.

5. SOLVING EQUATIONS

5.1. Simple equations.

5.1.1. *Using Theorems to Solve Simple Equations.* Most simple equations can be solved using the theorems from Table 6. The theorems on cancellation provide a way to obtain equations equivalent to a given equation by using the following equation solving principle.

5.1.2. *Basic Principle for Solving Simple Equations.* We obtain equivalent equations if on both sides of the equality sign in an equation we do the following:

- (i) add the same number
- (ii) subtract the same number
- (iii) multiply by the same number $\neq 0$
- (iv) divide by the same number $\neq 0$.

5.1.3. *Example Problem.*

$$\begin{array}{ll} 5x + 7 = 2x + 16 & \\ \Rightarrow 3x + 7 = 16 & \text{Subtract } 2x \text{ from both sides} \\ \Rightarrow 3x = 9 & \text{Subtract } 7 \text{ from both sides} \\ \Rightarrow x = 3 & \text{Divide both sides by } 3 \end{array}$$

5.1.4. *Practice Examples.*

(i) $3x + 10 = x + 4$

(ii) $6x - \frac{1}{2}(2x - 3) = 3(1 - x) - \frac{7}{6}(x + 2)$

(iii) $\frac{x+2}{x-2} - \frac{8}{x^2-2x} = \frac{2}{x}$

(iv) $\frac{x}{x-5} + \frac{1}{3} = \frac{-5}{5-x}$

5.2. Quadratic and cubic equations. An equation in which one or more of the variables is squared (or two variables are multiplied together) and there are no variables raised to the third power is called a quadratic equation.

An equation in which one or more of the variables is cubed (or three variables are multiplied together) and there are no variables raised to the fourth power is called a cubic equation.

5.2.1. *Some important identities involving squares and cubes.*

$$\mathbf{a:} (a + b)^2 = a^2 + 2ab + b^2$$

$$\mathbf{b:} (a - b)^2 = a^2 - 2ab + b^2$$

$$\mathbf{c:} (a + b)(a - b) = a^2 - b^2$$

$$\mathbf{d:} (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\mathbf{e:} (a + b)(a^2 - ab + b^2) = a^3 + b^3$$

$$\mathbf{f:} (a - b)(a^2 + ab + b^2) = a^3 - b^3$$

5.2.2. *Completing the square.* Consider the following two expressions:

$$x^2 + bx \tag{1a}$$

$$x^2 + 2hx + h^2 = (x + h)^2 \tag{1b}$$

An interesting question is “what do we have to do” to expression 1a so that it is equivalent to expression 1b. To find the answer write expression 1a as follows:

$$x^2 + bx + ? = x^2 + 2hx + h^2 \tag{2}$$

The first terms are equal and the second terms will be equal if $b = 2h$. This implies that $h = 1/2b$. It is then clear that $h^2 = \left(\frac{1}{2}b\right)^2$. So we can say:

Proposition 1 (completing the square). *To complete the square in the expression $x^2 + bx$, add $(1/2b)^2$ to it. Because we add $(1/2b)^2$ to the expression to create a perfect square we have to also subtract it to keep the expression the same.*

For example consider the expression $x^2 + 4x$. We add $\left(\left(\frac{1}{2}\right)(4)\right)^2$ to the expression to obtain $x^2 + 4x + 4$. We then subtract 4 to obtain $x^2 + 4x + 4 - 4$ and then write $x^2 + 4x + 4$ as $(x + 2)^2$ to obtain $(x + 2)^2 - 4$.

Here are some simple examples of expressions for practice in completing the square

$$\mathbf{1:} x^2 + 8x$$

$$\mathbf{2:} x^2 + 6x$$

$$\mathbf{3:} x^2 - 3x$$

$$\mathbf{4:} x^2 + \frac{b}{a}x$$

$$\mathbf{5:} x^2 + x$$

$$\mathbf{6:} x^2 + y^2 - 2x + 4y$$

$$\mathbf{7:} x^2 - \frac{2}{3}x$$

$$\mathbf{8:} x^2 - 10x$$

$$\mathbf{9:} x^2 - 10x + 20$$

$$\mathbf{10:} x^2 + \frac{1}{2}x$$

$$\mathbf{11:} x^2 + \frac{1}{2}x - \frac{1}{4}$$

$$\mathbf{12:} x^2 - 2x (\mu + \sigma^2t)$$

5.2.3. *Solving a quadratic equation by factoring.* The general form of a quadratic equation is

$$ax^2 + bx + c = 0, \quad (a \neq 0) \quad (3)$$

If the quadratic polynomial $ax^2 + bx + c$ can be factored, then the equation $ax^2 + bx + c = 0$ can be solved by using Theorem 6a: If the product $ab = 0$, then $a = 0$ or $b = 0$, or both and b equal zero.

Consider the equation $x^2 - 4 = 0$. We can factor it as

$$x^2 - 4 = (x - 2)(x + 2)$$

Rewriting the equation using this identity we obtain

$$\begin{aligned} (x - 2)(x + 2) &= 0 \\ \Rightarrow (x - 2) = 0 \text{ or } (x + 2) &= 0 \\ \Rightarrow x = 2 \text{ or } x &= -2 \end{aligned}$$

Now consider the equation $x^2 - 3x - 10 = 0$. We can factor it as

$$x^2 - 3x - 10 = (x - 5)(x + 2)$$

Rewriting the equation using this identity we obtain

$$\begin{aligned} (x - 5)(x + 2) &= 0 \\ \Rightarrow (x - 5) = 0 \text{ or } (x + 2) &= 0 \\ \Rightarrow x = 5 \text{ or } x &= -2 \end{aligned}$$

Now consider the equation $x^2 - 3.5x - 2 = 0$. We can factor it as

$$x^2 - 3.5x - 2 = (x - 4)\left(x + \frac{1}{2}\right)$$

Rewriting the equation using this identity we obtain

$$\begin{aligned} (x - 4)\left(x + \frac{1}{2}\right) &= 0 \\ \Rightarrow (x - 4) = 0 \text{ or } \left(x + \frac{1}{2}\right) &= 0 \\ \Rightarrow x = 4 \text{ or } x &= -\frac{1}{2} \end{aligned}$$

Finally consider the equation $12x^2 - 25x - 7 = 0$. We can factor it as

$$12x^2 - 25x - 7 = (4x + 1)(3x - 7)$$

Rewriting the equation using this identity we obtain

$$\begin{aligned} (4x + 1)(3x - 7) &= 0 \\ \Rightarrow (4x + 1) = 0 \text{ or } (3x - 7) &= 0 \\ \Rightarrow x = -\frac{1}{4} \text{ or } x &= \frac{7}{3} \end{aligned}$$

5.3. Completing the square to solve a quadratic equation.

5.3.1. *Solving a quadratic equation by completing the square.* Consider the quadratic equation $x^2 + 2x - 15 = 0$. We can factor it as follows.

$$\begin{aligned}x^2 + 2x - 15 &= 0 \\ \Rightarrow (x + 5)(x - 3) &= 0 \\ \Rightarrow (x + 5) = 0 \text{ or } (x - 3) &= 0 \\ \Rightarrow x = -5 \text{ or } x &= 3\end{aligned}$$

We can also solve it by completing the square as follows.

$$\begin{aligned}x^2 + 2x - 15 &= 0 \\ \Rightarrow x^2 + 2x + 1 - 1 - 15 &= 0 \\ \Rightarrow (x + 1)^2 - 1 - 15 &= 0 \\ \Rightarrow (x + 1)^2 - 16 &= 0 \\ \Rightarrow (x + 1)^2 &= 16 \\ \Rightarrow x + 1 = \pm \sqrt{16} &= \pm 4 \\ \Rightarrow x = -5 \text{ or } x &= 3\end{aligned}$$

We can check the answers as follows

$$\begin{aligned}x^2 + 2x - 15 &= 0 \\ \Rightarrow 3^2 + (2)(3) - 15 &= 0 \\ \Rightarrow 9 + 6 - 15 &= 0 \quad \text{and} \\ \Rightarrow (-5)^2 + (2)(-5) - 15 &= 0 \\ \Rightarrow 25 - 10 - 15 &= 0\end{aligned}$$

Now consider a different example $x^2 + 2x - 7 = 0$.

$$\begin{aligned}x^2 + 2x - 7 &= 0 \\ \Rightarrow (x + 1)^2 - 1 - 7 &= 0 \\ \Rightarrow (x + 1)^2 - 8 &= 0 \\ \Rightarrow (x + 1)^2 &= 8 \\ \Rightarrow x + 1 = \pm \sqrt{8} & \\ \Rightarrow x = \sqrt{8} - 1 \text{ or } -\sqrt{8} - 1 &\end{aligned}$$

We can check the answers as follows

$$\begin{aligned}
 x^2 + 2x - 7 &= 0 \\
 \Rightarrow (\sqrt{8} - 1)^2 + (2)(\sqrt{8} - 1) - 7 &= 0 \\
 \Rightarrow 8 - 2\sqrt{8} + 1 + 2\sqrt{8} - 2 - 7 &= 0 \\
 \Rightarrow 9 - 2 - 7 &= 0 \quad \text{and} \\
 \Rightarrow (-\sqrt{8} - 1)^2 + (2)(-\sqrt{8} - 1) - 7 &= 0 \\
 \Rightarrow 8 + 2\sqrt{8} + 1 - 2\sqrt{8} - 2 - 7 &= 0 \\
 \Rightarrow 9 - 2 - 7 &= 0
 \end{aligned}$$

Consider factoring $x^2 + 2x - 7 = 0$ as follows.

$$\begin{aligned}
 x^2 + 2x - 7 &= 0 \\
 \Rightarrow (x - (\sqrt{8} - 1))(x - (-\sqrt{8} - 1)) &= 0 \\
 \Rightarrow (x - (\sqrt{8} - 1))(x + (\sqrt{8} + 1)) &= 0 \\
 \Rightarrow (x^2 - \sqrt{8}x + x + \sqrt{8}x + x - (8 + \sqrt{8} - \sqrt{8} - 1)) &= 0 \\
 \Rightarrow x^2 + 2x - 8 + 1 &= 0 \\
 \Rightarrow x^2 + 2x - 7 &= 0
 \end{aligned}$$

We can always solve quadratic equations in this manner.

5.4. The quadratic formula. In many cases it is not easy to factor a given quadratic equation. But given the fact we can solve them by completing the square, we can derive a general *quadratic formula* for doing so. We do this in 7 steps as follows.

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

- (i) Divide both sides of the equation by $a \neq 0$. $x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad (a \neq 0)$
- (ii) Add $\frac{-c}{a}$ to each side to obtain $x^2 + \frac{b}{a}x = \frac{-c}{a} \quad (a \neq 0)$
- (iii) Complete the square by adding $\left(\frac{1}{2}\frac{b}{a}\right)^2$ giving $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \quad (a \neq 0)$
- (iv) Simplify the lhs as a perfect square. $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad (a \neq 0)$
- (v) Simplify the rhs by adding fractions. $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad (a \neq 0)$
- (vi) Take the square root of each side. $x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \quad (a \neq 0)$
- (vii) Add $\frac{-b}{2a}$ to both sides. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (a \neq 0)$

Notice that if $b^2 - 4ac$ is negative, we do not have a real solution to the equation.

5.5. Examples For Practicing Use of the Quadratic Formula.

1: $3x^2 + 11x - 4 = 0$

2: $4x^2 + 12x = 0$

3: $10x^2 + 13x + 3 = 0$

4: $4x^2 - 144 = 0$

5: $2x^2 - 4x - 7 = 0$

6: $20x + 100 = 0$

7: $6x^2 - 5x - 3 = 0$

6. FUNCTIONS OF A REAL VARIABLE

6.1. **Definition.** A function of a real variable x with **domain** D is a rule that assigns a unique real number to each number x in D . Functions are often given letter names such as f , g , F , or ϕ . We often call x the independent variable or the argument of f . If g is a function and x is a number in D , then $g(x)$ denotes the number that g assigns to x . We sometimes make the idea that F has an argument (we substitute a number for the variable in F) explicit by writing $F(\cdot)$. In the case of two variables we sometimes use $y = f(x)$ for the value of f evaluated at the number x . Note the difference between ϕ and $\phi(x)$.

6.2. **The domain of a function.** The domain is the set of all values that can be substituted for x in the function $f(\cdot)$. If a function f is defined using an algebraic formula, we normally adopt the convention that the domain consists of all values of the independent variable for which the function gives a meaningful value (unless the domain is explicitly mentioned).

6.3. **The range of a function.** Let g be a function with domain D . The set of all values $g(x)$ that the function assumes is called the range of g . To show that a number, say a , is in the range of a function

f, we must find a number x such that $f(x) = a$. Here are some example functions for which to find the domain and the range.

1: $f(x) = x, \quad 0 \leq x \leq 60$

The domain is $0 \leq x \leq 60$ and the range is $0 \leq y \leq 60$

2: $g(x) = x^2/20, \quad 0 \leq x \leq 60$

3: $h(x) = 12/x^2$

4: $\phi(x) = 3x^2$

The domain is all real numbers while the range is all non-negative real numbers.

5: $x = \frac{-1}{2}y^2 \quad y \geq 0$

6: $f(x) = \frac{3x}{x^2 - 4}$

7: $g(x) = +\sqrt{4 - 3x}$

The domain is $x < \frac{4}{3}$ and the range is all non-negative real numbers.

6.4. The graph of a function. When the rule that defines a function f is given by an equation in y and x , the graph of f is the graph of the equation, that is the set of points (x, y) in the xy -plane that satisfies the equation. Another way to say this is that the graph of the function g is the set of all point $(x, g(x))$, where x belongs to the domain of g .

6.5. The vertical line test. As set of points in the xy -plane is the graph of a function if and only if a vertical line intersects the graph in at most one point.

6.6. Linear functions.

6.6.1. *Definition.* A linear function of a real variable x is given by

$$y = f(x) = ax + b, \quad a \text{ and } b \text{ are constants.}$$

The graph of a linear equation is a straight line. The number a is called the slope of the function and the number b is called the y -intercept.

6.6.2. *The slope of a straight line given two distinct points on the line.* Consider two distinct points on a non-vertical straight line in the plane denoted by $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. Because the line is not vertical and P and Q are distinct, $x_1 \neq x_2$. The slope of the line is given by

$$a = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2. \quad (4)$$

6.6.3. *The point slope formula for a line.* Consider a point $P = (x_1, y_1)$ and any line with slope a . The line with slope a passing through the point can be determined as follows. Pick an arbitrary point on the line and denote it as (x, y) . Then use the formula for the slope of a line as follows.

$$a = \frac{y - y_1}{x - x_1}, \quad x \neq x_1 \quad (5)$$

Now solve the equation for either y or $y - y_1$ as follows.

$$\begin{aligned} a &= \frac{y - y_1}{x - x_1}, \quad x \neq x_1 \\ \Rightarrow y - y_1 &= a(x - x_1) \\ \Rightarrow y &= a(x - x_1) + y_1 \\ &= ax + (y_1 - ax_1). \end{aligned} \quad (6)$$

Consider the following example where the point is given by $P=(3,5)$ and the slope is given by $a = 2$.

$$\begin{aligned} 2 &= \frac{y - 5}{x - 3} \\ \Rightarrow y - 5 &= 2(x - 3) \\ \Rightarrow y &= 2(x - 3) + 5 \\ &= 2x - 6 + 5 \\ &= 2x - 1 \end{aligned} \quad (7)$$

6.6.4. *Point-point formula for a line.* If we are given two points on a line we can find the equation by first finding the slope and then using the point slope formula. Let the two points on the line be denoted (x_1, y_1) and (x_2, y_2) , $x_1 \neq x_2$. The slope is given by

$$a = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2. \quad (8)$$

If we substitute in the point-slope formula we obtain

$$\begin{aligned}
y - y_1 &= a(x - x_1) \\
&= \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x - x_1) \\
\Rightarrow y &= \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x - x_1) + y_1 \\
&= \left(\frac{y_2 - y_1}{x_2 - x_1} \right) x - \left(\frac{x_1 (y_2 - y_1)}{x_2 - x_1} \right) + y_1 \\
&= \left(\frac{y_2 - y_1}{x_2 - x_1} \right) x - \left(\frac{x_1 (y_2 - y_1)}{x_2 - x_1} \right) + \left(\frac{y_1 (x_2 - x_1)}{x_2 - x_1} \right) \\
&= \left(\frac{y_2 - y_1}{x_2 - x_1} \right) x + \left(\frac{-x_1 y_2 + x_1 y_1 + x_2 y_1 - x_1 y_1}{x_2 - x_1} \right) \\
&= \left(\frac{y_2 - y_1}{x_2 - x_1} \right) x + \left(\frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} \right).
\end{aligned} \tag{9}$$

7. COMPLEX NUMBERS

7.1. Definition of a complex number. A complex number is an ordered pair of real numbers denoted by (x_1, x_2) . The first member, x_1 , is called the real part of the complex number; the second member, x_2 , is called the imaginary part. We define equality, addition, subtraction, and multiplication so as to preserve the familiar rules of algebra for real numbers.

7.1.1. Equality of complex numbers. Two complex numbers $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are called equal iff

$$x_1 = y_1, \text{ and } x_2 = y_2. \tag{10}$$

7.1.2. Sum of complex numbers. The sum of two complex numbers $x + y$ is defined as

$$x + y = (x_1 + y_1, x_2 + y_2) \tag{11}$$

7.1.3. Difference of complex numbers. To subtract two complex numbers, the following rule applies.

$$x - y = (x_1 - y_1, x_2 - y_2) \tag{12}$$

7.1.4. Product of complex numbers. The product xy is defined as

$$xy = (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1). \tag{13}$$

The properties of addition and multiplication defined satisfy the commutative, associative and distributive laws.

7.2. The imaginary unit. The complex number $(0,1)$ is denoted by i and is called the imaginary unit. We can show that $i^2 = -1$ as follows.

$$i^2 = (0,1)(0,1) = (0 - 1, 0 + 0) = (-1, 0) = -1 \tag{14}$$

7.3. Representation of a complex number. A complex number $x = (x_1, x_2)$ can be written in the form

$$x = x_1 + ix_2. \quad (15)$$

Alternatively a complex number $z = (x, y)$ is sometimes written

$$z = x + iy. \quad (16)$$

Writing a complex number in this fashion we can see that

$$\begin{aligned} xy &= (x_1 + ix_2)(y_1 + iy_2) \\ &= x_1y_1 + ix_2y_1 + ix_1y_2 - x_2y_2 \\ &= x_1y_1 - x_2y_2 + i(x_2y_1 + x_1y_2) \\ &= (x_1y_1 - x_2y_2, x_2y_1 + x_1y_2) \end{aligned} \quad (17)$$

7.4. Modulus of a complex number. The modulus or absolute value of a complex number $x = (x_1, x_2)$ is the nonnegative real number $|x|$ given by

$$|x| = \sqrt{x_1^2 + x_2^2} \quad (18)$$

7.5. Complex conjugate of a complex number. For each complex number $z = x + iy$, the number $z_- = x - iy$ is called the complex conjugate of z . The product of a complex number and its conjugate is a real number. In particular, if $z = x + iy$ then

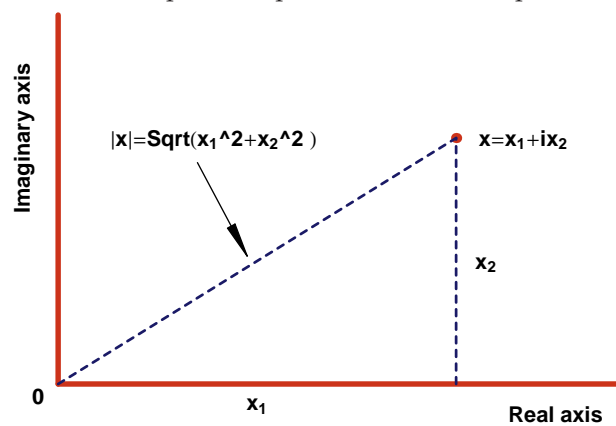
$$zz_- = (x, y)(x, -y) = (x^2 + y^2, -xy + yx) = (x^2 + y^2, 0) = x^2 + y^2 \quad (19)$$

Sometimes we will use the notation \bar{z} to represent the complex conjugate of a complex number. So $\bar{z} = x - iy$. We can then write

$$z\bar{z} = (x, y)(x, -y) = (x^2 + y^2, -xy + yx) = (x^2 + y^2, 0) = x^2 + y^2 \quad (20)$$

7.6. Graphical representation of a complex number. Consider representing a complex number in a two dimensional graph with the vertical axis representing the imaginary part. In this framework the modulus of the complex number is the distance from the origin to the point. This is seen clearly in figure 1.

FIGURE 1. Graphical Representation of Complex Number

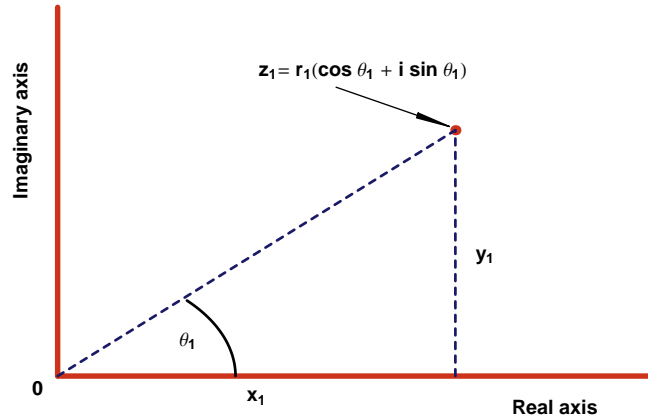


7.7. Polar form of a complex number. We can represent a complex number by its angle and distance from the origin. Consider a complex number $z_1 = x_1 + i y_1$. Now consider the angle θ_1 which the ray from the origin to the point z_1 makes with the x axis. Let the modulus of z be denoted by r_1 . Then $\cos \theta_1 = x_1/r_1$ and $\sin \theta_1 = y_1/r_1$. This then implies that

$$\begin{aligned} z_1 &= x_1 + i y_1 = r_1 \cos \theta_1 + i r_1 \sin \theta_1 \\ &= r_1 (\cos \theta_1 + i \sin \theta_1) \end{aligned} \tag{21}$$

Figure 2 shows how a complex number is represented in polar coordinates.

FIGURE 2. Graphical Representation of Complex Number in Polar Coordinates



7.8. Complex Exponentials. The exponential e^x is a real number. We want to define e^z when z is a complex number in such a way that the principle properties of the real exponential function will be preserved. The main properties of e^x , for x real, are the law of exponents, $e^{x_1}e^{x_2} = e^{x_1+x_2}$ and the equation $e^0 = 1$. If we want the law of exponents to hold for complex numbers, then it must be that

$$e^z = e^{x+iy} = e^x e^{iy} \tag{22}$$

We already know the meaning of e^x . We therefore need to define what we mean by e^{iy} . Specifically we define e^{iy} in equation 23

Definition 4.

$$e^{iy} = \cos y + i \sin y \tag{23}$$

With this in mind we can then define $e^z = e^{x+iy}$ as follows

Definition 5.

$$e^z = e^x e^{iy} = e^x (\cos y + i \sin y) \tag{24}$$

Obviously if $x = 0$ so that z is a pure imaginary number this yields

$$e^{iy} = (\cos y + i \sin y) \tag{25}$$

It is easy to show that $e^0 = 1$. If z is real then $y = 0$. Equation 24 then becomes

$$\begin{aligned} e^z &= e^x e^{i0} = e^x (\cos (0) + i \sin (0)) \\ &= e^x e^{i0} = e^x (1 + 0) = e^x \end{aligned} \tag{26}$$

So e^0 obviously is equal to 1.

To show that $e^x e^{iy} = e^{x+iy}$ or $e^{z_1} e^{z_2} = e^{z_1+z_2}$ we will need to remember some trigonometric formulas.

Theorem 1.

$$\sin(\phi + \theta) = \sin \phi \cos \theta + \cos \phi \sin \theta \quad (27a)$$

$$\sin(\phi - \theta) = \sin \phi \cos \theta - \cos \phi \sin \theta \quad (27b)$$

$$\cos(\phi + \theta) = \cos \phi \cos \theta - \sin \phi \sin \theta \quad (27c)$$

$$\cos(\phi - \theta) = \cos \phi \cos \theta + \sin \phi \sin \theta \quad (27d)$$

Now to the theorem showing that $e^{z_1} e^{z_2} = e^{z_1+z_2}$

Theorem 2. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then we have $e^{z_1} e^{z_2} = e^{z_1+z_2}$.

Proof.

$$\begin{aligned} e^{z_1} &= e^{x_1}(\cos y_1 + i \sin y_1), \quad e^{z_2} = e^{x_2}(\cos y_2 + i \sin y_2), \\ e^{z_1} e^{z_2} &= e^{x_1} e^{x_2} [\cos y_1 \cos y_2 - \sin y_1 \sin y_2 \\ &\quad + i(\cos y_1 \sin y_2 + \sin y_1 \cos y_2)]. \end{aligned} \quad (28)$$

Now $e^{x_1} e^{x_2} = e^{x_1+x_2}$, since x_1 and x_2 are both real. Also,

$$\cos y_1 \cos y_2 - \sin y_1 \sin y_2 = \cos(y_1 + y_2) \quad (29)$$

and

$$\cos y_1 \sin y_2 + \sin y_1 \cos y_2 = \sin(y_1 + y_2), \quad (30)$$

and hence

$$e^{z_1} e^{z_2} = e^{x_1+x_2} [\cos(y_1 + y_2) + i \sin(y_1 + y_2)] = e^{z_1+z_2}. \quad (31)$$

□

Now combine the results in equations 21, 24 and 25 to obtain the polar representation of a complex number

$$\begin{aligned} z_1 &= x_1 + iy_1 \\ &= r_1 (\cos \theta_1 + i \sin \theta_1) \\ &= r_1 e^{i\theta_1}, \quad (\text{by equation 23}) \end{aligned} \quad (32)$$

The usual rules for multiplication and division hold so that

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1+\theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)} \end{aligned} \quad (33)$$

8. APPENDIX A – PROOF THAT $\sqrt{2}$ IS NOT A RATIONAL NUMBER

We can show that $\sqrt{2} \notin Q$ (rationals) as follows.

Proof. The proof is by contradiction. Suppose that $\sqrt{2} \in Q$. Then there exist integers p and q such that $p/q = \sqrt{2}$, where we choose the smallest such p and q . Now $p^2/q^2 = 2$, or $p^2 = 2q^2$, so that p^2 must be an even number. Since the square of an odd number is always odd, p is even so we may write $p = 2r$, where r is also an integer. We then have

$$p^2 = 4r^2 = 2q^2$$

and so $q^2 = 2r^2$. Thus q^2 , and hence q , must be even. But if both p and q are even, they have a common factor of 2. This result contradicts the assumption that p and q were the smallest numbers to give $p/q = \sqrt{2}$. Thus, the statement $\sqrt{2} \in Q$ must be false. □

9. APPENDIX B – PROOFS OF THEOREMS ON PROPERTIES OF REAL NUMBERS AND EQUATIONS

Theorem 3.

a: If z and a are elements of \mathbf{R} such that $z + a = a$, then $z = 0$.

b: If u and $b \neq 0$ are elements of \mathbf{R} such that $u \cdot b = b$, then $u = 1$.

Proof of a. The hypothesis is that $z + a = a$. We add the element $-a$, whose existence is given in (A4), to both sides to get

$$(z + a) + (-a) = a + (-a).$$

If we use (A2), (A4), and (A3) in succession on the left side, we obtain

$$(z + a) + (-a) = z + (a + (-a)) = z + 0 = z;$$

If we use (A4) on the right side, we obtain

$$a + (-a) = 0.$$

Hence, we conclude that $z = 0$. □

The proof of (b) is left as an exercise. Note that the hypothesis $b \neq 0$ is crucial.

Theorem 4.

a: If a and b are elements of \mathbf{R} such that $a + b = 0$, then $b = -a$.

b: If $a \neq 0$ and b are elements of \mathbf{R} such that $a \cdot b = 1$, then $b = 1/a$.

Proof of a. If $a + b = 0$, then we add $-a$ to both sides to get

$$(-a) + (a + b) = (-a) + 0.$$

If we use (A2), (A4), and (A3) on the left side, we obtain

$$(-a) + (a + b) = ((-a) + a) + b = 0 + b = b;$$

if we use (A3) on the right side, we obtain

$$(-a) + 0 = -a.$$

Hence, we conclude that $b = -a$. □

The proof of (b) is left as an exercise. Note that the hypothesis $a \neq 0$ is used.

If we view the preceding properties in terms of solving equations, we note that (A4) and (M4) enable us to solve the equations $a + x = 0$ and $a \cdot x = 1$ (when $a \neq 0$) for x , and theorem 4 implies that the solutions are unique.

Theorem 5.

Let a, b be arbitrary elements of \mathbf{R} . Then:

- a:** the equation $a + x = b$ has the unique solution $x = (-a) + b$;
- b:** if $a \neq 0$, the equation $a \cdot x = b$ has the unique solution $x = (1/a) \cdot b$.

Proof of a. Using properties (A2), (A4), and (A3), we obtain

$$a + [(-a) + b] = [a + (-a)] + b = 0 + b = b.$$

which implies that $x = (-a) + b$ is a solution of the equation $a + x = b$. To show that it is the only solution, suppose that x_1 is any solution of the equation, then $a + x_1 = b$, and if we add $-a$ to both sides, we get

$$(-a) + (a + x_1) = (-a) + b.$$

If we now use (A2), (A4), and (A3) on the left side, we obtain

$$(-a) + (a + x_1) = (-a + a) + x_1 = 0 + x_1 = x_1.$$

Hence we conclude that $x_1 = (-a) + b$. □

The proof of part (b) is left as an exercise.

Theorem 6. If a and b are any elements of \mathbf{R} , then:

- a:** $a \cdot 0 = 0$,
- b:** $(-1) \cdot a = -a$,
- c:** $-(-a) = a$,
- d:** $(-1) \cdot (-1) = 1$,
- e:** $-(a+b) = (-a) + (-b)$.

Proof of theorem 6a. From (M3) we know that $a \cdot 1 = a$. Then adding $a \cdot 0$ and using (D) and (A3) we obtain

$$\begin{aligned} a + a \cdot 0 &= a \cdot 1 + a \cdot 0 \\ &= a \cdot (1 + 0) = a \cdot 1 = a. \end{aligned}$$

Thus, by theorem 3a we conclude that $a \cdot 0 = 0$. □

Proof of theorem 6b. We use (D), in conjunction with (M3), (A4), and part (a), to obtain

$$\begin{aligned} a + (-1) \cdot a &= 1 \cdot a + (-1) \cdot a \\ &= (1 + (-1)) \cdot a = 0 \cdot a = 0 \end{aligned}$$

Thus, from theorem 4a, we conclude that $(-1) \cdot a = -a$. □

Proof of theorem 6c. By (A4) we have $(-a) + a = 0$. Thus from theorem 4a, it follows that $a = -(-a)$. □

Proof of theorem 6d. In part (b), substitute $a = -1$. Then

$$(-1) \cdot (-1) = -(-1).$$

Hence, the assertion follows from part (c) with $a = 1$. □

Proof of theorem 6e. To prove part e we use the following two lemmas.

Lemma 1. *If $a, b,$ and c are any numbers, then*

$$a + b + c = a + c + b = b + a + c = b + c + a = c + a + b = c + b + a.$$

Proof left to the reader.

Lemma 2. *If $a, b, c,$ and d are numbers, then*

$$(a + c) + (b + d) = (a + b) + (c + d).$$
□

Proof. Using lemma 1 and the axioms, we have

$$\begin{aligned} (a + c) + (b + d) &= [(a + c) + b] + d \\ &= (a + c + b) + d = (a + b + c) + d \\ &= [(a + b) + c] + d = (a + b) + (c + d). \end{aligned}$$
□

Now to prove part e of the theorem.

Proof. We know from the definition of negative that

$$(a + b) + [-(a + b)] = 0.$$

Furthermore, using lemma 2 we have

$$(a + b) + [(-a) + (-b)] = a + (-a) + [b + (-b)] = 0 + 0 = 0.$$

The result follows from theorem 4a. □

Theorem 7. *Let x, y, z be arbitrary elements of R . If $x + y = x + z$ then $y = z$.*

Proof. If $x + y = x + z$, the axioms of addition give

$$\begin{aligned} y &= 0 + y = (-x + x) + y = -x + (x + y) \\ &= -x + (x + z) = (-x + x) + z = 0 + z = z. \end{aligned}$$
□

Theorem 8. *For any a and $b \in R$, we have*

a: $ab = 0$ if and only if $a = 0$ or $b = 0$ or both.

b: $a \neq 0$ and $b \neq 0$ if and only if $ab \neq 0$.

Proof. We must prove two statements in each of parts (a) and (b). To prove (a), observe that if $a = 0$ or $b = 0$, or both, then it follows from theorem 6a that $ab = 0$. Going the other way in (a), suppose that $ab = 0$. Then there are two cases, either $a = 0$ or $a \neq 0$. If $a = 0$, the result follows. If $a \neq 0$, then we see that

$$b = 1 \cdot b = (a^{-1}a)b = a^{-1}(ab) = a^{-1} \cdot 0 = 0.$$

Hence $b = 0$ and (a) is established. To prove (b), first suppose $a \neq 0$ and $b \neq 0$. Then $ab \neq 0$, because $a \neq 0$ and $b \neq 0$ is the negation of the statement “ $a = 0$ or $b = 0$ or both.” Thus (a) applies. For the second part of (b), suppose $ab \neq 0$. Then $a \neq 0$ and $b \neq 0$ for, if one of them were zero, theorem 6a would apply to give $ab = 0$. □

Theorem 9. Let a, b, c , be elements of \mathbf{R} .

a: If $a \neq 0$, then $1/a \neq 0$ and $1/(1/a) = a$.

b: If $a \cdot b = a \cdot c$ and $a \neq 0$, then $b = c$.

c: If $a \neq 0$ and $b \neq 0$, then $(a \cdot b)^{-1} = (a^{-1}) \cdot (b^{-1})$.

Proof of a. We are given $a \neq 0$, so that $1/a$ exists. If $1/a = 0$, then $1 = a \cdot (1/a) = a \cdot 0 = 0$, contrary to (M3). Thus $1/a \neq 0$ and since $(1/a) \cdot a = 1$, theorem 4b implies that $1/(1/a) = a$. □

Proof of b. If we multiply both sides of the equation $a \cdot b = a \cdot c$ by $1/a$ and apply the associative property (M2), we get

$$((1/a) \cdot a) \cdot b = ((1/a) \cdot a) \cdot c.$$

Thus, $1 \cdot b = 1 \cdot c$ which is the same as $b = c$. □

Proof of c. The proof of this theorem is like the proof of theorem 6 with addition replaced by multiplication, 0 replaced by 1, and $(-a)$, $(-b)$ replaced by a^{-1} , b^{-1} . The details are left to the reader. Note that if $a \neq 0$, then $a^{-1} \neq 0$ because $aa^{-1} = 1$ and $1 \neq 0$. Then theorem 8b may be used with $b = a^{-1}$. □

Theorem 10. If $b \neq 0$ and $d \neq 0$, then $b \cdot d \neq 0$ and $(\frac{a}{b}) \cdot (\frac{c}{d}) = \frac{a \cdot c}{b \cdot d}$.

In order to prove the theorem we will use the following lemmas, similar to lemma 1 and lemma 2. They can be proved by the reader. For the lemmas, we define abc as the common value of $(ab)c$ and $a(bc)$.

Lemma 3. If a, b , and c are numbers, then

$$abc = acb = bac = bca = cab = cba.$$

Lemma 4. If a, b, c and d are numbers, then

$$(ac) \cdot (bd) = (ab) \cdot (cd).$$

Proof. That $b \cdot d \neq 0$ follows from theorem 8. Using the notation for fractions, lemma 4 and theorem 9c, we find

$$\begin{aligned} \left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) &= (a \cdot b^{-1}) \cdot (cd^{-1}) \\ &= (a \cdot c) \cdot (b^{-1}d^{-1}) = (a \cdot c)(bd)^{-1} = \frac{a \cdot c}{b \cdot d}. \end{aligned}$$

□

Theorem 11.

a: If $b \neq 0$ and $c \neq 0$, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$.

b: If $c \neq 0$, then $\frac{a}{c} + \frac{b}{c} = \frac{(a+b)}{c}$.

c: If $b \neq 0$, then $-b \neq 0$ and $\frac{(-a)}{b} = \frac{a}{-b} = -\left(\frac{a}{b}\right)$.

d: If $b \neq 0$, $c \neq 0$, and $d \neq 0$, then $(c/d) \neq 0$ and $\frac{(a/b)}{(c/d)} = \frac{a \cdot d}{b \cdot c} = \left(\frac{a}{b}\right) \cdot \left(\frac{d}{c}\right)$.

e: If $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

Proof. Left to the reader.

□