COST FUNCTIONS

1. Introduction to the Cost Function

1.1. **Understanding and representing technology.** Economists are interested in the technology used by the firm. This is usually represented by a set or a function. For example we might describe the technology in terms of the input correspondence which maps output vectors in R^m_+ into subsets of $2^{R^n_+}$. These subsets are vectors of inputs that will produce the given output vector. This correspondence is given by

$$V: \; R^m_+ \rightarrow 2^{R^n_+} \tag{1}$$

Alternatively we could represent the technology by a function such as the production function y = f(x_1, x_2, . . . , x_n)

We typically postulate or assume certain properties for various representations of technology. For example we typically make the following assumptions on the input correspondence.

1.1.1. *V.1 No Free Lunch.*

a: V(0) = R^n_+
b: $0 \notin$ V(y), y > 0.

1.1.2. *V.2 Weak Input Disposability.* $\forall\, y \,\epsilon\, R^m_+,\; x \,\epsilon\, V(y)$ and $\lambda \geq 1 \;\Rightarrow\; \lambda x \,\epsilon\, V(y)$

1.1.3. *V.2.S Strong Input Disposability.* $\forall\, y \,\epsilon R^m_+ \; x \,\epsilon\, V(y)$ and $x' \;\geq\; x \;\Rightarrow x' \,\epsilon\, V(y)$

1.1.4. *V.3 Weak Output Disposability.* $\forall\, y \,\epsilon\, R^m_+ \, V(y) \;\subseteq V(\theta y),\; 0 \leq\; \theta \;\leq\; 1.$

1.1.5. *V.3.S Strong Output Disposability.* $\forall\, y,\, y' \,\epsilon R^m_+ ,\;\; y' \;\geq\; y \;\Rightarrow\; V(y') \;\subseteq\; V(y)$

1.1.6. *V.4 Boundedness for vector y.* If $\|y^\ell\| \rightarrow +\infty$ as $\ell \rightarrow +\infty$,

$$\cap_{\ell=1}^{+\infty} V(y^\ell) \;=\; \emptyset$$

If y is a scalar,

$$\bigcap_{y \epsilon (0, +\infty)} V(y) \;=\; \emptyset$$

1.1.7. *V.5 T(x) is a closed set.* V: $R^m_+ \rightarrow 2^R_{+^n}$ is a closed correspondence.

1.1.8. *V.6 Attainability.* If x ϵ V(y), y $\geq$ 0 and x $\geq$ 0, the ray $\{\lambda$x: $\lambda \geq 0\}$ intersects all V(θy), $\theta \geq 0$.

1.1.9. *V.7 Quasi-concavity.* V is quasi-concave on R^m_+ which means $\forall$ y, y′ ϵ R^m_+ , 0 $\leq \theta \leq$ 1, V(y) $\cap$ V(y′) $\subseteq$ V(θy + (1-θ)y′)

1.1.10. *V.8 Convexity of V(y).* V(y) is a convex set for all y ϵ R^m_+

1.1.11. *V.9 Convexity of T(x).* V is convex on R^m_+ which means $\forall$ y, y′ ϵ R^m_+ , 0 $\leq \theta \leq$ 1, θV(y)+(1-θ)V(y′) $\subseteq$ V(θy+ (1-θ)y′)

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1.2. **Understanding and representing behavior.** Economists are also very interested in the behavioral relationships that arise from firms' optimizing decisions. These decisions result in reduced form expressions such as supply functions $y^* = x(p, w)$, input demands $x^* = x(p, w)$, or Hicksian (cost minimizing) demand functions $x^* = x(w, y)$. One of the most basic of these decisions is the cost minimizing decision.

2. The Cost Function and its Properties

2.1. **Definition of cost function.** The cost function is defined for all possible output price vectors and all positive input price vectors $w = (w_1, w_2, \ldots, w_n)$. An output vector, y, is producible if y belongs to the effective domain of V(y), i.e,

$$\mathcal{D}om\, V \;=\; \{y \in R_+^m : V(y) \neq \emptyset\}$$

The cost function does not exist it there is no technical way to produce the output in question. The cost function is defined by

$$C(y,\ w) \;=\; \min_x \{wx : x \epsilon V(y)\}, \quad y \in \mathcal{D}om\, V, w > 0, \tag{2}$$

or in the case of a single output

$$C(w,\ y) \;=\; \min_x \{wx :\ f(x) \,\geq\, y\} \tag{3}$$

The cost function exists because a continuous function on a nonempty closed bounded set achieves a minimum in the set (Debreu [6, p. 16]). In figure 1,the set V(y) is closed and nonempty for y in the producible output set. The function wx is continuous. Because V(y) is non-empty if contains at least one input bundle x′. We can thus consider cost minimizing points that satisfy $wx \leq wx'$ But this set is closed and bounded given that w is strictly positive. Thus the function wx will attain a minimum in the set at x″.

2.2. **Solution to the cost minimization problem.** The solution to the cost minimization problem 2 is a vector x which depends on output vector y and the input vector w. We denote this solution by x(y,w). This demand inputs at for a fixed level of output and input prices is often called a Hicksian demand curve.

2.3. **Properties of the cost function.**

2.3.1. *C.1.1.* Non-negativity: $C(y, w) \geq 0$ for $w > 0$.

2.3.2. *C.1.2.* Nondecreasing in w: If $w \geq w'$ then $C(y, w) \geq C(y, w')$

2.3.3. *C.2.* Positively linearly homogenous in w

$$C(y,\ \lambda w) \;=\; \lambda\, C(y,\ w), \quad w > 0.$$

2.3.4. *C.3.* C is concave and continuous in w

2.3.5. *C.4.1.* No fixed costs: $C(0, w) = 0, \ \forall\, w > 0$. We sometimes assume we have a long run problem.

2.3.6. *C.4.2.* No free lunch: $C(y, w) > 0, w > 0, y > 0$.

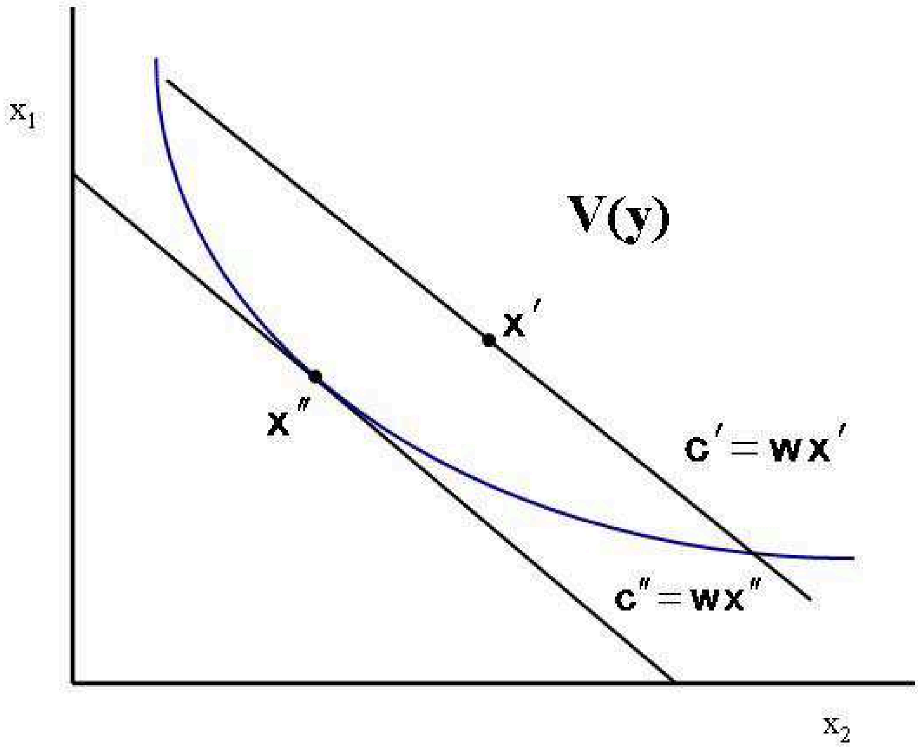
2.3.7. *C.5.* Nondecreasing in y (proportional): $C(\theta y, w) \leq C(y, w), 0 \leq \theta \leq 1, w > 0$.

2.3.8. *C.5.S.* Nondecreasing in y: $C(y', w) \leq C(y, w), y' \leq y, w > 0$.

2.3.9. *C.6.* For any sequence y^ℓ such that $\|\, y^\ell \,\| \to \infty$ as $\ell \to \infty$ and $w > 0$, $C(y^\ell, w) \to \infty$ as $\ell \to \infty$.

2.3.10. *C.7.* C(y,w) is lower semi-continuous in y, given $w > 0$.

FIGURE 1. Existence of the Cost Function



2.3.11. *C.8.* If the graph of the technology (GR) or T, is convex, C(y,w) is convex in y, w > 0.

2.4. **Discussion of properties of the cost function.**

2.4.1. *C.1.1.* Non-negativity: C(y, w) ≥ 0 for w > 0.

Because it requires inputs to produce an output and w is positive then C(y, w) = wx(y,w) > 0 (y > 0) where x(y,w) is the optimal level of x for a given y.

2.4.2. *C.1.2.* Nondecreasing in w: If w ≥ w′ then C(y, w) ≥ C(y, w′).

Let $w^1 \geq w^2$. Let x^1 be the cost minimizing input bundle with w^1 and x^2 be the cost minimizing input bundle with w^2. Then $w^2x^2 \leq w^2x^1$ because x^1 is not cost minimizing with prices w^2. Now $w^1x^1 \geq w^2x^1$ because $w^1 \geq w^2$ by assumption so that

$$C(w^1, y) = w^1\, x^1 \geq w^2\, x^1 \geq w^2\, x^2 \;=\; C(w^2, y)$$

2.4.3. *C.2.* Positively linearly homogenous in w

$$C(y,\,\lambda w)\;=\;\lambda\,C(y,\,w),\quad w>0.$$

Let the cost minimization problem with prices w be given by

$$C(y,\,w)\;=\;\min_{x}\,\{wx : x\epsilon V(y)\},\quad y\in\mathcal{D}om\,V, w>0, \tag{4}$$

The x vector that solves this problem will be a function of y and w, and is usually denoted x(y,w). The cost function is then given by

$$C(y,\,w)\;=\;w\,x(y,\,w) \tag{5}$$

Now consider the problem with prices tw (t >0)

$$\begin{aligned}\hat{C}(y,\,tw)\;&=\;\min_{x}\,\{twx : x\epsilon V(y)\},\quad y\in\mathcal{D}om\,V, w>0\\ &=t\,\min_{x}\,\{wx : x\epsilon V(y)\},\quad y\in\mathcal{D}om\,V, w>0\end{aligned} \tag{6}$$

The x vector that solves this problem will be the same as the vector which solves the problem in equation 4, i.e., x(y,w). The cost function for the revised problem is then given by

$$\hat{C}(y,\,tw)\;=\;tw\,x(y,\,w)\;=\;tC(y,w) \tag{7}$$

2.4.4. *C.3.* C is concave and continuous in w

To demonstrate concavity let (w, x) and (w′, x′) be two cost-minimizing price-factor combinations and let w″= tw + (1-t)w′ for any $0 \leq t \leq 1$. Concavity implies that C(w″ y) $\geq$ tC(w, y) + (1-t) C(w′, y). We can prove this as follows. We have that C(w″ y) = w″· x″= tw · x″+ (1-t)w′ · x″ where x″ is the optimal choice of x at prices w″ Because x″ is not necessarily the cheapest way to produce y at prices w′ or w,we have w · x″$\geq$ C(w, y) and w′· x″ $\geq$ C(w′ y) so that by substitution C(w″ y) $\geq$ tC(w, y) + (1-t) C(w′, y). The point is that if w · x″ and w′· x″ are each larger than the corresponding term is the linear combination then C(w″,y) is larger than the linear combination. Because C(y, w)is concave it will be continuous by the property of concavity.

Consider figure 2. Let x* be the cost minimizing bundle at prices w*. Let the price of x_i^* change. At input prices (w*), costs are at the level C(w*). If we hold input levels fixed at (x*) and change w_i, we move along the tangent line denoted by $\tilde{C}(w_i, \bar{w}, y)$, where $\bar{w}^*$ and $\bar{x}^*$ represent all the prices and inputs other than the ith. Costs are higher along this line than along the cost function because we are not adjusting inputs. Along the cost function, as the price of input i increases, we probably use less of input x_i and more of other inputs.

Rockafellar [14, p. 82] shows that a concave function defined on an open set (w $>$ 0) is continuous.

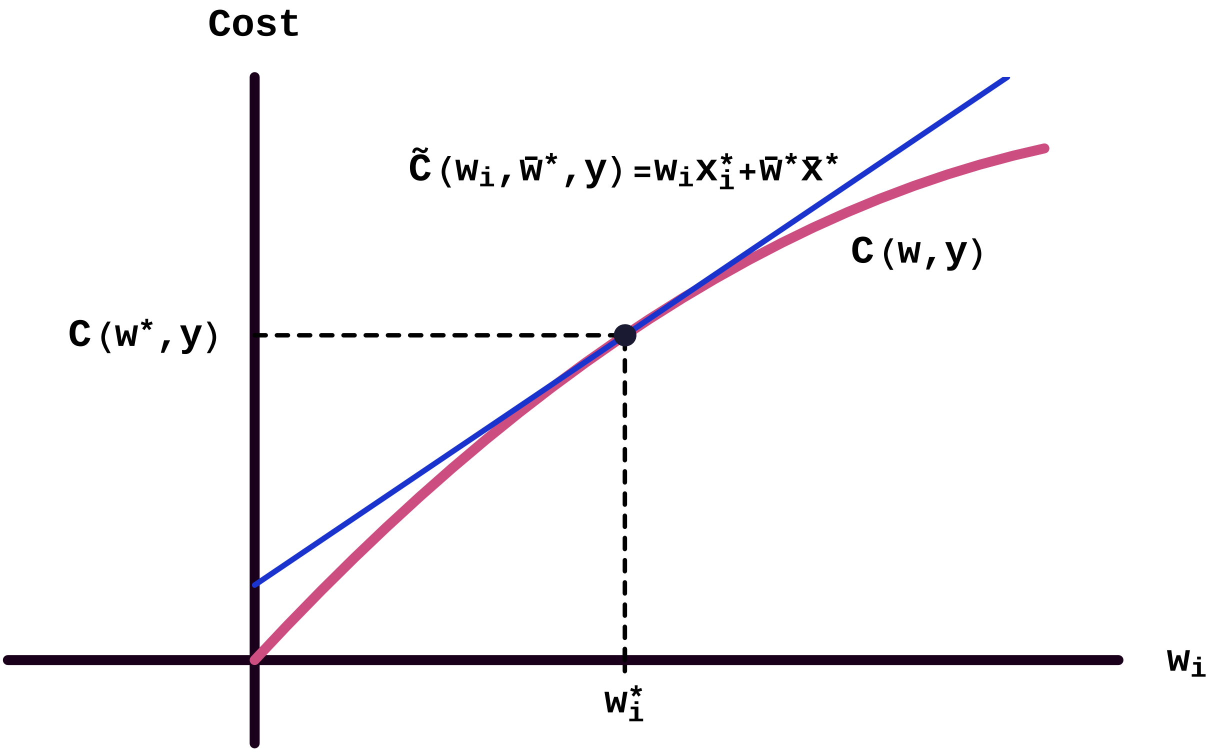
2.4.5. *C.4.1.* No fixed costs: C(0, w) = 0, $\forall$ w $>$ 0.

We assume this axiom if the problem is long run, in the short run fixed costs may be positive with zero output. Specifically V.1a implies that to produce zero output, any input vector in R_+^n will do, including the zero vector with zero costs.

2.4.6. *C.4.2.* No free lunch: C(y, w) $>$ 0, w $>$ 0, y $>$ 0.

Because we cannot produce outputs without inputs (V.1b: no free lunch with the technology), costs for any positive output are positive for strictly positive input prices.

FIGURE 2. The Cost Function in Concave



2.4.7. *C.5.* Nondecreasing in y (proportional): $C(\theta y, w) \leq C(y, w)$, $0 \leq \theta \leq 1$, $w > 0$.

If outputs go down proportionately, costs cannot rise. This is clear because $V(y_1)$ is a subset of $V(y_2)$ if $y_1 \geq y_2$ from V.3, then $C(y_1, w) = \min wx|x{\in}V(y_1) \geq \min wx \mid x{\in} V(y_2) = C(y_2, w)$. The point is that if we have a smaller set of possible x's to choose from then cost must increase.

2.4.8. *C.5.S.* Nondecreasing in y: $C(y', w) \leq C(y, w)$, $y' \leq y$, $w > 0$.

If any output goes down, costs cannot increase.

2.4.9. *C.6.* For any sequence y^ℓ such that $\|\, y^\ell \,\| \to \infty$ as $\ell \to \infty$ and $w > 0$, $C(y^\ell, w) \to \infty$ as $\ell \to \infty$.

This axiom implies that if outputs increase without bound, so will costs.

2.4.10. *C.7.* C(y,w) is lower semi-continuous in y, given $w > 0$.

The cost function may not be continuous in output as it is in input prices, but if there are any jump points, it will take the lower value at these points.

2.4.11. *C.8.* If the graph of the technology (GR) or T, is convex, C(y,w) is convex in y, $w > 0$.

If the technology is convex (more or less decreasing returns to scale as long as V.1 holds), costs will rise at an increasing rate as y increases.

2.5. **Shephard's Lemma.**

2.5.1. *Definition of Shephard's lemma.* In the case where V is strictly quasi-concave and V(y) is strictly convex the cost minimizing point is unique. Rockafellar [14, p. 242] shows that the cost function is differentiable in w, w > 0 at (y,w) if and only if the cost minimizing set of inputs at (y,w) is a singleton, i.e., the cost minimizing point is unique. In such cases.

$$\frac{\partial C(y,\ w)}{\partial w_i} = x_i(y, w) \tag{8}$$

which is the Hicksian Demand Curve. As with Hotelling's lemma in the case of the profit function, this lemma allows us to obtain the input demand functions as derivatives of the cost function.

2.5.2. *Constructive proof of Shephard's lemma in the case of a single output.* For a single output, the cost minimization problem is given by

$$C(y, w) \ = \ \min_x \ wx : f(x) \ - \ y \ = \ 0 \tag{9}$$

The associated Lagrangian is given by

$$\mathcal{L} \ = wx - \lambda(f(x) \ - \ y) \tag{10}$$

The first order conditions are as follows

$$\frac{\partial \mathcal{L}}{\partial x_i} \ = \ w_i \ - \ \lambda \, \frac{\partial f}{\partial x_i} \ = 0, \quad i \ = \ 1, \, \ldots, \, n \tag{11a}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} \ = \ - \, (f(x) \ - \ y) \ = 0 \tag{11b}$$

Solving for the optimal x's yields

$$x_i(y,\ w) \tag{12}$$

with C(y,w) given by

$$C(w, y) = wx(y, w) \tag{13}$$

If we now differentiate 13 with respect to w we obtain

$$\frac{\partial C}{\partial w_i} \ = \ \Sigma_{j=1}^{n} \ w_j \ \frac{\partial x_j(y, w)}{\partial w_i} \ + \ x_i\,(y, w) \tag{14}$$

From the first order conditions in equation 11a (assuming that the constraint is satisfied as an equality) we have

$$w_j \ = \ \lambda \, \frac{\partial f}{\partial x_j} \tag{15}$$

Substitute the expression for w_j from equation 15 into equation 14 to obtain

$$\frac{\partial C}{\partial w_i} \ = \ \Sigma_{j=1}^{n} \, \lambda \, \frac{\partial f(x)}{\partial x_j} \, \frac{\partial x_j\,(y, w)}{\partial w_i} \ + \ x_i\,(y, w) \tag{16}$$

If $\lambda > 0$ then equation 11b implies(f(x)-y) = 0. Now differentiate equation 11b with respect to w_i to obtain

$$\Sigma_{j=1}^{n} \frac{\partial f(x(y, w))}{\partial x_j} \, \frac{\partial x_j\,(y, w)}{\partial w_i} \ = \ 0 \tag{17}$$

which implies that the first term in equation 16 is equal to zero and that

$$\frac{\partial C(y,w)}{\partial w_i} \;=\; x_i\,(y,w) \tag{18}$$

2.5.3. *A Silberberg* [17] *type proof of Shephard's lemma.* Set up a function L as follows

$$L(y,w,\hat{x}) \;=\; w\,\hat{x} \;-\; C(w,y) \tag{19}$$

where $\hat{x}$ is the cost minimizing choice of inputs for prices $\hat{w}$. Because C(w, y) is the cheapest way to produce y at prices w, L $\geq$ 0. If w = $\hat{w}$, then L will be equal to zero. Because this is the minimum value of L, the derivative of L at this point is zero so

$$\begin{aligned} \frac{\partial L(y,\,\hat{w},\hat{x})}{\partial w_i} &= \hat{x}_i \;-\; \frac{\partial C(\hat{w},y)}{\partial w_i} \;=\; 0 \\ \Rightarrow \hat{x}_i \;&=\frac{\partial C(\hat{w},\;y)}{\partial w_i} \end{aligned} \tag{20}$$

The second order necessary conditions for minimizing L imply that

$$\left[\frac{-\partial^2 C}{\partial w_i \partial w_j}\right] \tag{21}$$

is positive semi-definite so that

$$\frac{\partial^2\, C}{\partial w_i\, \partial w_j} \tag{22}$$

is negative semi-definite which implies that C is concave.

2.5.4. *Graphical representation of Shephard's lemma.* In figure 3 we hold all input prices except the jth fixed at $\hat{w}$. We assume that when the jth price is $\hat{w}_j$, the optimal input vector is ($\hat{x}_1$, $\hat{x}_2$, . . . , $\hat{x}_j$, . . . ,$\hat{x}_n$). The cost function lies below the tangent line in figure 3 but conincides with the tangent line at $\hat{w}_j$ By differentiability the slope of the cost function at this point is the slope of its tangent i.e.,

$$\frac{\partial C}{\partial w_j} \;=\; \frac{\partial (tangent)}{\partial w_j} \;=\; \hat{x}_j \tag{23}$$

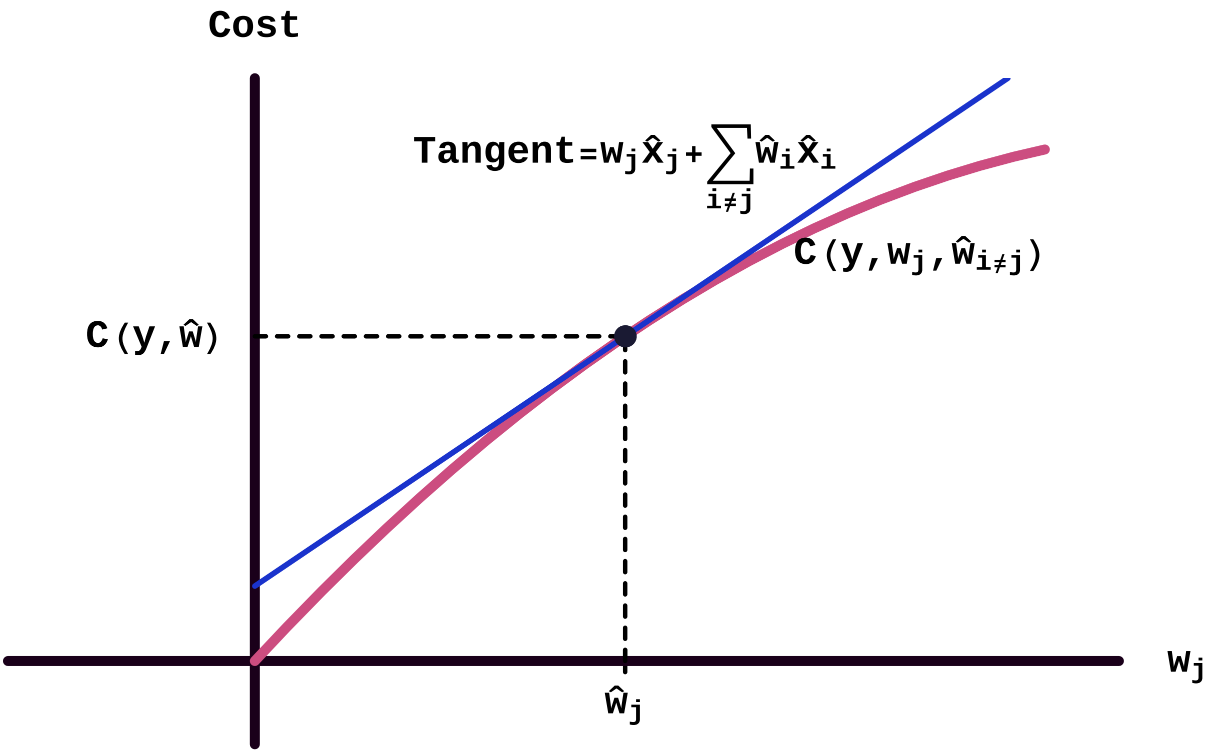
2.6. **Sensitivity analysis.**

2.6.1. *Demand slopes.* If the Hessian of the cost function is negative semi-definite, the diagonal elements all must be non-positive(Hadley [11, p. 260-262]) so we have

$$\frac{\partial^2\, C(y,w)}{\partial w_i^2} \;=\; \frac{\partial x_i\,(y,\;w)}{\partial w_i} \;\leq\; 0\,\forall_i \tag{24}$$

This implies then that Hicksian demand curves slope down because the diagonal elements of the Hessian of the cost function are just the derivatives in input demands with with respect to their own prices.

FIGURE 3. Shephard's Lemma



2.6.2. *Cross price effects.* By homogeneity of degree zero of input demands and Euler's theorem we have

$$\Sigma_{j=1}^{n} \frac{\partial x_i}{\partial w_j}\, w_j \;=\; \Sigma_{j\neq i} \frac{\partial x_i}{\partial w_j}\, w_j \;+\; \frac{\partial x_i}{\partial w_i}\, w_i \;=\; 0 \tag{25}$$

And we know that

$$\frac{\partial x_i}{\partial w_i} \;\leq\; 0 \tag{26}$$

by the concavity of C. Therefore

$$\Sigma_{j\neq i} \frac{\partial x_i}{\partial w_j}\, w_j \geq\; 0 \tag{27}$$

.

This implies that at least one cross price effect is positive.

2.6.3. *Symmetry of input demand response to input prices.* By Young's theorem

$$\begin{aligned}\frac{\partial^2 C}{\partial w_i\, \partial w_j} &= \frac{\partial^2 C}{\partial w_j\, \partial w_i} \\ \Rightarrow \frac{\partial x_i}{\partial w_j} &= \frac{\partial x_j}{\partial w_i}\end{aligned} \tag{28}$$

So cross derivatives in input prices are symmetric.

2.6.4. *Marginal cost.* Consider the first order conditions for cost minimization. If we differentiate 13 with respect to y we obtain

$$\frac{\partial C}{\partial y} = \Sigma_{j=1}^{n} w_j \frac{\partial x_j(y,w)}{\partial y} \tag{29}$$

From the first order conditions in equation 11a (assuming that the constraint is satisfied as an equality) we have

$$w_j = \lambda \frac{\partial f}{\partial x_j} \tag{30}$$

Substitute the expression for w_j from equation 30 into equation 29 to obtain

$$\begin{aligned} \frac{\partial C}{\partial y} &= \Sigma_{j=1}^{n} \lambda \frac{\partial f(x)}{\partial x_j} \frac{\partial x_j(y,w)}{\partial y} \\ &= \lambda \Sigma_{j=1}^{n} \frac{\partial f(x)}{\partial x_j} \frac{\partial x_j(y,w)}{\partial y} \end{aligned} \tag{31}$$

If $\lambda > 0$ then equation 11b implies(f(x)-y) = 0. Now differentiate equation 11b (ignoring λ) with respect to y to obtain

$$\begin{aligned} &\Sigma_{j=1}^{n} \frac{\partial f(x(y,w))}{\partial x_j} \frac{\partial x_j(y,w)}{\partial y} - 1 = 0 \\ \Rightarrow\ &\Sigma_{j=1}^{n} \frac{\partial f(x(y,w))}{\partial x_j} \frac{\partial x_j(y,w)}{\partial y} = 1 \end{aligned} \tag{32}$$

This then implies that the first term in equation 16 that

$$\frac{\partial C(y,w)}{\partial y} = \lambda(y,w) \tag{33}$$

The Lagrangian multiplier from the cost minimization problem is equal to marginal cost or the increase in cost from increasing the targeted level of y in the constraint.

2.6.5. *Symmetry of input demand response to changes in output and changes in marginal cost with respect to input prices.* Remember that

$$\frac{\partial^2 C}{\partial y\, \partial w_i} = \frac{\partial^2 C}{\partial w_i\, \partial y} \tag{34}$$

This then implies that

$$\frac{\partial x_i}{\partial y} = \frac{\partial \lambda}{\partial w_i} \tag{35}$$

The change in $x_i(y,w)$ from an increase in y equals the change in marginal cost due to an increase in w_i.

2.6.6. *Marginal cost is homogeneous of degree one in input prices.* Marginal cost is homogeneous of degree one in input prices.

$$\frac{\partial C(ty,\, w)}{\partial y} = t \frac{\partial C(y,\, w)}{\partial y}, \quad t > 0. \tag{36}$$

We can show using the Euler equation. If a function is homogeneous of degree one, then

$$\Sigma_{i=1}^{n} \frac{\partial f}{\partial x_i}\, x_i \;=\; f(x) \tag{37}$$

Applying this to marginal cost we obtain

$$\begin{aligned}
\frac{\partial MC(y,w)}{\partial w_i} \;&= \frac{\partial \lambda(y,w)}{\partial w_i} \;=\; \frac{\partial^2 C(y,w)}{\partial w_i\, \partial y} \;=\; \frac{\partial^2 C(y,w)}{\partial y\, \partial w_i} \;=\; \frac{\partial\, x_i(y,w)}{\partial y} \\
\Rightarrow\; \Sigma_i \frac{\partial \lambda(y,w)}{\partial w_i}\, w_i &= \Sigma_i \frac{\partial^2 C(y,w)}{\partial w_i\, \partial y}\, w_i \;=\; \Sigma_i \frac{\partial^2 C(y,w)}{\partial y\, \partial w_i}\, w_i \\
&= \frac{\partial}{\partial y} \left(\Sigma_i \frac{\partial\, C(y,w)}{\partial w_i}\, w_i \right) \\
&= \frac{\partial}{\partial y}\, C(y,\, w\,), \quad \text{by homogeneity of C(y,w) in w} \\
&= MC(y,\, w) \;=\; \lambda(y,\, w)
\end{aligned} \tag{38}$$

3. Traditional Approach to the Cost Function

3.1. **First order conditions for cost minimization.** In the case of a single output, the cost function can be obtained by carrying out the maximization problem

$$C(y,w) \;=\; \min_x \; wx : f(x) \,-\, y \;=\; 0 \tag{39}$$

with associated Lagrangian function

$$\mathcal{L} \;= wx - \lambda(f(x) \,-\, y) \tag{40}$$

The first order conditions are as follows

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= w_1 - \quad \lambda \frac{\partial f}{\partial x_1} \;=\; 0\\
\frac{\partial \mathcal{L}}{\partial x_2} &= w_2 - \quad \lambda \frac{\partial f}{\partial x_2} \;=\; 0\\
&\vdots \qquad\qquad \vdots\\
\frac{\partial \mathcal{L}}{\partial x_n} &= w_n - \quad \lambda \frac{\partial f}{\partial x_n} \;=\; 0
\end{aligned} \tag{41a}$$

$$\frac{\partial \lambda}{\partial \lambda} \;=\; -\,(f(x) \,-\, y) \;=\; 0 \tag{41b}$$

If we take the ratio of any of the first order conditions we obtain

$$\frac{\frac{\partial f}{\partial x_j}}{\frac{\partial f}{\partial x_i}} \;=\; \frac{w_j}{w_i} \tag{42}$$

This implies that the RTS between inputs i and j is equal to the negative inverse factor price ratio because

$$RTS \;=\; \frac{\partial x_i}{\partial x_j} \;=\; \frac{-\,\frac{\partial f}{\partial x_j}}{\frac{\partial f}{\partial x_i}} \tag{43}$$

Substituting we obtain

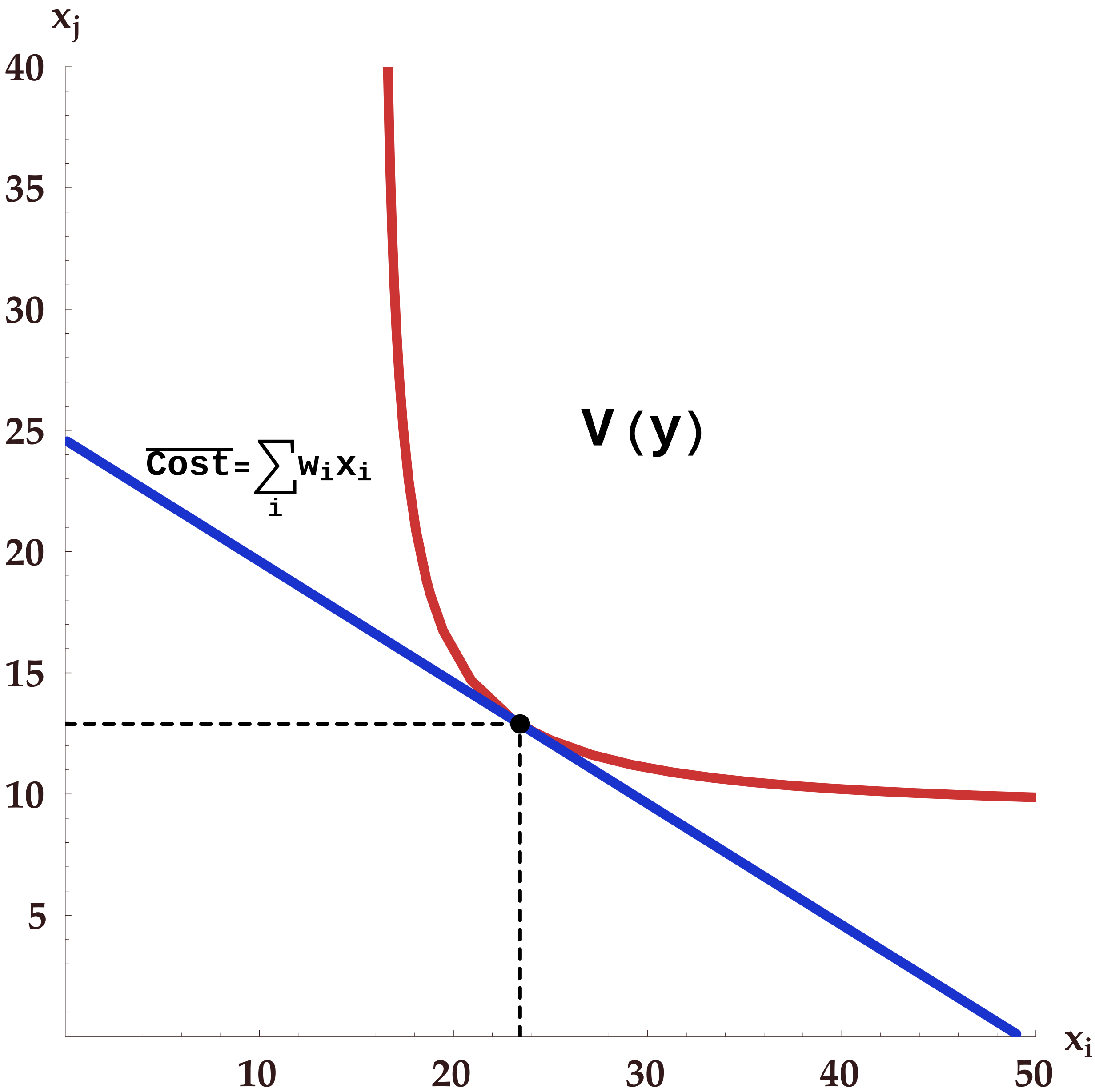
$$RTS \;=\; \frac{\partial x_i}{\partial x_j} \;=\; -\frac{w_j}{w_i} \tag{44}$$

Graphically this implies that the slope of an isocost line is equal to the slope of the lower boundary of V(y). Note that an isocost line is given by

$$\begin{aligned}
\overline{cost} \;&= w_1\, x_1 \;+\; w_2\, x_2 \;+\; \ldots \;+\; w_n\, x_n\\
\Rightarrow\; w_i\, x_i \;&= \overline{cost} \;-\; w_1\, x_1 \;-\; w_2\, x_2 \;-\; \ldots \;-\; w_{i-1}\, x_{i-1} \;-\; w_{i+1}\, x_{i+1} \;-\; \ldots \;-\; w_j\, x_j \;-\; \ldots \;-\; w_n\, x_n\\
\Rightarrow\; x_j \;&= \frac{\overline{cost}}{w_i} \;-\; \frac{w_1}{w_i}\, x_1 \;-\; \frac{w_2}{w_i}\, x_2 \;-\; \ldots \;-\; \frac{w_{i-1}}{w_i}\, x_{i-1} \;-\; \frac{w_{i+1}}{w_i}\, x_{i+1} \;-\; \ldots \;-\; \frac{w_j}{w_i}\, x_j \;-\; \ldots \;-\; \frac{w_n}{w_i}\, x_n\\
\Rightarrow\; \text{Slope of isocost} \;&=\; -\;\frac{w_j}{w_i}
\end{aligned} \tag{45}$$

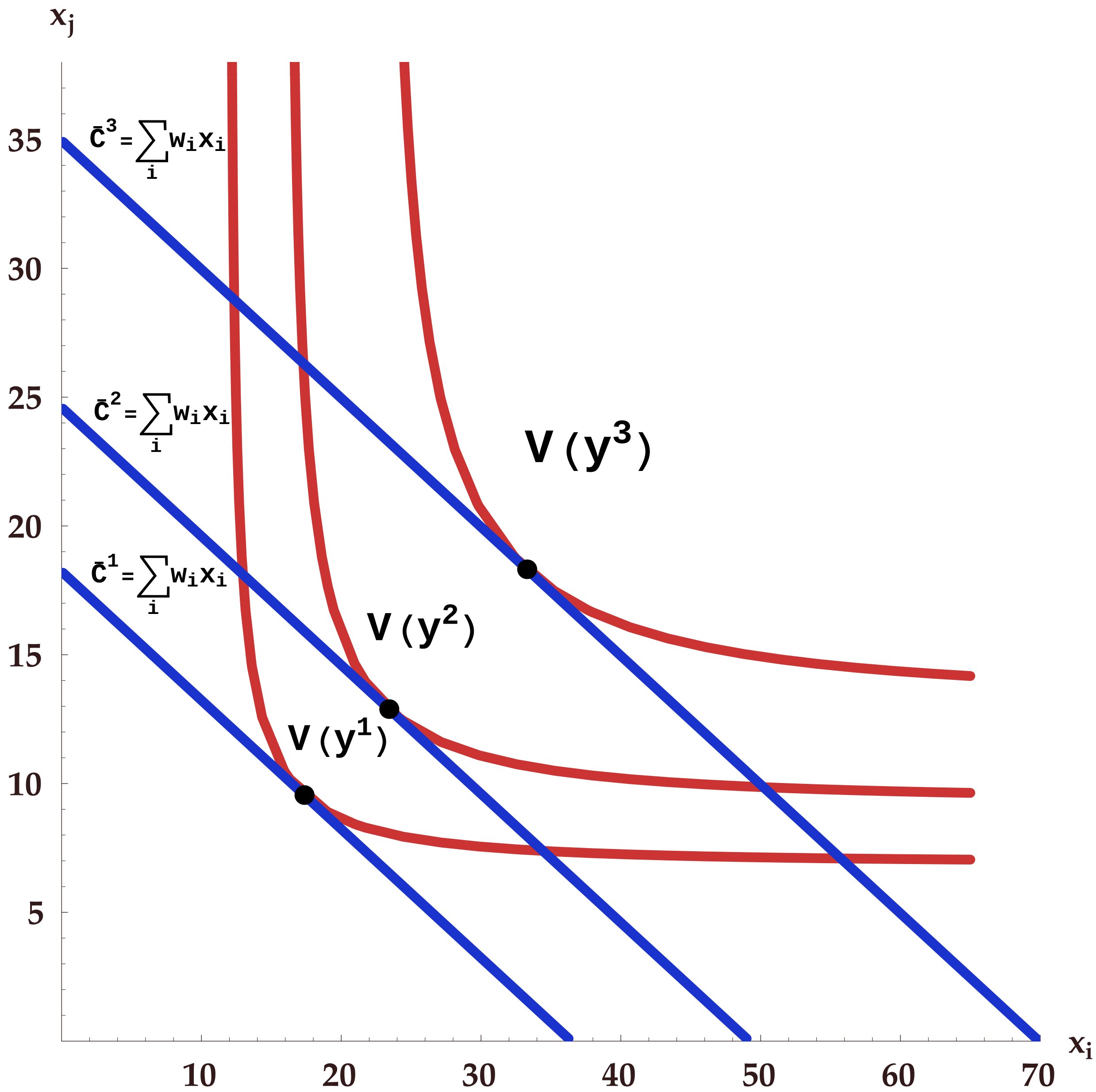
In figure 4, we can see that slope of the isocost line is tangent to the the slope of the lower boundary of V(y) at the cost minimizing point.

FIGURE 4. Cost Minimizing Input Combination



In figure 5, we can see that slope of the isocost line is tangent to the the slope of the lower boundary of V(y) at different levels of output.

FIGURE 5. Cost Minimizing Input Combinations at Alternative Output Levels



3.2. **Notes on quasiconcavity.**

Definition 1. A real valued function f, defined on a convex set $X \subset R^n$, a said to be **quasiconcave** if

$$f(\lambda x^1 \; + \; +(1-\lambda)\, x^2) \; \geq \; min[f(x^1),\; f(x^2)\,] \tag{46}$$

A function f is said to be quasiconvex if - f is quasiconcave.

Theorem 1. *Let f be a real valued function defined on a convex set $X \subset R^n$. The upper contour sets $\{(x, y)$: $x \in S$, $\alpha \leq f(x)\}$ of f are convex for every $\alpha \in R$ if and only if f is a quasiconcave function.*

Proof. Suppose that S(f,α) is a convex set for every $\alpha \in$ R and let $x^1 \in$ X, $x^2 \in$ X, $\bar{\alpha}$ = min[f(x^1), f(x^2)]. Then $x^1 \in$ S(f,$\bar{\alpha}$) and $x^2 \in$ S(f,$\bar{\alpha}$), and because S(f,$\bar{\alpha}$) is convex, (λx^1 + (1-λ)x^2) $\in$ S(f,$\bar{\alpha}$) for arbitrary λ. Hence

$$f(\lambda\, x^1 \;+\; (1-\lambda)\, x^2) \;\geq\; \bar{\alpha} = min[f(x^1),\, f(x^2)] \tag{47}$$

Conversely, let S(f,α) be any level set of f. Let $x^1 \in$ S(f,α) and $x^2 \in$ S(f,α). Then

$$f(x^1) \;\geq\; \alpha, \quad f(x^2) \;\geq\; \alpha \tag{48}$$

and because f is quasiconcave, we have

$$f(\lambda\, x^1 \;+\; (1-\lambda)\, x^2) \;\geq\; \alpha \tag{49}$$

and (λx^1 + (1-λ) x^2) $\in$ S(f,α).

□

Theorem 2. *Let f be differentiable on an open convex set $X \subset R^n$. Then f is quasiconcave if and only if for any $x^1 \in$ X, $x^2 \in$ X such that*

$$f(x^1) \;\geq\; f(x^2) \tag{50}$$

we have

$$(x^1 \;-\; x^2)'\, \nabla f(x^2) \;\geq\; 0 \tag{51}$$

Definition 2. The kth-ordered bordered determinant D_k(f,x) of a twice differentiable function f at at point x $\in R^n$ is defined as

$$D_k(f,x) \;=\; det \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_k} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_k} & \frac{\partial f}{\partial x_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_k \partial x_1} & \frac{\partial^2 f}{\partial x_k \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_k^2} & \frac{\partial f}{\partial x_k} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_k} & 0 \end{bmatrix} \quad k \;=\; 1,\, 2,\, \ldots,\, n \tag{52}$$

Definition 3. Some authors define the kth-ordered bordered determinant D_k(f,x) of a twice differentiable function f at at point x $\in R^n$ in a different fashion where the first derivatives of the function f border the Hessian of the function on the top and left as compared to in the bottom and right as in equation 52.

$$D_k(f,x) \;=\; det \begin{bmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_k} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_k} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_k} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial f}{\partial x_k} & \frac{\partial^2 f}{\partial x_k \partial x_1} & \frac{\partial^2 f}{\partial x_k \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_k^2} \end{bmatrix} \quad k \;=\; 1,\, 2,\, \ldots,\, n \tag{53}$$

The determinant in equation 52 and the determinant in equation 53 will be the same. If we interchange any two rows or any two columns of a determinant, the determinant will change sign but keep its absolute value. A certain number of row exchanges will be necessary to move the bottom row to the top. A like number of column exchanges will be necessary to move the rightmost column to the left. Given that equations 52and 53 are the same except for this even number of row and column exchanges, the determinants will be the same. You can illustrate this to yourself using the following three variable example.

$$\tilde{H}_B = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \frac{\partial f}{\partial x_2} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & 0 \end{bmatrix} \quad \hat{H}_B = \begin{bmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial f}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

3.2.1. *Quasi-concavity and bordered hessians.*

a.: If f(x) is quasi-concave on a solid (non-empty interior) convex set X ⊂ R^n, then

$$(-1)^k D_k(f, x) \geq 0, \quad k = 1, 2, \ldots, n \tag{54}$$

for every x ∈ X.

b.: If

$$(-1)^k D_k(f, x) > 0, \quad k = 1, 2, \ldots, n \tag{55}$$

for every x ∈ X,then f(x) is quasi-concave on X (Avriel [3, p.149], Arrow and Enthoven [2, p. 781-782]

If f is quasiconcave, then when k is odd, D_k(f,x) will be negative and when k is even, D_k(f,x) will be positive. Thus D_k(f,x)will alternate in sign beginning with positive in the case of two variables.

3.2.2. *Relationship of quasi-concavity to signs of minors (cofactors) of a matrix.* Let

$$F = \begin{vmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial f}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{vmatrix} = det\, H_B \tag{56}$$

where det H_B is the determinant of the bordered Hessian of the function f. Now let F_{ij} be the cofactor of $\frac{\partial^2 f}{\partial x_i \partial x_j}$ in the matrix H_B. It is clear that F_{nn} and F have opposite signs because F includes the last row and column of H_B and F_{nn} does not. If the $(-1)^n$ in front of the cofactors is positive then F_{nn} must be positive with F negative and vice versa. Since the ordering of rows since is arbitrary it is also clear that F_{ii} and F have opposite signs. Thus when a function is quasi-concave $\frac{F_{ii}}{F}$ will have a negative sign.

3.3. Second order conditions for cost minimization.

3.3.1. *Note on requirements for a minimum of a constrained problem.* Consider a general constrained optimization problem with one constraint.

$$\underset{x}{\text{maximize}}\ f(x)$$
$$\text{subject to } g(x) = 0$$

where $g(x) = 0$ denotes a constraint, We can also write this as

$$\max_{x_1,\, x_2, \ldots x_n} f(x_1,\, x_2, \ldots,\, x_n)$$

subject to

$$g\,(x_1,\, x_2, \ldots,\, x_n) \;=\; 0 \tag{57}$$

The solution can be obtained using the Lagrangian function

$$\mathcal{L}(x;\, \lambda) \;=\; f(x_1,\, x_2, \ldots) - \lambda\, g(x) \tag{58}$$

Notice that the gradient of $\mathcal{L}$ will involve a set of derivatives, i.e.

$$\nabla_x \mathcal{L} = \nabla_x f(x) - \lambda\, \nabla_x g(x)$$

There will be one equation for each x. There will also an equation involving the derivative of $\mathcal{L}$ with respect to λ. The necessary conditions for an extremum of f with the equality constraints $g(x) = 0$ are that

$$\nabla \mathcal{L}(x^*,\, \lambda^*) = 0 \tag{59}$$

where it is implicit that the gradient in (59) is with respect to both x and λ. The typical sufficient conditions for a maximum or minimum of f(x_1, x_2, . . . , x_n) subject to g(x_1, x_2,. . . , x_n) = 0 require that f and g be twice continuously differentiable real-valued functions on R^n. Then if there exist vectors $x^* \in R^n$, and $\lambda^* \in R^1$ such that

$$\nabla \mathcal{L}(x^*,\, \lambda^*) = 0 \tag{60}$$

and for every non-zero vector $z \in R^n$ satisfying

$$z' \nabla g(x^*) = 0 \tag{61}$$

it follows that

$$z' \nabla^2_x \mathcal{L}(x^*,\, \lambda^*) z \;>\; 0 \tag{62}$$

then f has a strict local minimum at x^*, subject to $g(x) = 0$. If the inequality in (62) is reversed, then f has strict local maximum at x^*.

For the cost minimization problem the Lagrangian is given by

$$\mathcal{L} \;= wx - \lambda(f(x) \,-\, y) \tag{63}$$

where the objective function is wx and the constraint is f(x) - y = 0. Differentiating equation 63 twice with respect to x we obtain

$$\begin{aligned} \nabla^2_x\, \mathcal{L}(x^*,\, \lambda^*) \;&=\; \left[-\lambda\, \frac{\partial^2 f(x)}{\partial x_i\, \partial x_j} \right] \\ &=\; -\,\lambda\, \left[\frac{\partial^2 f(x)}{\partial x_i\, \partial x_j} \right] \end{aligned} \tag{64}$$

And so the condition in equation 62 will imply that

$$
\begin{aligned}
-\lambda\, z' \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right] z \;&>\; 0 \\
\Rightarrow\; z' \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right] z \;&<\; 0
\end{aligned}
\tag{65}
$$

for all z satisfying z′∇f(x) = 0. This is also the condition for f to be a quasi-concave function. (Avriel [3, p.149]. Thus these sufficient conditions imply that f must be quasi-concave.

3.3.2. *Checking the sufficient conditions for cost minimization.* Consider the general constrained minimization problem where f and g are twice continuously differentiable real valued functions. If there exist vectors $x^* \,\epsilon\, R^n,\ \lambda^* \,\epsilon\, R^m$, such that

$$\nabla L(x^*,\, \lambda^*) = 0 \tag{66}$$

and if

$$
D(p) \;=\; (-1)\det \begin{bmatrix}
\frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_p} & \frac{\partial g(x^*)}{\partial x_1} \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_p \partial x_1} & \cdots & \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_p \partial x_p} & \frac{\partial g(x^*)}{\partial x_p} \\
\frac{\partial g(x^*)}{\partial x_1} & \cdots & \frac{\partial g(x^*)}{\partial x_p} & 0
\end{bmatrix} > 0 \tag{67}
$$

for $p = 2, 3, \ldots, n$, then f has a strict local minimum at x^*, such that

$$g(x^*) = 0 \tag{68}$$

We can also write this as follows where after multiplying both sides by negative one.

$$
D(p) \;=\; \det \begin{bmatrix}
\frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_p} & \frac{\partial g(x^*)}{\partial x_1} \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_p \partial x_1} & \cdots & \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_p \partial x_p} & \frac{\partial g(x^*)}{\partial x_p} \\
\frac{\partial g(x^*)}{\partial x_1} & \cdots & \frac{\partial g(x^*)}{\partial x_p} & 0
\end{bmatrix} \;<\; 0 \tag{69}
$$

for $p = 2, 3, \ldots, n$, then f has a strict local minimum at x^*, such that

$$g(x^*) \;=\; 0 \tag{70}$$

We check the determinants in (69) starting with the one that has 2 elements in each row and column of the Hessian and 1 element in each row or column of the derivative of the constraint with respect to x.

For the cost minimization problem, D(p) is given by

$$D(p) = \begin{vmatrix} \frac{-\lambda\partial^2 f}{\partial x_1^2} & \frac{-\lambda\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{-\lambda\partial^2 f}{\partial x_1 \partial x_p} & \frac{\partial f}{\partial x_1} \\ \frac{-\lambda\partial^2 f}{\partial x_2 \partial x_1} & \frac{-\lambda\partial^2 f}{\partial x_2^2} & \cdots & \frac{-\lambda\partial^2 f}{\partial x_2 \partial x_p} & \frac{\partial f}{\partial x_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{-\lambda\partial^2 f}{\partial x_p \partial x_1} & \frac{-\lambda\partial^2 f}{\partial x_p \partial x_2} & \cdots & \frac{-\lambda\partial^2 f}{\partial x_p^2} & \frac{\partial f}{\partial x_p} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_p} & 0 \end{vmatrix} \tag{71}$$

In order to factor -λ out of the determinant and write the second order conditions in terms of the bordered Hessian of the production function multiply last row and last column of D(p) by -λ, multiply out front by $\frac{-1}{\lambda^2}$, and then compute the determinant.

$$D(p) = (-\frac{1}{\lambda})^2 \begin{vmatrix} \frac{-\lambda\partial^2 f}{\partial x_1^2} & \frac{-\lambda\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{-\lambda\partial^2 f}{\partial x_1 \partial x_p} & -\lambda\frac{\partial f}{\partial x_1} \\ \frac{-\lambda\partial^2 f}{\partial x_2 \partial x_1} & \frac{-\lambda\partial^2 f}{\partial x_2^2} & \cdots & \frac{-\lambda\partial^2 f}{\partial x_2 \partial x_p} & -\lambda\frac{\partial f}{\partial x_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{-\lambda\partial^2 f}{\partial x_p \partial x_1} & \frac{-\lambda\partial^2 f}{\partial x_p \partial x_2} & \cdots & \frac{-\lambda\partial^2 f}{\partial x_p^2} & -\lambda\frac{\partial f}{\partial x_p} \\ -\lambda\frac{\partial f}{\partial x_1} & -\lambda\frac{\partial f}{\partial x_2} & \cdots & -\lambda\frac{\partial f}{\partial x_p} & 0 \end{vmatrix} \tag{72}$$

Now factor -λ out of all p+1 rows of D(p).

$$D(p) = (-\lambda)^{p+1}(-\frac{1}{\lambda})^2 \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_p} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_p} & \frac{\partial f}{\partial x_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_p \partial x_1} & \frac{\partial^2 f}{\partial x_p \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_p^2} & \frac{\partial f}{\partial x_p} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_p} & 0 \end{vmatrix} \tag{73}$$

Because f(x) must be greater than or equal to y in the cost minimization problem, we can assume $\lambda > 0$ so that the sign of the D(p) in equation 73 is equal to the sign of

$$(-1)^{p+1} \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_p} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_p} & \frac{\partial f}{\partial x_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_p \partial x_1} & \frac{\partial^2 f}{\partial x_p \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_p^2} & \frac{\partial f}{\partial x_p} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_p} & 0 \end{vmatrix} \tag{74}$$

For a minimum we want $|D| < 0$ for p = 2,...,n. Therefore for a minimum, a sufficient condition is that

$$(-1)^{p+1}\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_p} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_p} & \frac{\partial f}{\partial x_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_p \partial x_1} & \frac{\partial^2 f}{\partial x_p \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_p^2} & \frac{\partial f}{\partial x_p} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_p} & 0 \end{vmatrix} < 0 \tag{75}$$

or

$$(-1)^{p}\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_p} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_p} & \frac{\partial f}{\partial x_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_p \partial x_1} & \frac{\partial^2 f}{\partial x_p \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_p^2} & \frac{\partial f}{\partial x_p} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_p} & 0 \end{vmatrix} > 0 \tag{76}$$

This condition in equation 76 is the condition for the quasi-concavity of the production function f from equation 55. (Avriel [3, p. 149].

3.3.3. *Example problem with two variable inputs.* The Lagrangian function is given by

$$\mathcal{L} = w_1\, x_1 + w_2\, x_2 - \lambda(f(x_1,\, x_2) - y) \tag{77}$$

The first order conditions are as follows

$$\frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \lambda\, \frac{\partial f}{\partial x_1} = 0 \tag{78a}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = w_2 - \lambda\, \frac{\partial f}{\partial x_2} = 0 \tag{78b}$$

$$-\,\lambda\,(f(x) - y) = 0 \tag{78c}$$

The bordered Hessian for the Lagrangian is given by

$$H_B = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(x^*,\, \lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(x^*,\, \lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 \mathcal{L}(x^*,\, \lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}(x^*,\, \lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0 \end{bmatrix} \tag{79}$$

$$= \begin{bmatrix} \frac{-\lambda \partial^2 f}{\partial x_1^2} & \frac{-\lambda \partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1} \\ \frac{-\lambda \partial^2 f}{\partial x_2 \partial x_1} & \frac{-\lambda \partial^2 f}{\partial x_2^2} & \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & 0 \end{bmatrix}$$

The determinant of this matrix must be negative for this solution to be a minimum. To see how this relates to the bordered Hessian of the production function multiply the last row and column by -λ and the whole determinant by $\frac{-1}{\lambda^2}$ as follows

$$\left(\frac{-1}{\lambda^2}\right)\begin{vmatrix} \frac{-\lambda\partial^2 f}{\partial x_1^2} & \frac{-\lambda\partial^2 f}{\partial x_1 \partial x_2} & -\lambda\,\frac{\partial f}{\partial x_1} \\ \frac{-\lambda\partial^2 f}{\partial x_2 \partial x_1} & \frac{-\lambda\partial^2 f}{\partial x_2^2} & -\lambda\,\frac{\partial f}{\partial x_2} \\ -\lambda\,\frac{\partial f}{\partial x_1} & -\lambda\,\frac{\partial f}{\partial x_2} & 0 \end{vmatrix} = (-\lambda)^3\left(\frac{-1}{\lambda^2}\right)\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & 0 \end{vmatrix} \tag{80}$$

With $\lambda > 0$ this gives

$$(-1)^3\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & 0 \end{vmatrix} \tag{81}$$

so that

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & 0 \end{vmatrix} > 0 \tag{82}$$

for a minimum. This is the condition for a quasi-concave function with two variables. If there were three variables, the determinant of the next bordered Hessian would be negative.

3.4. **Input demands.** If we solve the equations in 41 for x_j, j = 1, 2, ..., n, and λ, we obtain the optimal values of x for a given y and w. As a function of w for a fixed y, this gives the vector of factor demands for x and the optimal Lagrangian multiplier λ(y,w).

$$
\begin{aligned}
\tilde{x} &= \tilde{x}(y,\, w) \;=\; (\tilde{x}_1(y,w), \tilde{x}_2(y,w), \ldots, \tilde{x}_n(y,w)) \\
\tilde{\lambda} &= \tilde{\lambda}(y,\, w)
\end{aligned}
\tag{83}
$$

3.5. **Sensitivity analysis.** We can investigate the properties of $\tilde{x}$(y,w) by substituting $\tilde{x}$(y,w) for x in equation 41 and then treating it as an identity.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= w_1 - \quad \lambda\, \frac{\partial f(\tilde{x}_1(y,w), \tilde{x}_2(y,w), \ldots, \tilde{x}_n(y,w))}{\partial x_1} \;=\; 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= w_2 - \quad \lambda\, \frac{\partial f(\tilde{x}_1(y,w), \tilde{x}_2(y,w), \ldots, \tilde{x}_n(y,w))}{\partial x_2} \;=\; 0 \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
\frac{\partial \mathcal{L}}{\partial x_n} &= w_n - \quad \lambda\, \frac{\partial f(\tilde{x}_1(y,w), \tilde{x}_2(y,w), \ldots, \tilde{x}_n(y,w))}{\partial x_n} \;=\; 0
\end{aligned}
\tag{84a}
$$

$$
-\,\lambda\,(f((\tilde{x}_1(y,w), \tilde{x}_2(y,w), \ldots, \tilde{x}_n(y,w)) \;-\; y) \;=\; 0 \tag{84b}
$$

Where it is obvious we write x(y,w) for $\tilde{x}$(y,w) and x_j(y,w) for $\tilde{x}_j$(y,w). If we differentiate the first equation in 84a with respect to w_j we obtain

$$
\begin{aligned}
&0 - \lambda \left(\frac{\partial^2 f(x(y,\, w))}{\partial x_1^2} \frac{\partial x_1(y,\, w)}{\partial w_j} + \frac{\partial^2 f(x(y,w))}{\partial x_2\, \partial x_1} \frac{\partial x_2(y,w)}{\partial w_j} + \frac{\partial^2 f(x(y,w))}{\partial x_3 \partial x_1} \frac{\partial\, x_3(y,w)}{\partial w_j} + \cdots \right) - \frac{\partial f(x(y,\, w))}{\partial x_1} \frac{\partial \lambda(y,\, w)}{\partial w_j} \;\equiv\; 0 \\
&\Rightarrow \lambda \begin{pmatrix} \frac{\partial^2 f(x(y,w))}{\partial x_1^2} & \frac{\partial^2 f(x(y,w))}{\partial x_2\, \partial x_1} & \frac{\partial^2 f(x(y,w))}{\partial x_3 \partial x_1} & \cdots & \frac{\partial^2 f(x(y,w))}{\partial x_n \partial x_1} & \frac{\partial f(x(y,w))}{\partial x_1} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(y,w)}{\partial w_j} \\ \frac{\partial x_2(y,w)}{\partial w_j} \\ \frac{\partial x_3(y,w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_n(y,w)}{\partial w_j} \\ \frac{1}{\lambda} \frac{\partial \lambda(y,w)}{\partial w_j} \end{pmatrix} \;\equiv\; 0
\end{aligned}
\tag{85}
$$

If we differentiate the second equation in 84a with respect to w_j we obtain

$$0 - \lambda \left(\frac{\partial^2 f(x(y,\,w))}{\partial x_1 \partial x_2} \frac{\partial x_1(p,\,w)}{\partial w_j} + \frac{\partial^2 f(x(y,w))}{\partial x_2^2} \frac{\partial x_2(y,w)}{\partial w_j} + \frac{\partial^2 f(x(y,w))}{\partial x_3 \partial x_2} \frac{\partial\, x_3(y,w)}{\partial w_j} + \cdots \right) - \frac{\partial f(x(y,\,w))}{\partial x_2} \frac{\partial \lambda(y,\,w)}{\partial w_j} \;\equiv\; 0$$

$$\Rightarrow \lambda \begin{pmatrix} \frac{\partial^2 f(x(y,\,w))}{\partial x_1\, \partial x_2} & \frac{\partial^2 f(x(y,w))}{\partial x_2^2} & \frac{\partial^2 f(x(y,w))}{\partial x_3 \partial x_2} \cdots \frac{\partial^2 f(x(y,w))}{\partial x_n \partial x_2} & \frac{\partial f(x(y,\,w))}{\partial x_2} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(y,\,w)}{\partial w_j} \\ \frac{\partial x_2(y,\,w)}{\partial w_j} \\ \frac{\partial x_3(y,\,w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_n(y,\,w)}{\partial w_j} \\ \frac{1}{\lambda} \frac{\partial \lambda(y,\,w)}{\partial w_j} \end{pmatrix} \;\equiv\; 0 \tag{86}$$

If we differentiate the jth equation in 84a with respect to w_j we obtain

$$1 - \lambda \left(\frac{\partial^2 f(x(y,\,w))}{\partial x_1 \partial x_j} \frac{\partial x_1(y,\,w)}{\partial w_j} + \frac{\partial^2 f(x(y,\,w))}{\partial x_2\, \partial x_j} \frac{\partial x_2(y,w)}{\partial w_j} + \cdots + \frac{\partial^2 f(x(y,\,w))}{\partial x_j^2} \frac{\partial\, x_j(y,w)}{\partial w_j} + \cdots \right) - \frac{\partial f(x(y,\,w))}{\partial x_j} \frac{\partial \lambda(y,\,w)}{\partial w_j} \;\equiv\; 0$$

$$\Rightarrow \lambda \begin{pmatrix} \frac{\partial^2 f(x(y,\,w))}{\partial x_1\, \partial x_j} & \frac{\partial^2 f(x(y,w))}{\partial x_2\, \partial x_j} & \cdots & \frac{\partial^2 f(x(y,w))}{\partial x_j^2} \cdots \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_j} & \frac{\partial f(x(y,\,w))}{\partial x_j} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(y,\,w)}{\partial w_j} \\ \frac{\partial x_2(y,\,w)}{\partial w_j} \\ \frac{\partial x_3(y,\,w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_n(y,\,w)}{\partial w_j} \\ \frac{1}{\lambda} \frac{\partial \lambda(y,\,w)}{\partial w_j} \end{pmatrix} \;\equiv\; 1 \tag{87}$$

Continuing in the same fashion we obtain

$$\lambda \begin{pmatrix} \frac{\partial^2 f(x(y,w))}{\partial x_1^2} & \frac{\partial^2 f(x(y,w)}{\partial x_2\, \partial x_1} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j \partial x_1} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_1} & \frac{\partial f(x(y,w)}{\partial x_1} \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x(y,w)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j \partial x_2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_2} & \frac{\partial f(x(y,w)}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_j} & \frac{\partial^2 f(x(y,w)}{\partial x_2 \partial x_j} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j^2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_j} & \frac{\partial f(x(y,w)}{\partial x_j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_n} & \frac{\partial^2 f(x(y,w)}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j \partial x_n} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n^2} & \frac{\partial f(x(y,w)}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(y,w)}{\partial w_j} \\ \frac{\partial x_2(y,w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_j(y,w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_n(y,w)}{\partial w_j} \\ \frac{1}{\lambda} \frac{\partial \lambda(y,w)}{\partial w_j} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \tag{88}$$

This is a system of n equations in n+1 unknowns. Equation 84b implies that -λ(y,w) $(f((\tilde{x}_1(y,w), \tilde{x}_2(y,w), \ldots, \tilde{x}_n(y,$
If we differentiate equation 84b with respect to w_j we obtain

$$-\lambda \left(\frac{\partial f(x(y,\,w))}{\partial x_1} \frac{\partial x_1(y,\,w)}{\partial w_j} + \frac{\partial f(x(y,w)}{\partial x_2} \frac{\partial x_2(y,w)}{\partial w_j} + \cdots + \frac{\partial f(x(y,w)}{\partial x_j} \frac{\partial\, x_j(y,w)}{\partial w_j} + \cdots + \frac{\partial f(x(y,w)}{\partial x_j} \frac{\partial\, x_n(y,w)}{\partial w_j} \right)$$
$$-\; (\, f(x(y,w) \;-\; y\,)\, \frac{\partial \lambda(y,\,w)}{\partial w_j} \;\equiv\; 0 \tag{89}$$

If the solution is such that λ(y,w) is not zero, then $(\, f(x(y,w) \;-\; y\,)$ is equal to zero and we can write

$$\lambda \begin{pmatrix} \frac{\partial f(x(y,w))}{\partial x_1} & \frac{\partial f(x(y,w)}{\partial x_2} & \cdots & \frac{\partial f(x(y,w)}{\partial x_j} & \cdots & \frac{\partial f(x(y,w)}{\partial x_n} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(y,w)}{\partial w_j} \\ \frac{\partial x_2(y,w)}{\partial w_j} \\ \frac{\partial x_3(y,w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_n(y,w)}{\partial w_j} \\ \frac{1}{\lambda}\frac{\partial \lambda(y,w)}{\partial w_j} \end{pmatrix} \equiv 0 \tag{90}$$

Combining equations 88 and 90 we obtain

$$\lambda \begin{pmatrix} \frac{\partial^2 f(x(y,w))}{\partial x_1^2} & \frac{\partial^2 f(x(y,w)}{\partial x_2\,\partial x_1} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j \partial x_1} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_1} & \frac{\partial f(x(y,w)}{\partial x_1} \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x(y,w)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j \partial x_2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_2} & \frac{\partial f(x(y,w)}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_j} & \frac{\partial^2 f(x(y,w)}{\partial x_2 \partial x_j} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j^2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_j} & \frac{\partial f(x(y,w)}{\partial x_j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_n} & \frac{\partial^2 f(x(y,w)}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j \partial x_n} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n^2} & \frac{\partial f(x(y,w)}{\partial x_n} \\ \frac{\partial f(x(y,w))}{\partial x_1} & \frac{\partial f(x(y,w)}{\partial x_2} & \cdots & \frac{\partial f(x(y,w)}{\partial x_j} & \cdots & \frac{\partial f(x(y,w)}{\partial x_n} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(y,w)}{\partial w_j} \\ \frac{\partial x_2(y,w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_j(y,w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_n(y,w)}{\partial w_j} \\ \frac{1}{\lambda}\frac{\partial \lambda(y,w)}{\partial w_j} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{91}$$

If we then consider derivatives with respect to each of the w_j we obtain

$$\lambda \begin{pmatrix} \frac{\partial^2 f(x(y,w))}{\partial x_1^2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_1} & \frac{\partial f(x(y,w)}{\partial x_1} \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_2} & \frac{\partial f(x(y,w)}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n^2} & \frac{\partial f(x(y,w)}{\partial x_n} \\ \frac{\partial f(x(y,w))}{\partial x_1} & \cdots & \frac{\partial f(x(y,w)}{\partial x_n} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(y,w)}{\partial w_1} & \frac{\partial x_1(y,w)}{\partial w_2} & \cdots & \frac{\partial x_1(y,w)}{\partial w_n} \\ \frac{\partial x_2(y,w)}{\partial w_1} & \frac{\partial x_2(y,w)}{\partial w_2} & \cdots & \frac{\partial x_2(y,w)}{\partial w_n} \\ \vdots & & & \\ \frac{\partial x_n(y,w)}{\partial w_1} & \frac{\partial x_n(y,w)}{\partial w_2} & \cdots & \frac{\partial x_n(y,w)}{\partial w_n} \\ \frac{1}{\lambda}\frac{\partial \lambda(y,w)}{\partial w_1} & \frac{1}{\lambda}\frac{\partial \lambda(y,w)}{\partial w_n} & \cdots & \frac{1}{\lambda}\frac{\partial \lambda(y,w)}{\partial w_n} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{92}$$

The matrix on the leftmost side of equation 92 is the just the bordered Hessian of the production function from equation 76 which is used in verifying that the extreme point is a minimum where p = n. We repeat it there for convenience.

$$(-1)^p \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_p} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_p} & \frac{\partial f}{\partial x_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_p \partial x_1} & \frac{\partial^2 f}{\partial x_p \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_p^2} & \frac{\partial f}{\partial x_p} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_p} & 0 \end{vmatrix} > 0$$

3.5.1. *Obtaining derivatives using Cramer's rule.* We can obtain the various derivatives using Cramer's rule. For example to obtain $\frac{\partial x_1(y,w)}{\partial w_1}$ we would replace the first column of the bordered Hessian with the first column of matrix on the right of equation 91. To obtain $\frac{\partial x_2(y,w)}{\partial w_1}$ we would replace the second column of the bordered Hessian with the first column of matrix on the right of equation 91. To obtain $\frac{\partial x_2(y,w)}{\partial w_j}$, we would replace the second column of the bordered Hessian with the jth column of matrix on the right of equation 91 and so forth.

Consider for example the case of two variable inputs. The bordered Hessian is given by

$$H_B = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & 0 \end{bmatrix} \tag{93}$$

The determinant of the H_B in equation 93 is often referred to as F and is given by

$$\begin{aligned} F &= 0 \cdot (-1)^6 \cdot \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} + \frac{\partial f}{\partial x_2} \cdot (-1)^5 \cdot \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial f}{\partial x_2} \end{vmatrix} + \frac{\partial f}{\partial x_1} \cdot (-1)^4 \cdot \begin{vmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial f}{\partial x_2} \end{vmatrix} \\ &= \frac{\partial f}{\partial x_1} \cdot \begin{vmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial f}{\partial x_2} \end{vmatrix} - \frac{\partial f}{\partial x_2} \cdot \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial f}{\partial x_2} \end{vmatrix} \\ &= \frac{\partial f}{\partial x_1} \cdot \left[\frac{\partial^2 f}{\partial x_1 \partial x_2} \frac{\partial f}{\partial x_2} - \frac{\partial f}{\partial x_1} \frac{\partial^2 f}{\partial x_2^2} \right] - \frac{\partial f}{\partial x_2} \cdot \left[\frac{\partial^2 f}{\partial x_1^2} \frac{\partial f}{\partial x_2} - \frac{\partial f}{\partial x_1} \frac{\partial^2 f}{\partial x_2 \partial x_1} \right] \\ &= 2 \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \frac{\partial^2 f}{\partial x_1 \partial x_2} - \left(\frac{\partial f}{\partial x_1} \right)^2 \frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial f}{\partial x_2} \right)^2 \frac{\partial^2 f}{\partial x_1^2} \\ &= 2\, f_1\, f_2\, f_{12} - f_1^2 f_{22} - f_2^2\, f_{11} \end{aligned} \tag{94}$$

We know that this expression is positive by the quasiconcavity of f or the second order conditions for cost minimization. We can determine $\frac{\partial x_1}{\partial w_2}$ by replacing the first column of F with the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Doing so we obtain

$$\begin{bmatrix} 0 & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1} \\ 1 & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial f}{\partial x_2} \\ 0 & \frac{\partial f}{\partial x_2} & 0 \end{bmatrix} \tag{95}$$

The determinant of the matrix in equation 95 is easy to compute and is given by

$$\begin{aligned} &\frac{\partial f}{\partial x_2} \cdot (-1)^5 \cdot \begin{vmatrix} 0 & \frac{\partial f}{\partial x_1} \\ 1 & \frac{\partial f}{\partial x_2} \end{vmatrix} \\ &= -\frac{\partial f}{\partial x_2} \cdot \left(-\frac{\partial f}{\partial x_1} \right) = \frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial x_2} \geq 0 \end{aligned} \tag{96}$$

Given that F is positive, this implies that $\frac{\partial x_1}{\partial w_2}$ is positive. To find $\frac{\partial x_2}{\partial w_2}$ we replace the second column of F with the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & 0 & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & 1 & \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & 0 & 0 \end{bmatrix} \tag{97}$$

The determinant of the matrix in equation 97 is also easy to compute and is given by

$$\frac{\partial f}{\partial x_1} \cdot (-1)^4 \cdot \begin{vmatrix} 0 & \frac{\partial f}{\partial x_1} \\ 1 & \frac{\partial f}{\partial x_2} \end{vmatrix} = \frac{\partial f}{\partial x_1} \cdot \left(- \frac{\partial f}{\partial x_1} \right) = - \left(\frac{\partial f}{\partial x_1} \right)^2 \leq 0 \tag{98}$$

Given that F is positive, this implies that $\frac{\partial x_2}{\partial w_2}$ is negative.

3.5.2. *Obtaining derivatives by matrix inversion.* We can also solve equation 92 directly for the response function derivatives by inverting the bordered Hessian matrix. This then implies that

$$\begin{pmatrix} \frac{\partial x_1(y,w)}{\partial w_1} & \frac{\partial x_1(y,w)}{\partial w_2} & \cdots & \frac{\partial x_1(y,w)}{\partial w_n} \\ \frac{\partial x_2(y,w)}{\partial w_1} & \frac{\partial x_2(y,w)}{\partial w_2} & \cdots & \frac{\partial x_2(y,w)}{\partial w_n} \\ \vdots & & & \\ \frac{\partial x_n(y,w)}{\partial w_1} & \frac{\partial x_n(y,w)}{\partial w_2} & \cdots & \frac{\partial x_n(y,w)}{\partial w_n} \\ \frac{1}{\lambda}\frac{\partial \lambda(y,w)}{\partial w_1} & \frac{1}{\lambda}\frac{\partial \lambda(y,w)}{\partial w_2} & \cdots & \frac{1}{\lambda}\frac{\partial \lambda(y,w)}{\partial w_n} \end{pmatrix} \equiv \frac{1}{\lambda} \begin{pmatrix} \frac{\partial^2 f(x(y,w))}{\partial x_1^2} & \frac{\partial^2 f(x(y,w)}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_1} & \frac{\partial f(x(y,w)}{\partial x_1} \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x(y,w))}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_2} & \frac{\partial f(x(y,w)}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_n} & \frac{\partial^2 f(x(y,w))}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n^2} & \frac{\partial f(x(y,w)}{\partial x_n} \\ \frac{\partial f(x(y,w))}{\partial x_1} & \frac{\partial f(x(y,w))}{\partial x_2} & \cdots & \frac{\partial f(x(y,w)}{\partial x_n} & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{99}$$

For example $\frac{\partial x_2}{\partial w_2}$ would be the (2,2) element of the inverse of the bordered Hessian. We can determine this element using the formula for the inverse of a matrix involving the determinant of the matrix and the adjoint of the matrix. The **adjoint** of the matrix A denoted adj (A) or A^+ is the transpose of the matrix obtained from A by replacing each element a_{ij} by its cofactor A_{ij}. For a square nonsingular matrix A, its inverse is given by

$$A^{-1} = \frac{1}{|A|} A^+ \tag{100}$$

We can compute (2,2) element of the inverse of the bordered Hessian from the cofactor of that element divided by the determinant of the bordered Hessian which we have denoted F. It is clear from section 3.2.2 that F_{22} (the cofactor of the (2,2) element) and F have opposite signs so that the partial derivative will be negative. Given that the Hessian is a symmetric matrix, it is also clear that $\frac{\partial x_i}{\partial w_j} = \frac{\partial x_j}{\partial w_i}$.

Consider the two variable case with bordered Hessian

$$H_B = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & 0 \end{bmatrix} \tag{101}$$

We found the determinant of H_B in equation 94, i.e,

$$\begin{aligned} det\, H_B = F &= 2 \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \frac{\partial^2 f}{\partial x_1 \partial x_2} - \left(\frac{\partial f}{\partial x_1} \right)^2 \frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial f}{\partial x_2} \right)^2 \frac{\partial^2 f}{\partial x_1^2} \\ &= 2\, f_1\, f_2\, f_{12} - f_1^2 f_{22} - f_2^2\, f_{11} \end{aligned} \tag{102}$$

Each element of the inverse will have this expression as a denominator. To compute the numerator of the (1,1) element of the inverse we compute the cofactor of the (1,1) element of the bordered Hessian. This is given by

$$\begin{aligned} & (-1)^2 \cdot \begin{vmatrix} \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_2} & 0 \end{vmatrix} \\ & = -\left(\frac{\partial f}{\partial x_2}\right)^2 \end{aligned} \tag{103}$$

To compute the numerator of the (1,2) element of the inverse we compute the cofactor of the (2,1) element of the bordered Hessian. This is given by

$$\begin{aligned} & (-1)^3 \cdot \begin{vmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} & 0 \end{vmatrix} \\ & = \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \end{aligned} \tag{104}$$

To compute the numerator of the (3,2) element of the inverse we compute the cofactor of the (2,3) element of the bordered Hessian. This is given by

$$\begin{aligned} & (-1)^5 \cdot \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{vmatrix} \\ & = (-1) \left(\frac{\partial^2 f}{\partial x_1^2} \frac{\partial f}{\partial x_2} - \frac{\partial^2 f}{\partial x_1 \partial x_2} \frac{\partial f}{\partial x_1} \right) \\ & = \frac{\partial f}{\partial x_1} \frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial^2 f}{\partial x_1^2} \end{aligned} \tag{105}$$

To compute the numerator of the (3,3) element of the inverse we compute the cofactor of the (3,3) element of the bordered Hessian. This is given by

$$\begin{aligned} & (-1)^6 \cdot \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} \\ & = \frac{\partial^2 f}{\partial x_1^2} \frac{\partial f}{\partial x_2^2} - \frac{\partial^2 f}{\partial x_1 \partial x_2} \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ & = \frac{\partial^2 f}{\partial x_1^2} \frac{\partial f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} \right)^2 \end{aligned} \tag{106}$$

This expression will be positive or zero for a concave function. We can now start to fill in the inverse matrix of the bordered Hessian.

$$(H_B)^{-1} = \begin{bmatrix} \frac{-\left(\frac{\partial f}{\partial x_2}\right)^2}{2\frac{\partial f}{\partial x_1}\,\frac{\partial f}{\partial x_2}\,\frac{\partial^2 f}{\partial x_1 \partial x_2} - \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial f}{\partial x_2}\right)^2 \frac{\partial^2 f}{\partial x_1^2}} & \frac{\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_1}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} & \frac{?}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} \\ \frac{\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_1}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} & \frac{-f_{11}^2}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} & \frac{\frac{\partial f}{\partial x_1}\,\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial f}{\partial x_2}\,\frac{\partial^2 f}{\partial x_1^2}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} \\ \frac{?}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} & \frac{\frac{\partial f}{\partial x_1}\,\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial f}{\partial x_2}\,\frac{\partial^2 f}{\partial x_1^2}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} & \frac{\frac{\partial^2 f}{\partial x_1^2}\,\frac{\partial f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} \end{bmatrix}$$

$$= \frac{\begin{bmatrix} -f_2^2 & f_1\,f_2 & f_2\,f_{12} - f_1\,f_{22} \\ f_1\,f_2 & -f_1^2 & f_1\,f_{12} - f_2\,f_{11} \\ ? & f_1\,f_{12} - f_2\,f_{11} & f_{11}\,f_{22} - f_{12}^2 \end{bmatrix}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} \tag{107}$$

If we postmultiply this matrix by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ we can obtain the comparative statics matrix.

$$\frac{1}{\lambda}\begin{bmatrix} \frac{-\left(\frac{\partial f}{\partial x_2}\right)^2}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} & \frac{\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_1}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} & \frac{\frac{\partial f}{\partial x_2}\,\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial f}{\partial x_1}\,\frac{\partial^2 f}{\partial x_2^2}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} \\ \frac{\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_1}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} & \frac{-f_{11}^2}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} & \frac{\frac{\partial f}{\partial x_1}\,\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial f}{\partial x_2}\,\frac{\partial^2 f}{\partial x_1^2}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} \\ \frac{\frac{\partial f}{\partial x_2}\,\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial f}{\partial x_1}\,\frac{\partial^2 f}{\partial x_2^2}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} & \frac{\frac{\partial f}{\partial x_1}\,\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial f}{\partial x_2}\,\frac{\partial^2 f}{\partial x_1^2}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} & \frac{\frac{\partial^2 f}{\partial x_1^2}\,\frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} \end{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} \frac{\partial x_1(y,w)}{\partial w_1} & \frac{\partial x_1(y,w)}{\partial w_2} \\ \frac{\partial x_2(y,w)}{\partial w_1} & \frac{\partial x_2(y,w)}{\partial w_2} \\ \frac{1}{\lambda}\frac{\partial \lambda(y,w)}{\partial w_1} & \frac{1}{\lambda}\frac{\partial \lambda(y,w)}{\partial w_2} \end{pmatrix} \tag{108}$$

For example, $\frac{\partial x_2}{\partial w_1}$ is equal to

$$\frac{1}{\lambda}\left[\frac{\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_1}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} \quad \frac{-f_{11}^2}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} \quad \frac{\frac{\partial f}{\partial x_1}\,\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial f}{\partial x_2}\,\frac{\partial^2 f}{\partial x_1^2}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} \right] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\lambda}\frac{\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_1}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}} \tag{109}$$

or $\frac{\partial \lambda}{\partial w_1}$ is equal to $\frac{\frac{\partial f}{\partial x_2}\,\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial f}{\partial x_1}\,\frac{\partial^2 f}{\partial x_2^2}}{2f_1\,f_2\,f_{12} - f_1^2\,f_{22} - f_2^2\,f_{11}}$.

3.6. **The impact of output on input demands.** If we differentiate the first order conditions in equation 84 with respect to y instead of with respect to w_j the system in equation 91 would be as follows.

$$\lambda \begin{pmatrix} \frac{\partial^2 f(x(y,w))}{\partial x_1^2} & \frac{\partial^2 f(x(y,w)}{\partial x_2\,\partial x_1} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j \partial x_1} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_1} & \frac{\partial f(x(y,w)}{\partial x_1} \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x(y,w)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j \partial x_2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_2} & \frac{\partial f(x(y,w)}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_j} & \frac{\partial^2 f(x(y,w)}{\partial x_2 \partial x_j} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j^2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_j} & \frac{\partial f(x(y,w)}{\partial x_j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_n} & \frac{\partial^2 f(x(y,w)}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j \partial x_n} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n^2} & \frac{\partial f(x(y,w)}{\partial x_n} \\ \frac{\partial f(x(y,w))}{\partial x_1} & \frac{\partial f(x(y,w)}{\partial x_2} & \cdots & \frac{\partial f(x(y,w)}{\partial x_j} & \cdots & \frac{\partial f(x(y,w)}{\partial x_n} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(y,w)}{\partial y} \\ \frac{\partial x_2(y,w)}{\partial y} \\ \vdots \\ \frac{\partial x_j(y,w)}{\partial y} \\ \vdots \\ \frac{\partial x_n(y,w)}{\partial y} \\ \frac{1}{\lambda}\frac{\partial \lambda(y,w)}{\partial y} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{110}$$

Consider the impact of an increase in output on marginal cost. To investigate this response replace the last column of the bordered Hessian with the right hand side of equation 110. This will yield

$$\begin{pmatrix} \frac{\partial^2 f(x(y,w))}{\partial x_1^2} & \frac{\partial^2 f(x(y,w)}{\partial x_2\,\partial x_1} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j \partial x_1} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_1} & 0 \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x(y,w)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j \partial x_2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_j} & \frac{\partial^2 f(x(y,w)}{\partial x_2 \partial x_j} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j^2} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n \partial x_j} & 0 \\ \frac{\partial^2 f(x(y,w))}{\partial x_1 \partial x_n} & \frac{\partial^2 f(x(y,w)}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_j \partial x_n} & \cdots & \frac{\partial^2 f(x(y,w)}{\partial x_n^2} & 0 \\ \frac{\partial f(x(y,w))}{\partial x_1} & \frac{\partial f(x(y,w)}{\partial x_2} & \cdots & \frac{\partial f(x(y,w)}{\partial x_j} & \cdots & \frac{\partial f(x(y,w)}{\partial x_n} & 1 \end{pmatrix} \tag{111}$$

We can then find $\frac{1}{\lambda}\,\frac{\partial \lambda(y,w)}{\partial y}$ using Cramer's rule noting that $\frac{1}{\lambda}$ appears on both sides of the expression for $\frac{\partial \lambda(y,w)}{\partial y}$ and so can be ignored.

An alternative way to investigate the impact of output on marginal cost is to totally differentiate the first order conditions as follows.

$$\begin{aligned} -\lambda\, \Sigma_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\, dx_j - \frac{\partial f}{\partial x_i}\, d\lambda \;&=\; 0 \\ \Rightarrow \frac{\partial f}{\partial x_i} \;&=\; -\Sigma\, \lambda \frac{\partial^2 f}{\partial x_i \partial x_j}\, \frac{dx_j}{d\lambda} \end{aligned} \tag{112}$$

Now substitute into the differential of the constraint by replacing $\frac{\partial f}{\partial x_i}$ in

$$dy \;=\; \Sigma\, \frac{\partial f}{\partial x_i}\, dx_i \tag{113}$$

with the above expression to obtain

$$dy\, d\lambda \;=\; -\lambda\, \Sigma_i \Sigma_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\, dx_j\, dx_i \;>\; 0 \tag{114}$$

because $\lambda > 0$ and the summation is just the quadratic form for the concave function f which is always negative. Because changes in marginal cost and output have the same signs, marginal cost is increasing as a function of output. Thus with concave production, marginal cost is always upward sloping (Shephard [16, p. 83; 88-92]). For further discussion of this traditional approach see Allen [1, p.502-504], Ferguson [10, p. 109; 137-140], or Samuelson [15, p. 64-65].

4. Cost Function Duality

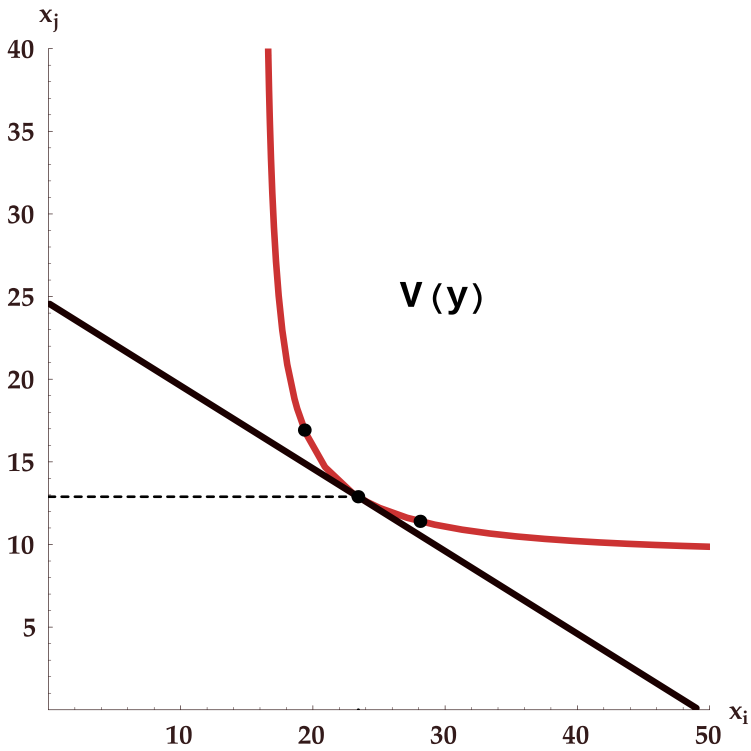
We can show that if we have an arbitrarily specified cost function that satisfies the conditions in section 2.3, we can construct from it a reasonable technology (specifically the input requirement function) that satisfies the conditions we specified in section 1.1.1 and that would generate the specified cost function.

4.1. **Constructing the input requirement set from the cost function.** Let $V^*(y)$ be defined as

$$\begin{aligned} V^*(y) &\equiv \{x : wx \geq C(y,w), \forall w > 0\} \\ &\equiv \cap_{\,w\,>\,0}\, \{x :\ w\,x\ \geq\ C(y,\ w)\} \end{aligned} \tag{115}$$

To see the intuition of why we can construct V(y) in this way consider figure 6

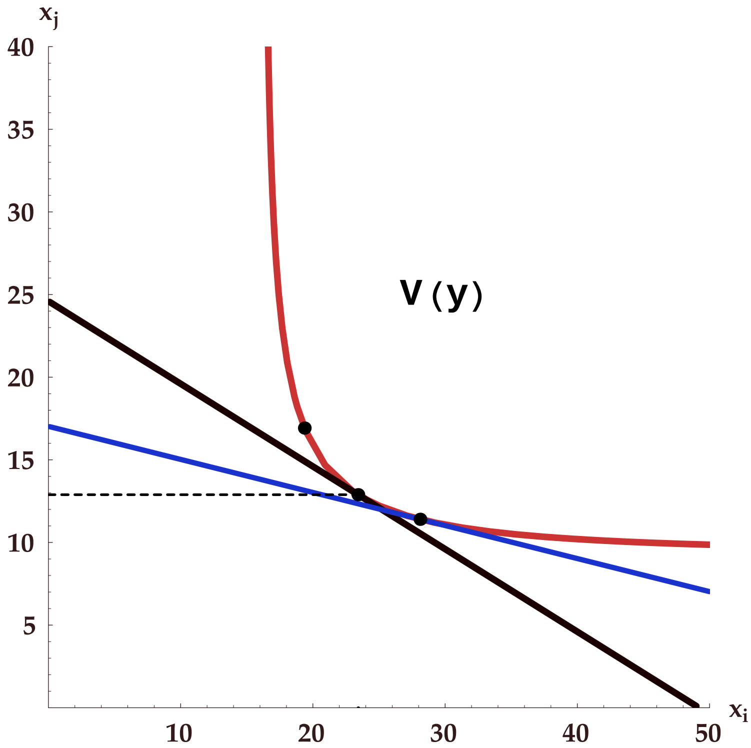
FIGURE 6. Half-Spaces and Input Requirement Set



If we pick a particular set of prices then $V^*(y)$ consists of all points above the line that is tangent to V(y) at the optimal input combination. The equation $\{x) : wx \geq C(y,w)$ for a particular set of prices defines a line in R^2 or a hyperplane in R^n. Points above the line are considered to be in $V^*(y)$. Now pick a different set of prices with w_2 higher and construct a different line as in figure 7. The number of points in $V^*(y)$ is now less than before. If we add a third set of prices with w_2 lower than in the original case as in figure 8, we can reduce the size of $V^*(y)$ even more. Continuing in this fashion we can recover V(y). This is an application of what is often called Minkowski's theorem.

Theorem 3. *Suppose V(y) satisfies properties 1, 2 (strong), 5 and 7. If $y \geq 0$ and is an element of the domain of V then V(y) = $V^*(y)$*

FIGURE 7. Two Half-Spaces and Input Requirement Set



Proof. First we prove that if $x^i \in V(y)$ then $x^i \in V^*(y)$

Let $x^i \epsilon V(y)$, then $w\, x^i \geq C(y, w)\ \forall\ w$ ¿ 0, thus $x^i \epsilon V^*(y)$

Now we need to show that if $x^i \notin V(y)$ then $x^i \notin V^*(y)$. First note that if $y > 0$ then $0_n \notin V(y)$. Also note that R^n_+ excluding V(y) is not an empty set. Therefore there exist $x^i \notin V(Y)$ but which are elements of R^n_+ . Assume $x^i \notin V(y)$ but $x^i \in R^n_+$ Now since V(y) is a closed (convex) set; by the strict separation theorem there exists a $w^i \in R^n$, $w^i \neq 0$ and an $\in$ ¿ 0 such that

$$w^i x^i\ <\ \alpha\ \leq\ wx\ for\ each\ x\ \epsilon\ V(y)$$

Since this is true for any $x \in V(y)$ it is true for the cost minimizing x, i.e.,

$$(w^i x^i\ +\ \epsilon)\ <\ \min_x\ [w^{i\,x:}\ x\ \epsilon\ V\ (y)\,]. \tag{116}$$

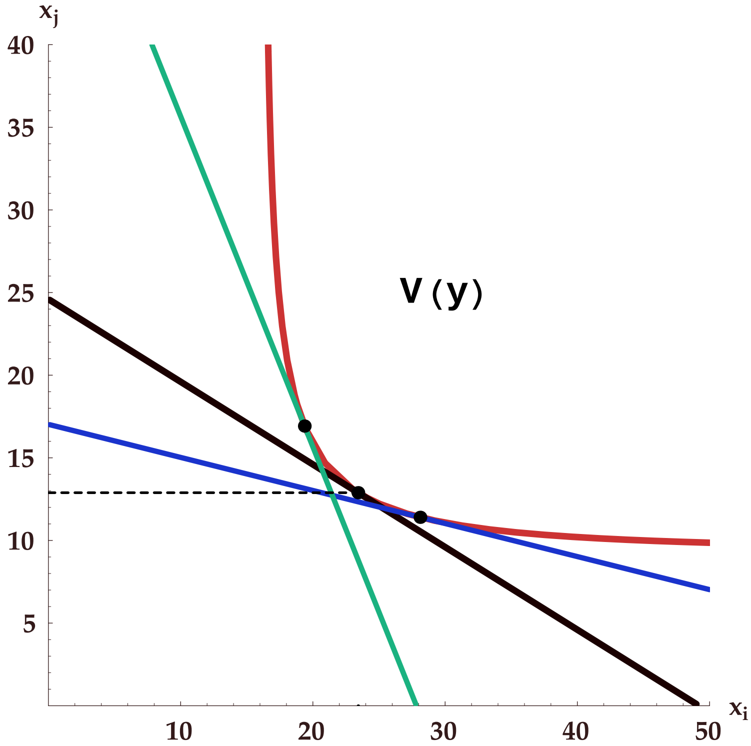
The second property of V(y) of monotonicity in x implies that w^i must be an element of R^n_+ and thus $w^i \geq 0$, i.e. it cannot be negative. To see this pick an x^j which is greater than the cost minimizing $x(w^i\ , y)$ which is in V(y). Then if elements of w^i ¡ 0, $w^i x^j$ could be less than $w^i x^i$ since we are multiplying some elements of x^j by a negative number. This is not allowed since $w^i x^i + \in$ is already less than the cost minimizing x which is in V(y).

Furthermore we can show that w^i can be taken to be strictly positive as in the definition of $V^*(y)$. Since $\in$ ¿ 0 and $x^i \geq 0$ there exists w^k ¿ 0 such that $w^k\ x^i$ ¡ $\in/2$. This w^k may have very small but positive elements in each position. Now define

$$\hat{w}\ =\ (w^i\ +\ w^k) \tag{117}$$

It is obvious that w is greater than zero since $w^i \geq 0$ and w^k is strictly positive.

FIGURE 8. Three Half-Spaces and Input Requirement Set



Now we can show that this new price vector $\hat{w}$ also satisfies the previous inequality. i.e.,

$$\hat{w}x^i \;<\; \min_x [w^{i\,x:}\; x \,\epsilon\, V(y)] \tag{118}$$

Proof of E.4
Start with the original inequality based on the separating hyperplane theorem

$$(w^i\, x^i \;+\; \epsilon) \;<\; \min_x\, [w^{i\,x:}\; x \;\epsilon\; V\,(y)\,]. \tag{119}$$

From equation E.3, w = w^i + w'^k so if we multiply this by x^i we obtain

$$\hat{w}\; x^i \;=\; w^{i\,x^i} \;+\; w^{k\;x^i} \tag{120}$$

By assumption w^k x^i ¡ ∈/2. If we rewrite E.5 we obtain

$$\hat{w}\; x^i \;=\; w^{i\,x^i} \;+\; w^{k\;x^i\,<\,w^i\,x^i} \;+\; \epsilon/2 \tag{121}$$

Since ∈/2 ¡ ∈, it is obvious that we can also write

$$\begin{aligned} \hat{w}\; x^i \;=\; w^{i\,x^i} \;+\; w^{k\;x^i\,<}\; & w^{i\,x^i} \;+\; \epsilon \\ & < \min_x\, [w^{i\,x:}\; x \;\epsilon\; V\,(y)\,] \end{aligned} \tag{122}$$

by E.2.

Now relate the cost minimizing bundle with prices w^i to the one with prices $\hat{w}$

$$\min_{x\ (w^{i}\,x:\ x} \epsilon\, V(y)) \ \leq\ \min_{x}\{\hat{w}\, x :\ x\, \epsilon\, V(y)\} \tag{123}$$

since $\hat{w}$ ¿ w^i by definition.
Combining the E.7 and E.8 and the definition of $C(y,\ \hat{w})$ implies that
or specifically

$$\hat{w}\, x^{i\, <\, C}(y,\ \hat{w})$$

Now remember that $C(y, \hat{w})$ is equal to min $\hat{w}$ x: $x \in V(y)$. Therefore that x^i is not in the set x: $wx \geq C(y, w)\ \forall\ w$ since it is not in this set for $\hat{w}$. Thus $x^i \notin V^*(y)$ and so $V(y) = V^*(y)$.

□

4.2. Using the properties of the cost function to recover properties of the technology.

4.2.1. *Strong disposability of inputs.* $\forall\ y \in R^m_+\ x \in V^*(y)$ and $x' \geq x \Rightarrow x \in V^*(y)$

To show this, let $x \in V^*(y)$. If this is the case then, let $wx \geq C(y,w)$. But $w\, x' \geq C(y,w)$ because $x' \geq x$. But this, of course, implies that $x' \in V^*(y)$.

4.2.2. *Weak essentiality.* $0_n \notin V^*(y)$, $y > 0$.
To show this note that $C(w,y) > 0$ if $y > 0$. But $w \cdot 0_n = 0 < C(y,w)$. So $0_n \notin V^*(y)$, $y > 0$.

REFERENCES

[1] Allen, R.G.D., *Mathematical Analysis for Economists*. New York: St. Martins's Press, 1938
[2] Arrow, K. J. and R. C. Enthoven. "Quasi-Concave Programming." *Econometrica*29 (1961):779-800.
[3] Avriel, M. *Nonlinear Programming*. Englewood Cliffs, NJ: Prentice-Hall, Inc., 1976.
[4] Chambers, R. G. *Applied Production Analysis*. Cambridge: Cambridge University Press, 1988.
[5] Bazaraa, M. S. H.D. Sherali, and C. M. Shetty. *Nonlinear Programming* 2$^{\text{nd}}$ Edition. New York: John Wiley and Sons, 1993.
[6] Debreu, G. *Theory of Value*. New Haven, CN: Yale University Press, 1959
[7] Diewert, W.E. "Duality Approaches to Microeconomic Theory" in *Handbook of Mathematical Economics,* Vol 2, ed. K.J. Arrow and M.D. Intriligator. Amsterdam: North Holland, 1982
[8] Fare, R., *Fundamentals of Production Theory*. New York: Springer-Verlag, 1988.
[9] Fare, R. and D. Primont. *Multi-output Production and Duality: Theory and Applications.* Boston: Kluwer Academic Publishers, 1995.
[10] Ferguson, C. E., *The Neoclassical Theory of Production and Distribution*. Cambridge: Cambridge University Press, 1969.
[11] Hadley, G. *Linear Algebra*. Reading, MA: Addison-Wesley, 1961
[12] Hanoch, G. "The Elasticity of Scale and the Shape of Average Costs." *American Economic Review* 65 (1975): 429-97.
[13] McFadden, D. "Duality of Production, Cost and Profit Functions" in *Production Economics: A Dual Approach to Theory and Applications*, ed. M. Fuss and D. McFadden, Amsterdam: North-Holland, 1978.
[14] Rockafellar, R. T. *Convex Analysis*. Princeton University Press, 1970.
[15] Samuelson, P. A. *Foundations of Economic Analysis* .Cambridge, MA: Harvard University Press, 1983.
[16] Shephard, R. W., *Theory of Cost and Production Functions*. Princeton: Princeton University Press, New Jersey, 1970.
[17] Silberberg, E. "A Revision of Comparative Statics Methodology in Economics, or, How to Do Comparative Statics on the Back of an Envelope." *Journal of Economic Theory*, 7(1974): 159-72.
[18] Silberberg , E. *The Structure of Economics*. New York: McGraw-Hill, 1990
[19] Varian, H. R. *Microeconomic Analysis.* 3$^{\text{rd}}$ Edition. New York: Norton, 1992.