Matrix Algebra

March 5, 2008

Systems of Equations

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$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & \cdots & + & a_{3n}x_n & = & b_3 \\ \vdots & + & \vdots & + & \cdots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{1}$$

Matrix Form

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$$Ax = b$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \tag{2}$$

Augmented Coefficient Matrix

We define the augmented coefficient matrix for the system as

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix} \tag{3}$$

Example Matrix and Vector

Consider the following matrix A and vector b

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -3 & -4 & -2 \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 8 \\ -4 \end{pmatrix} \tag{4}$$

Example System

We can then write

$$
\begin{aligned}
Ax &= b \\
\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -3 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 3 \\ 8 \\ -4 \end{pmatrix}
\end{aligned} \tag{5}
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\end{aligned} \tag{5}
$$

for the linear equation system

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 3 \\
2x_1 + 5x_2 + 2x_3 &= 8 \\
-3x_1 - 4x_2 - 2x_3 &= -4
\end{aligned} \tag{6}
$$

Row-echelon form of a matrix

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Leading zeroes in the row of a matrix

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A row of a matrix is said to have k leading zeroes if the first k elements of the row are all zeroes and the (k+1)st element of the row is not zero.

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Row echelon form of a matrix

A matrix is in row echelon form if each row has more leading zeroes than the row preceding it.

Examples of row echelon matrices

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

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$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{pmatrix} \tag{7}$$

Pivots

The first non-zero element in each row of a matrix in row-echelon form is called a **pivot**. For the matrix A the pivots are 3,5,4.

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For the matrix B the pivots are 1,2.

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For the matrix C the pivots are 1,4,3.

$$C = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Reduced Row Echelon Form

A row echelon matrix in which **each pivot is a 1** and in which each column containing a pivot **contains no other nonzero entries**, is said to be in **reduced row echelon form**. This implies that columns containing pivots are columns of an identity matrix.

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In the matrix E in equation (9) the first two columns are pivot columns. The matrices D and E are both in reduced row echelon form.

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9}$$

Pivots

In matrix F in equation (10), columns 2, 4 and 5 are pivot columns. The matrix F is in row echelon form but not in reduced row echelon form because of the 1 in the a_{25} and the 3 in the a_{15} elements of the matrix. The 5 in the a_{13} element is irrelevant because column 3 is not a pivot column.

$$F = \begin{pmatrix} 0 & 1 & 5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

Reducing a matrix to row echelon or reduced row echelon form

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These operations are the same ones that we used when solving a linear system using the method of Gaussian elimination. There are three types of **elementary row operations**. Performing these operations does not change the basic nature of the system or its solution.

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1 Changing the order in which the equations or rows are listed produces an equivalent system.

2 Multiplying both sides of a single equation of the system (or a single row of $\tilde{A}$) by a nonzero number (leaving the other equations unchanged) results in a system of equations that is equivalent to the original system.

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2 Multiplying both sides of a single equation of the system (or a single row of $\tilde{A}$) by a nonzero number (leaving the other equations unchanged) results in a system of equations that is equivalent to the original system.
3 Adding a multiple of one equation or row to another equation or row (leaving the other equations unchanged) results in a system of equations that is equivalent to the original system.

Rank

Definition 1 (Rank). The number of non-zero rows in the row echelon form of an $m \times n$ matrix A produced by elementary operations on A is called the rank of A. Alternatively, the rank of an $m \times n$ matrix A is the number of pivots we obtain by reducing a matrix to row echelon form.

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Matrix D in equation (8) has rank 3.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (8)$$

Rank

Definition 3 (Rank)**.** The number of non-zero rows in the row echelon form of an m × n matrix A produced by elementary operations on A is called the rank of A. Alternatively, the rank of an m × n matrix A is the number of pivots we obtain by reducing a matrix to row echelon form.

Matrix D in equation (8) has rank 3.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

Matrix E has rank 2.

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9}$$

Rank

Matrix F in (10) has rank 3.

$$F = \begin{pmatrix} 0 & 1 & 5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (10)$$

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Definition 4 (Full column rank)**.** We say that an $m \times n$ matrix A has full column rank if $r = n$. If $r = n$, all columns of A are pivot columns.

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Definition 5 (Full column rank)**.** We say that an $m \times n$ matrix A has full column rank if $r = n$. If $r = n$, all columns of A are pivot columns.

The matrix L is of full column rank

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Rank

Definition 6 (Full row rank)**.** We say that an $m \times n$ matrix A has full row rank if $r = m$. If $r = m$, all rows of A have pivots and the reduced row echelon form has no zero rows.

Rank

Definition 7 (Full row rank)**.** We say that an $m \times n$ matrix A has full row rank if $r = m$. If $r = m$, all rows of A have pivots and the reduced row echelon form has no zero rows.

The matrix M is of full row rank

$$M = \begin{pmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -7 \end{pmatrix}$$

Solutions to equations (stated without proof)

a A system of linear equations with coefficient matrix A which is m × n, a right hand side vector b which is m × 1, and augmented matrix $\hat{A}$ has a solution if and only if

$$rank\,(A) \;=\; rank\,(\hat{A})$$

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b A linear system of equations must have either no solution, one solution, or infinitely many solutions.

c If the rank of A = r = n < m, the linear system Ax = b has no solution or exactly one solution. In other words, if the matrix A is of full column rank and has less columns than rows, the system will be inconsistent, or will have exactly one solution.

Solutions to equations

d If the rank of $A = r = m < n$, the linear system $Ax = b$ will always have a solution and there will be an infinite number of solutions. In other words, if the matrix A is of full row rank and has more columns than rows, the system will always have an infinite number of solutions.

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e If the rank of $A = r$ where $r < m$ and $r < n$, the linear system $Ax = b$ will either have no solutions or there will be an infinite number of solutions. In other words, if the matrix A has neither full row nor column rank, the system will be inconsistent or will have an infinite number of solutions.

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f If the rank of A = r = m = n, the linear system Ax = b will exactly one solution. In other words, if the matrix A is square and has full rank, the system will have a unique solution.

Nonsingular Coefficient Matrices

A coefficient matrix is said to be **nonsingular**, that is, the corresponding linear system has one and only one solution for every choice of right hand side b_1, b_2, ... , b_m, if and only if

$$number\ of\ rows\ of\ A\ =\ number\ of\ columns\ of\ A\ =\ rank\,(A)$$

Solving Systems of Equations via Elimination

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If the matrix is in row echelon we can solve it by back substitution.

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If the matrix is in row echelon we can solve it by back substitution.

If the matrix is in reduced row echelon form we can read the coefficients off directly from the matrix.

Example System

Consider matrix equation 5 with its augmented matrix $\tilde{A}$.

$$Ax = b$$

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$$
\begin{aligned}
Ax &= b \\
\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -3 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 3 \\ 8 \\ -4 \end{pmatrix} \qquad (5) \\
\tilde{A} &= \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ -3 & -4 & -2 & -4 \end{pmatrix}
\end{aligned}
$$

Steps to Solution

Step 1. Consider the first row and the first column of the matrix. Element a_{11} is a non-zero number and so is appropriate for the first row of the row echelon form.

$$\tilde{A} = \begin{pmatrix} \textcircled{1} & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ -3 & -4 & -2 & -4 \end{pmatrix}$$

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$$\tilde{A} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ -3 & -4 & -2 & -4 \end{pmatrix}$$

If element a_{11} were zero, we would exchange row 1 with a row of the matrix with a non-zero element in column 1.

Steps to Solution

Step 2. Use row operations to "knock out" a_{21} by turning it into a zero. This would then make the second row have one more leading zero than the first row. We can turn the a_{21} element into a zero by adding negative 2 times the first row to the second row.

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$$\tilde{A} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ \textcircled{2} & 5 & 2 & 8 \\ -3 & -4 & -2 & -4 \end{pmatrix}$$

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$$\Rightarrow \begin{pmatrix} -2 & -4 & -2 & -6 \end{pmatrix}$$

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$$\begin{array}{cccc} -2 & -4 & -2 & -6 \\ 2 & 5 & 2 & 8 \\ -- & -- & -- & -- \\ 0 & 1 & 0 & 2 \end{array}$$

The New Matrix $\tilde{A}_1$

This will give a new matrix on which to operate. Call it $\tilde{A}_1$.

$$\tilde{A}_1 = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ -3 & -4 & -2 & -4 \end{pmatrix}$$

The Elimination Matrix

We can obtain $\tilde{A}_1$ by premultiplying $\tilde{A}$ by what is called the a_{21} elimination matrix denoted by E_{21} where E_{21} is given by

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$\overbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{E_{21}} \overbrace{\begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ -3 & -4 & -2 & -4 \end{pmatrix}}^{\tilde{A}}$$

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Steps to Solution

Step 3. Use row operations to "knock out" a_{31} by turning it into a zero. This would then make the third row have one more leading zero than the first row. We can turn the a_{31} element into a zero by adding 3 times the first row to the third row.

$$\tilde{A}_1 = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ \textcircled{-3} & -4 & -2 & -4 \end{pmatrix}$$

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$$3\times \begin{pmatrix} 1 & 2 & 1 & 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 6 & 3 & 9 \end{pmatrix}$$

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$$\begin{array}{cccc} 3 & 6 & 3 & 9 \\ -3 & -4 & -2 & -4 \\ -- & -- & -- & -- \\ 0 & 2 & 1 & 5 \end{array}$$

The Elimination Matrix

We can obtain $\tilde{A}_2$ by premultiplying $\tilde{A}_1$ by what is called the a_{31} elimination matrix denoted by E_{31} where E_{31} is given by

$$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

The Elimination Matrix

We can obtain $\tilde{A}_2$ by premultiplying $\tilde{A}_1$ by what is called the a_{31} elimination matrix denoted by E_{31} where E_{31} is given by

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$$\overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}}^{E_{31}} \overbrace{\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ -3 & -4 & -2 & -4 \end{pmatrix}}^{\tilde{A}_1} = \overbrace{\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 5 \end{pmatrix}}^{\tilde{A}_2}$$

Steps to Solution

Step 4. Use row operations to "knock out" a_{32} by turning it into a zero. We can turn the a_{32} element into a zero by adding negative 2 times the second row of $\tilde{A}_2$ to the third row of $\tilde{A}_2$.

$$\tilde{A}_2 = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 5 \end{pmatrix}$$

Steps to Solution

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$$\tilde{A}_2 = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & \textcircled{2} & 1 & 5 \end{pmatrix}$$

$$-2\times \begin{pmatrix} 0 & 1 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -2 & 0 & -4 \end{pmatrix}$$

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$$\begin{array}{cccc} 0 & -2 & 0 & -4 \\ 0 & 2 & 1 & 5 \\ -- & -- & -- & -- \\ 0 & 0 & 1 & 1 \end{array}$$

The Elimination Matrix

We can obtain $\tilde{A}_3$ by premultiplying $\tilde{A}_2$ by what is called the a_{32} elimination matrix denoted by E_{32} where E_{32} is given by

$$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

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$$\overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}}^{E_{32}} \overbrace{\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 5 \end{pmatrix}}^{\tilde{A}_2} = \overbrace{\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}}^{\tilde{A}_3}$$

Solution by Backsubstitution

Consider the matrix $\tilde{A}_3$. It is in row-echelon form.

$$\tilde{A}_3 = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

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This implies

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 0x_1 + x_2 + 0x_3 &= 2 \\ 0x_1 + 0x_2 + x_3 &= 1 \end{aligned}$$

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This system is easily solved for x_3, x_2 and then x_1.

Solution by Converting to Reduced Row Echelon Form

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 3 \\ 0x_1 & + & x_2 & + & 0x_3 & = & 2 \\ 0x_1 & + & 0x_2 & + & x_3 & = & 1 \end{array}$$

Solution by Converting to Reduced Row Echelon Form

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 0x_1 + x_2 + 0x_3 &= 2 \\ 0x_1 + 0x_2 + x_3 &= 1 \end{aligned}$$

We would like to have the following system.

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= ? \\ 0x_1 + x_2 + 0x_3 &= 2 \\ 0x_1 + 0x_2 + x_3 &= 1 \end{aligned}$$

Steps to Solution

Step 5. The a_{23} element is already zero, so we can proceed to the a_{13} element. We use row operations to "knock out" a_{13} by turning it into a zero. We can turn the a_{13} element into a zero by adding negative 1 times the third row to the first row.

$$\tilde{A}_3 = \begin{pmatrix} 1 & 2 & \textcircled{1} & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Steps to Solution

Step 5. The a_{23} element is already zero, so we can proceed to the a_{13} element. We use row operations to "knock out" a_{13} by turning it into a zero. We can turn the a_{13} element into a zero by adding negative 1 times the third row to the first row.

$$\tilde{A}_3 = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$-1 \times \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix}$$

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$$\Rightarrow \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix}$$

$$\begin{array}{cccc} 0 & 0 & -1 & -1 \\ 1 & 2 & 1 & 3 \\ -- & -- & -- & -- \\ 1 & 2 & 0 & 2 \end{array}$$

The Elimination Matrix

We can obtain $\tilde{A}_4$ by premultiplying $\tilde{A}_3$ by what is called the a_{13} elimination matrix denoted by E_{13} where E_{13} is given by

$$E_{13} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$\overbrace{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{E_{13}} \overbrace{\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}}^{\tilde{A}_3} = \overbrace{\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}}^{\tilde{A}_4}$$

Steps to Solution

Step 6. Use row operations to "knock out" a_{12} by turning it into a zero. We can turn the a_{12} element into a zero by adding negative 2 times the second row to the first row.

$$\tilde{A}_4 = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Steps to Solution

Step 6. Use row operations to "knock out" a_{12} by turning it into a zero. We can turn the a_{12} element into a zero by adding negative 2 times the second row to the first row.

$$\tilde{A}_4 = \begin{pmatrix} 1 & \textcircled{2} & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$-2 \times \begin{pmatrix} 0 & 1 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -2 & 0 & -4 \end{pmatrix}$$

Steps to Solution

Step 6. Use row operations to "knock out" a_{12} by turning it into a zero. We can turn the a_{12} element into a zero by adding negative 2 times the second row to the first row.

$$\tilde{A}_4 = \begin{pmatrix} 1 & \textcircled{2} & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$-2 \times \begin{pmatrix} 0 & 1 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -2 & 0 & -4 \end{pmatrix}$$

$$\begin{array}{cccc} 0 & -2 & 0 & -4 \\ 1 & 2 & 0 & 2 \\ -- & -- & -- & -- \\ 1 & 0 & 0 & -2 \end{array}$$

The Elimination Matrix

We can obtain $\tilde{A}_5$ by premultiplying $\tilde{A}_4$ by what is called the a_{12} elimination matrix denoted by E_{12} where E_{12} is given by

$$E_{12} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Elimination Matrix

We can obtain $\tilde{A}_5$ by premultiplying $\tilde{A}_4$ by what is called the a_{12} elimination matrix denoted by E_{12} where E_{12} is given by

$$E_{12} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overbrace{\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{E_{12}} \overbrace{\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}}^{\tilde{A}_4} = \overbrace{\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}}^{\tilde{A}_5}$$

The Elimination Matrix

We can obtain $\tilde{A}_5$ by premultiplying $\tilde{A}_4$ by what is called the a_{12} elimination matrix denoted by E_{12} where E_{12} is given by

$$E_{12} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overbrace{\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{E_{12}} \overbrace{\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}}^{\tilde{A}_4} = \overbrace{\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}}^{\tilde{A}_5}$$

It is obvious that $x_1 = -2$, $x_2 = 2$, and $x_3 = 1$.