THE NEOCLASSICAL FIRM AND TECHNOLOGY

1. Definition of a neoclassical firm

A neoclassical firm is an organization that **controls** the transformation of **inputs** (resources it owns or purchases) into **outputs or products** (valued products that it sells) and **earns** the difference between what it receives in revenue and what it spends on inputs.

A technology is a description of process by which inputs are converted in outputs. There are a myriad of ways to describe a technology, but all of them in one way or another specify the outputs that are feasible with a given choice of inputs. Specifically, a production technology is a description of the set of outputs that can be produced by a given set of factors of production or inputs using a given method of production or production process.

We assume that neoclassical firms exist to make money. Such firms are called *for-profit* firms. We then set up the firm level decision problem as maximizing the returns from the technologies controlled by the firm taking into account the demand for final consumption products, opportunities for buying and selling products from other firms, and the actions of other firms in the markets in which the firm participates. In perfectly competitive markets this means the firm will take prices as given and choose the levels of inputs and outputs that maximize profits.

If the firm controls more than one production technology it takes into account the interactions between the technologies and the overall profits from the group of technologies.

The neoclassical definition of the firm treats the firm as synonymous with the technology. The firm is a engineering construct that specifies how inputs and outputs are related, assumes a decision rule for choosing the inputs and outputs subject to the technology and earns any returns that come from this process. In reality, firms must deal with many complex human challenges, such as creating incentives, and coping with incomplete information. The neoclassical model of a firm views labor as an input like any other. However, labor is different, since the workers must be motivated to work effectively (supply the input purchased). Supplying effective incentives may be difficult, because the employer cannot have complete information about the effort a worker is exerting. Such issues will be ignored for the present.

2. Descriptions of Technology

There are many ways to describe the technology of a firm. In all that follows y = (y_1, y_2, ... , y_m) $\epsilon\ R^m_+$ is a vector of net outputs for the firm while x = (x_1, x_2, ... ,

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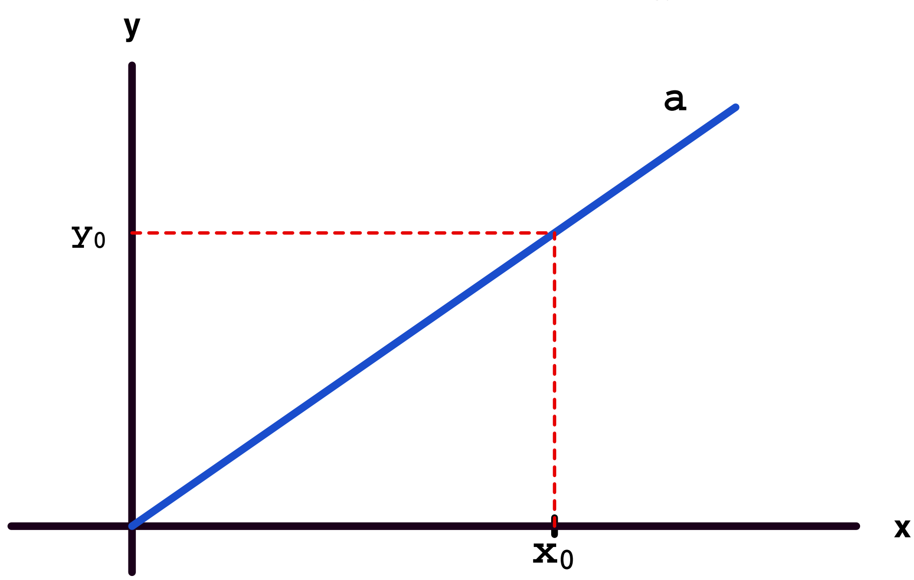
$x_n) \,\epsilon R^n_+$ is a vector of net inputs for the firm.

This lecture will concentrate on descriptions based on sets. One of the most common ways to describe a production technology is with a *production set*.
The technology set for a given production process is defined as

$$T \;=\; \{(x,\, y) : \; x\,\epsilon\, R^n_+, \; y\,\epsilon\, R^{m:}_+ \; x \text{ can produce } y \;\}$$

where x is a vector of inputs and y is a vector of outputs. The set consists of those combinations of x and y such that y can be produced from the given x. For the case of a single input and single output the production set T is represented by the area bounded by the x-axis and **a** in figure 1.

FIGURE 1. Representation of Technology Set



As an example consider the production technology for producing pancakes on a weekend camp-out. The input vector might be as follows:

$$x \;=\; \begin{pmatrix} powdered\ milk & water & eggs & oil & flour \\ baking\ powder & salt & bowl & whip & measuring\ devices \\ small\ griddle & camp\ stove & white\ gas & spatula & semi-skilled\ labor \\ butter & maple\ syrup & plate & knife & fork \end{pmatrix}$$

Let the output in this case be a single product consisting of buttered pancakes covered with syrup, ready to eat. The technology set then consists of different numbers of pancakes along with all the various input combinations that could produce them. For later reference denote this as technology 1. One element of this set might be as follows:

$$\left[\begin{pmatrix} 1/3\,c\ powdered\ milk & 15/16\,c\ water & 1\,egg & 2\,T\ oil & 1\,c\,flour \\ 2\,t\ baking\ powder & 1/4\,t\ salt & 1\,bowl & 1\,whip & 1\,measuring\ set \\ 1\,small\,griddle & 1\,camp\ stove & 1/4\,c\ white\,gas & 1\,spatula & 1/4\,h\,semi-skilled\,labor \\ 3\,T\ butter & 1/2\,c\,maple\ syrup & 1\,plate & 1\,knife & 1\,fork \end{pmatrix} \left(\,10\,pancakes\,\right)\right]$$

Of course, the same inputs with an output of 6 pancakes is also possible since we can always throw the extras to the wild creatures (assuming we are not particularly environmentally conscious). Other combinations are also possible. A particular element of the production set is called a **production plan.** The production process for pancakes can, of course, be defined in different ways depending on which parts we want to consider. If the output is pancakes hot off the griddle, then the inputs butter, maple syrup, plate, knife and fork can be eliminated. We also might consider dividing the process up into steps where the first step is the production of "pancake mix". In this case the technology for hot off the griddle pancakes might be

$$\left[\begin{pmatrix} pancake\,mix & water & bowl \\ whip & measuring\ set & small\ griddle \\ camp\ stove & white\ gas & spatula \\ semi-skilled\ labor & & \end{pmatrix} \left(\,pancakes\ \ \right)\right]$$

where the pancakes are assumed to be off the grill only. This might be denoted technology 2. We could also consider a more primitive process denoted technology 3 that does not use the manufactured input flour but considers wheat, a grindstone and grinding labor as additional inputs replacing flour.

The firm may choose to organize the technologies that it controls in a variety of ways. Consider again the example of the pancakes. The firm may choose to use technology 1 or technology 2. In the case of technology 2 the firm could produce its own pancake mix, or it could purchase it on the market. The **vertical boundaries** of a firm in a vertical chain define the activities that a firm performs for itself as opposed to purchasing them from independent firms in the market. Activities closer to the beginning in a vertical chain are called upstream in the chain while those closer to the finished goods are called downstream. Thus a firm's vertical boundaries deal with how many stages up or downstream from a given process the firm chooses to control.

3. Factors of production

3.1. **Definition of a factor of production.** A **factor of production (input)** is a product or service that is employed in the production process. The factors of production used by a firm fall into two general classes, those that are used up in the production process and those that simply contribute a service to the process. For example the flour that goes into pancakes is gone once the pancakes are made and sold, while the mixing bowl is still available for future use. Thus we categorize inputs into two categories, expendables and capital.

3.2. **Expendable factors of production. Expendable factors of production** are raw materials, or produced factors that are completely used up or consumed during a single production period. Examples might include gasoline, seed, iron ore, thread and cleaning fluid.

3.3. **Capital. Capital** is a stock that is not used up during a single production period, provides services over time, and retains a unique identity. Examples include machinery, buildings, equipment, land, stocks of natural resources, production rights, and human capital.

3.4. **Capital services. Capital services** are the flow of productive services that can be obtained from a given capital stock during a production period. They arise from a specific item of capital rather than from a production process. It is usually possible to separate the right to use services from ownership of the capital good. For example, one may hire the services of a backhoe to dig a trench, a laborer (with embodied human capital) to flip burgers, or land to grow corn.

3.5. **Examples.** A number of examples will illustrate the argument. Land is considered a capital asset but the right to use the land for a specific period is an expendable service flow. A laborer and the embodied human capital is considered capital, but the service available from that laborer is considered an expendable capital service. Shares in an water district are considered capital, but the acre feet available for use in a given period are an expendable input.

4. THE OUTPUT CORRESPONDENCE,OUTPUT SETS,AND EFFICIENT USE OF INPUTS

4.1. **Notation.** We will often use the following mathematical symbols.

(1) $\in$ means is an element of, as in a $\in$ S.
(2) $\subseteq$ is the symbol for subset. B is a subset of A (written B $\subseteq$ A) iff every member of B is a member of A.
(3) $\subset$ is the symbol for proper subset. If B is a proper subset of A(i.e., a subset other than the set itself), this is written B $\subset$ A .
(4) $\forall$ means for every
(5) $\iff$ means if and only if
(6) $\exists$ means there exists
(7) $\sum_{i=1}^{n}$ x$_i$ means the sum of the terms labeled x$_1$,x$_2$, . . . ,x$_n$
(8) $\prod_{i=1}^{n}$ x$_i$ means the products of the terms labeled x$_1$,x$_2$, . . . ,x$_n$
(9) $\bigcap_{i=1}^{n}$ x$_i$ means the intersection of the terms labeled x$_1$,x$_2$, . . . ,x$_n$
(10) $\bigcup_{i=1}^{n}$ x$_i$ means the union of the terms labeled x$_1$,x$_2$, . . . ,x$_n$

4.2. **Definitions.** Rather than representing a firm's technology with the technology set T, it is often convenient to define a production correspondence and the associated output set.

1: The output correspondence P, maps inputs x ϵ R_+^n into subsets of outputs, i.e., P: $R_+^n \rightarrow 2^{R_+^m}$. A correspondence is different from a function in that a given domain is mapped into a set as compared to a single real variable (or number) as in a function.

2: The output set for a given technology, P(x), is the set of all output vectors y ϵ R^m_+ that are obtainable from the input vector x ϵ R^n_+. P(x) is then the set of all output vectors y ϵ R^m_+ that are obtainable from the input vector x ϵ R^n_+. We often write P(x) for both the set based on a particular value of x, and the rule (correspondence) that assigns a set to each vector x.

4.3. **Relationship between P(x) and T(x,y).**

$$P(x) = (y : (x, y) \epsilon T)$$

In the case of two outputs the output set is a region of the plane, the set of all combinations of y_1 and y_2 that can be produced with given levels of the x variables. Figure 2 shows P(x) for the case of two outputs and a fixed input bundle.

FIGURE 2. P(x) for Two Outputs and a Fixed Input Bundle

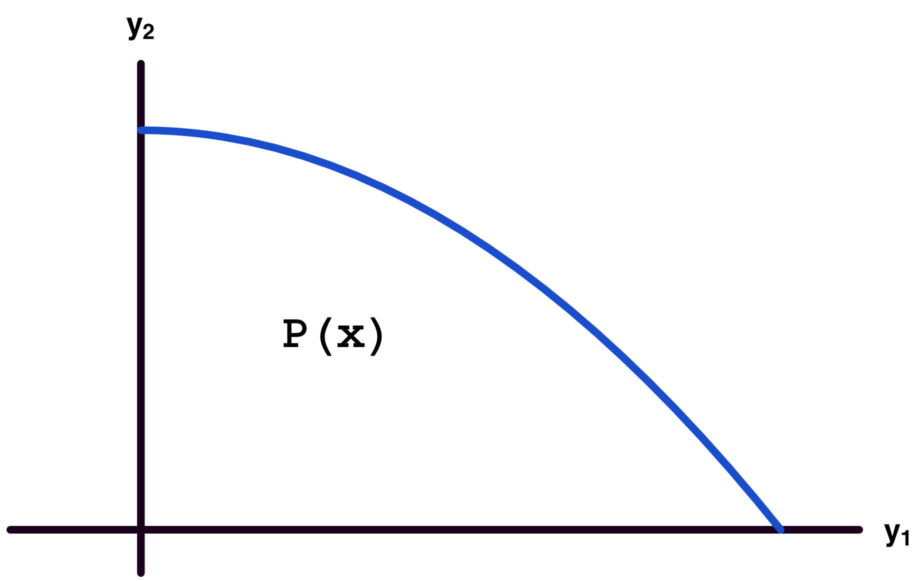
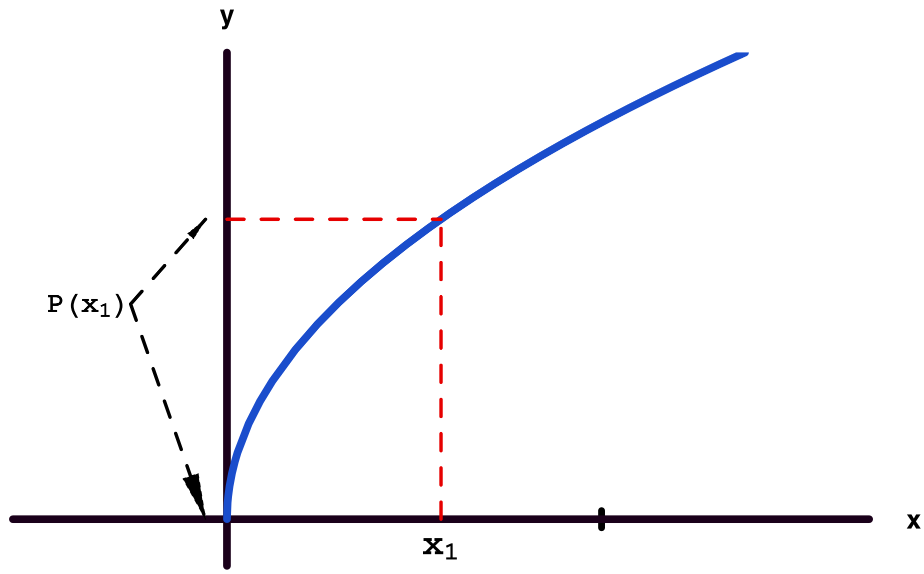


Figure 3 shows P(x) for the case of one input and one output. In this case, P(x) is a vertical line segment starting at 0 for each x. Production is "efficient" only along the curve which lies above all of these line segments. Points below the curve represent less output with the same level of input.

If there is only one output, then max *P(x)* is the maximum level of y that can be produced using a given level of x. The firm figures out how to "optimally" use the level of resources x and no more output can be obtained by combining them in another way. Each input is being used in such a way it cannot produce more output.

4.4. **Properties of P(x).** The following are a set of axioms proposed for the output correspondence.

FIGURE 3. P(x) for One Input and One Output



4.4.1. *P.1 Inaction and No Free Lunch.*

a: 0ϵ P(x) $\forall$ x ϵ R^n_+.
b: y $\notin$ P(0), y > 0

4.4.2. *P.2 Input Disposability.* $\forall$ x ϵ R^n_+ , P(x) $\subseteq$ P(θx), $\theta \geq 1$.

4.4.3. *P.2.S Strong Input Disposability.* $\forall$ x, x′ ϵ R^n_+ , x′ $\geq$ x $\Rightarrow$ P(x) $\subseteq$ P(x′)

4.4.4. *P.3 Output Disposability.* $\forall$ x ϵ R^n_+ , y ϵ P(x) and $0 \leq \lambda \leq 1 \Rightarrow \lambda$y ϵ P(x)

4.4.5. *P.3.S Strong Output Disposability.* $\forall$ x ϵ R^n_+ , y ϵ P(x) $\Rightarrow$ y′ ϵ P(x), $0 \leq$ y′ $\leq$ y

4.4.6. *P.4 Boundedness.* P(x) is bounded for all x ϵ $R^n{}_+$

4.4.7. *P.5 T is a closed set.* P: $R^n_+ \rightarrow 2^{R^m_+}$ is a closed correspondence, i.e., if [$x^\ell \rightarrow x^0$, $y^\ell \rightarrow y^0$ and y^ℓ ϵ P(x^ℓ), $\forall$ ℓ] then y^0 ϵ P(x^0)

4.4.8. *P.6 Attainability.* If y ϵ P(x), y $\geq$ 0 and x $\geq$ 0, then $\forall$ $\theta \geq 0$, $\exists$ $\lambda_\theta \geq 0$ such that θy ϵ P(λ_θx)

4.4.9. *P.7 P(x) is convex.* P(x) is convex for all x ϵ R^n_+ .

4.4.10. *P.8 P is quasi-concave.* The correspondence P is quasi-concave on R^n_+ which means $\forall$ x, x′ ϵ R^n_+ , $0 \leq \theta \leq 1$, P(x) $\cap$ P(x′) $\subseteq$ P(θx + (1-θ)x′)

4.4.11. *P.9 Convexity of T.* P is concave on R^n_+ which means $\forall$ x, x′ ϵ R^n_+ , $0 \leq \theta \leq 1$, θP(x)+(1-θ)P(x′) $\subseteq$ P(θx + (1-θ)x′)

4.5. **Discussion of properties of P(x).**

4.5.1. *P.1 Inaction and No Free Lunch.*

a: 0ϵ P(x) $\forall$ x ϵ R^n_+.

This implies that it is possible to produce a zero level of output, no matter what the input level.

b: y $\notin$ P(0), y $>$ 0

If there are no inputs, there can be no output. Parts a and b together imply that P(0) = 0.

4.5.2. *P.2 Input Disposability.* $\forall$ x ϵ R^n_+ , P(x) $\subseteq$ P(θx), $\theta \geq 1$.

If inputs are proportionately increased, outputs do not decrease.

4.5.3. *P.2.S Strong Input Disposability.* $\forall$ x, x′ ϵ R^n_+ , x′ $\geq$ x $\Rightarrow$ P(x) $\subseteq$ P(x′)

If some inputs are increased, outputs do not decrease. P.2.S implies P.2.

4.5.4. *P.3 Output Disposability.* $\forall$ x ϵ R^n_+ , y ϵ P(x) and $0 \leq \lambda \leq 1 \Rightarrow \lambda$y ϵ P(x)

Weak disposability of outputs implies that a proportional reduction in outputs is feasible. Suppose the input vector x^1 can produce the output vector y^1={ y_1^1, y_2^1 }. Then even if the technology cannot produce λ y^1 = { λ y_1^1, λ y_2^1 } where $0 \leq \lambda \leq$ 1, the firm can always produce y^1={ y_1^1, y_2^1 } and throw the extra levels of y away in a proportionate fashion.

4.5.5. *P.3.S Strong Output Disposability.* $\forall$ x ϵ R^n_+ , y ϵ P(x) $\Rightarrow$ y′ ϵ P(x), 0 $\leq$ y′ $\leq$ y
Any output can be disposed of without affecting inputs. This may not always be the case. If laws require that pollution output be disposed of properly, the initial level of inputs may not be be able to produce the same level of a "good" output and less of the "bad" output. Alternatively, two products may be produced in more or less fixed proportions so that output combinations along the positively sloped line from **0** to **a** in figures 4 and 5shows that as y_1 is increased there is also an increase in y_2. Figure 4 can be used to differentiate P.3 and P.3.S. The weakly disposable technology is bounded by (0abc0). The output vector may be proportionately decreased while holding inputs constant.

Consider the point q (or any other point in P(x) in figure 5). The radial contraction of it will always be in P(x).

If outputs are strongly disposable, the output set P(x) is augmented to (0dabc0). The output vector may be decreased in only one component while maintaining the output of the other component. Consider y_1 to be a "bad" output and assume the firm is producing at point a in figure 4. The firm can throw away (0,a′) of y_1 without reducing the output level of y_2. In figure 6 the positively sloping sections of the boundary of P(x) would be eliminated with strong disposability.

In another sense, any point within P(x) can be extended to the axis is the sense that one of the outputs can be tossed. If one is producing at point q in figure 7, one can reduce y_1 to zero and maintain the level of y_2.

FIGURE 4. Disposability of Output

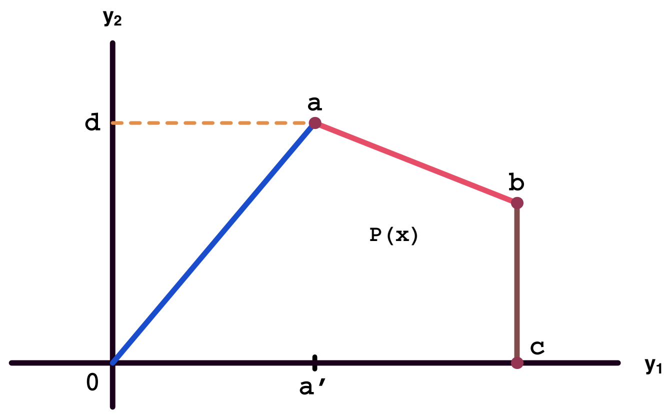
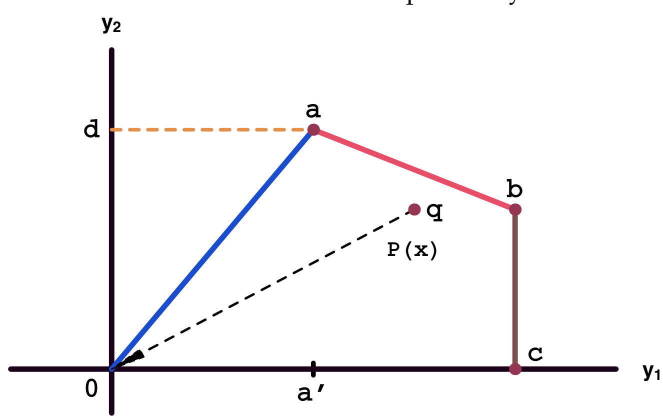


FIGURE 5. Radial Disposability



4.5.6. *P.4 Boundedness.* P(x) is bounded for all x ϵ $R^n{}_+$

Boundedness implies that finite inputs only yield finite outputs.

4.5.7. *P.5 T is a closed set.* P: $R^n_+ \rightarrow 2^{R^m_+}$ is a closed correspondence, i.e., if [$x^\ell \rightarrow x^0$, $y^\ell \rightarrow y^0$ and $y^\ell \;\epsilon\; P(x^\ell), \forall\; \ell$] then $y^0 \;\epsilon\; P(x^0)$

The implication is that the production set T = (x, y) is closed. This means that sequences in T(x,y) that converge do so within T(x,y). It also means that every point outside T(x,y) has a neighborhood disjoint from T(x,y). P.5 also means that P(x) is a closed set. P.4 and P.5 together imply that P(x) is compact for all x $\in R^n_+$. This implies that the set P(x) contains its boundary.

Figures 8 and 9 demonstrate the difference between P(x) being a closed and an open set. In figure 8 the boundary of P(x) is part of P(x) while in figure 9, P(x) does not contain its boundary.

FIGURE 6. Strong Disposability Eliminates Positively Sloped Sections of the Boundary of P(x)

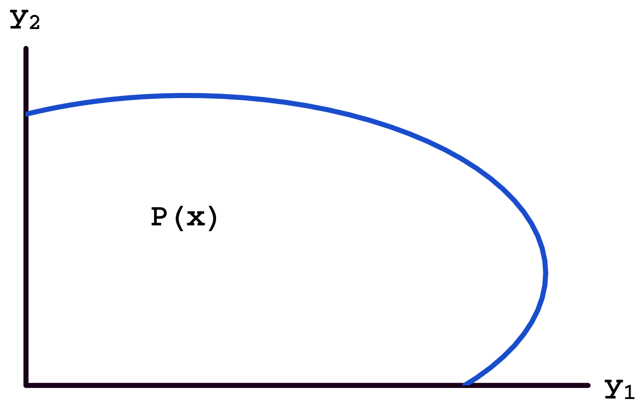
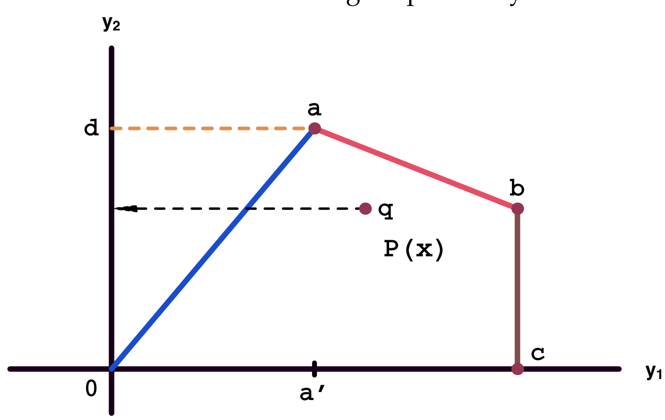


FIGURE 7. Strong Disposability



4.5.8. *P.6 Attainability.* If y ϵ P(x), y $\geq$ 0 and x $\geq$ 0, then $\forall\ \theta \geq 0$, $\exists\ \lambda_\theta \geq 0$ such that θy ϵ P(λ_θx)

This implies that in an unconstrained environment, if a given output vector is attainable, then any scalar multiplication of it is obtainable by proportional scaling of inputs.

4.5.9. *P.7 P(x) is convex.* P(x) is a convex set for all x ϵ R^n_+. The output sets are convex if

$$\forall\, x \in \Re^n_+,\ \forall\, y^1 \text{ and } y^2 \in P(x) \text{ and } 0 \leq \theta \leq 1,\ \theta y^1\ +\ (1-\theta)y^2 \in P(x)$$

This implies that if a set of inputs x will produce the output vector y^1 and also produce y^2, then that same level of x will produce a linear combination of y^1 and y^2. We can see this in figure

FIGURE 8. P(x) is a Closed Set

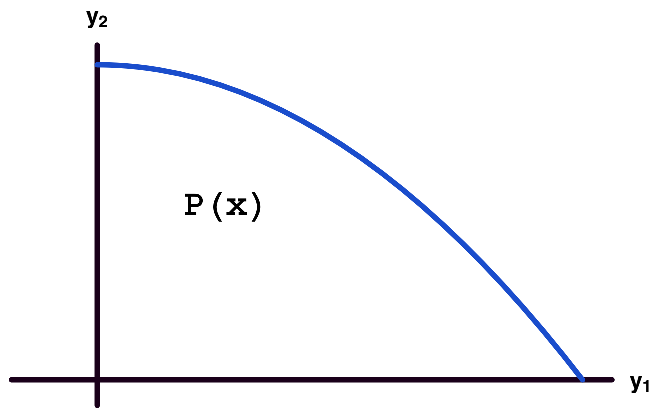
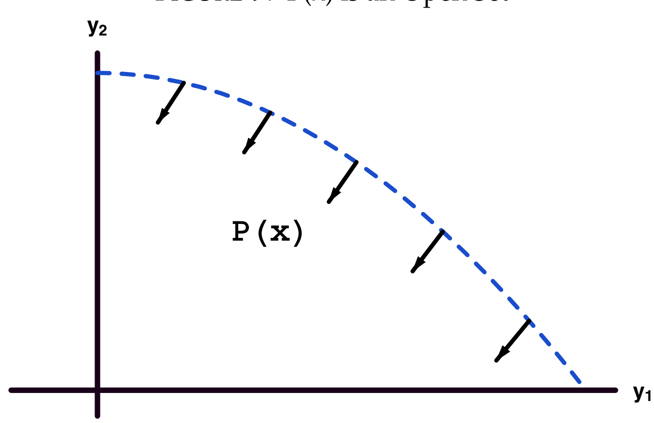


FIGURE 9. P(x) is an Open Set



4.5.10. *P.8 P is quasi-concave.* The correspondence P is quasi-concave on R^n_+ which means ∀ x, x′ ϵ R^n_+ , $0 \leq \theta \leq 1$, P(x) ∩ P(x′) ⊆ P(θx + (1-θ)x′) P.8 implies that the set P(x) is a convex set. The set P(x) in figure 11 in not convex.

4.5.11. *P.9 Convexity of T.* P is concave on R^n_+ which means ∀ x, x′ ϵ R^n_+ , $0 \leq \theta \leq 1$, θP(x)+(1-θ)P(x′) ⊆ P(θx + (1-θ)x′)

P.9 implies that the set T(x,y) is a convex set. The technology in figure 12 is not convex.

4.6. **The efficient subset of P(x).** The efficient output subset of P(x) is defined as follows:

$$EffP(x) = y : y \in P(x),\ y' \geq y \text{ and } y \neq y' \Rightarrow y' \notin P(x)$$

FIGURE 10. The Producible Output Set is Convex

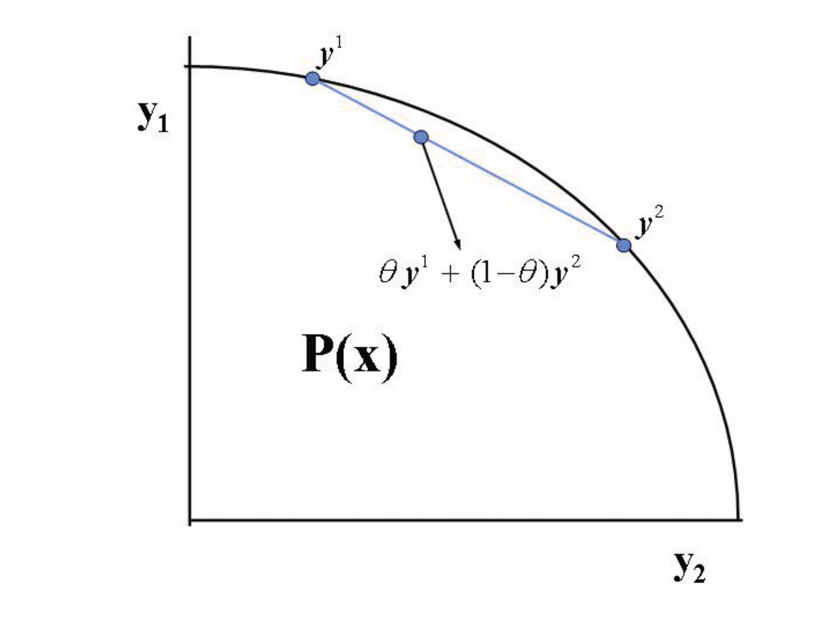
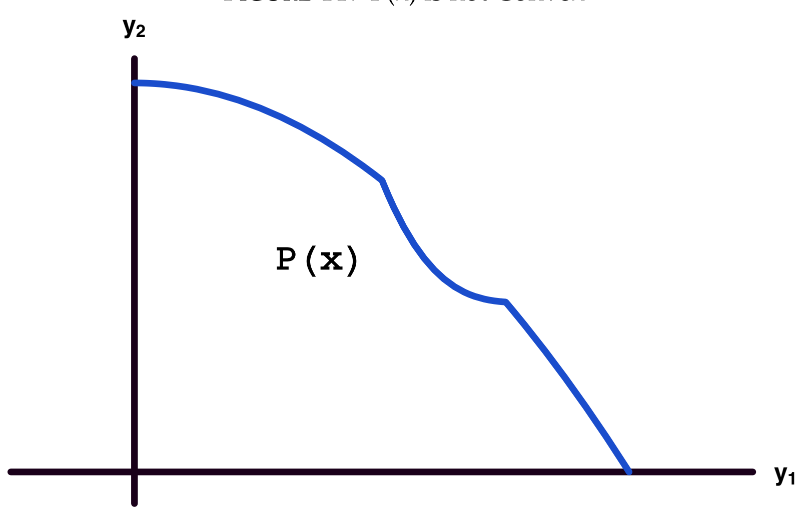


FIGURE 11. P(x) is not Convex



An efficient element of P(x) is an output level that cannot be exceeded with the set of inputs x. In essence, the efficient set is elements of P(x) such that any expansion in any element in the output y will remove it from P(x). The boundary in figure 13 is the efficient subset of P(x).

If there is only one output, then Eff P(x) = max P(x)

FIGURE 12. T(x) is not Convex

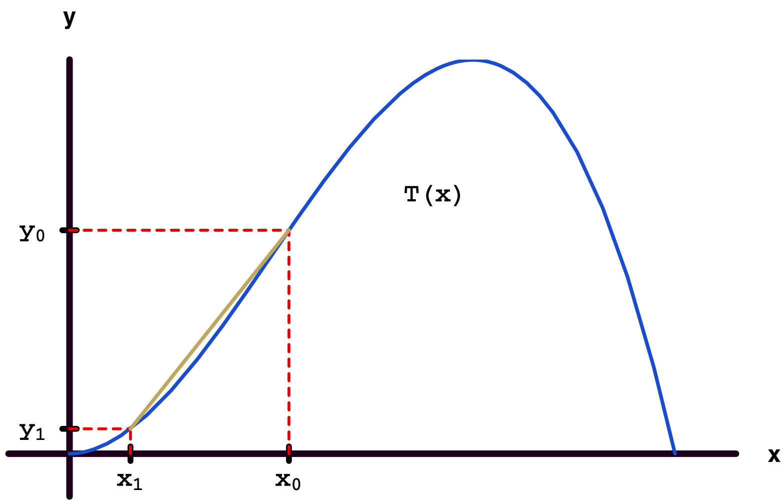
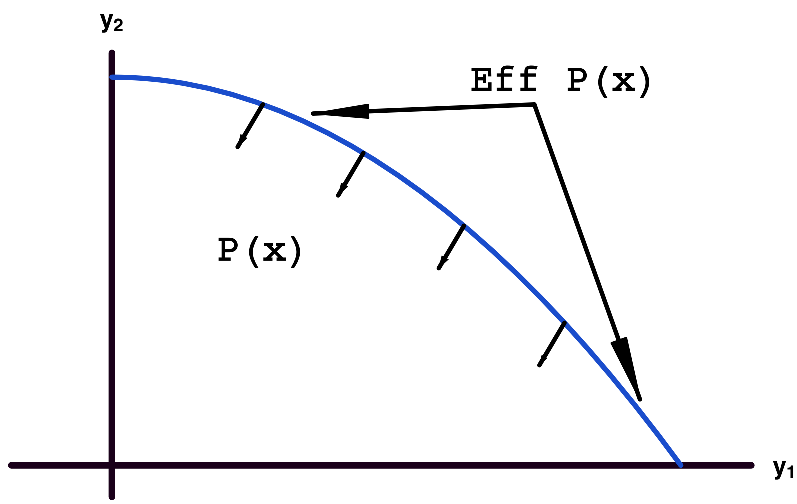


FIGURE 13. Efficient Subset of P(x) is not Convex



4.7. **Optimal use of inputs.** A firm uses engineering, agronomic, accounting, economic and other principles in order to insure that it is on the boundary of the output set. The optimal organization of inputs is sometimes called "technical efficiency."

5. THE INPUT CORRESPONDENCE AND INPUT (REQUIREMENT) SETS

5.1. **Definitions.** Rather than representing a firm's technology with the technology set T or the production set P(x), it is often convenient to define an input correspondence and the associated input requirement set.

1: The input correspondence maps outputs y ϵ R^m_+ into subsets of inputs, V: $R^m_+ \rightarrow 2^{R^n_+}$. A correspondence is different from a function in that a given domain is mapped into a set as compared to a single real variable (or number) as in a function.

2: The input requirement set V(y) of a given technology is the set of all combinations of the various inputs x $\epsilon\, R^n_+$ that will produce at least the level of output y $\epsilon\, R^m_+$. V(y) is then the set of all input vectors x $\epsilon\, R^n_+$ that will produce the output vector y $\epsilon\, R^m_+$. We often write V(y) for both the set based on a particular value of y, and the rule (correspondence) that assigns a set to each vector y.

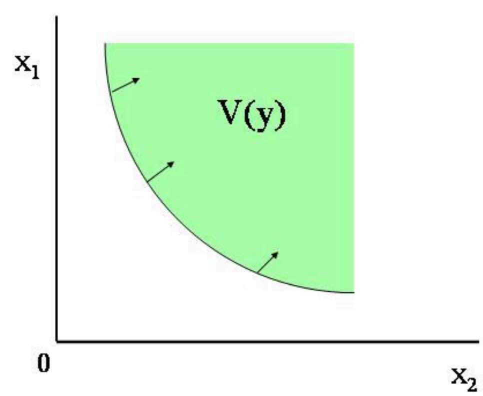
Varian [9, p. 2-10] provides a nice discussion of input requirement sets and their relation to various functional representations of technology.

5.2. **Relationship between V(y) and T(x,y).**

$$V(y) \;=\; (x:\; (x,\; y)\,\epsilon\, T)$$

In the case of a single output and two inputs V(y) is the set of all input levels that will produce at least the output level y. This can be seen graphically in figure 14

FIGURE 14. V(y)



The set of all points above the curve represents those combinations of x_1 and x_2 that will produce at least the level of output y. As an example with more inputs, consider the various combinations of corn, corn silage, soybean meal, milo, hay, molasses, and a mineral supplement that can be used to produce 5 tons of cattle feed with specific protein and net energy content.

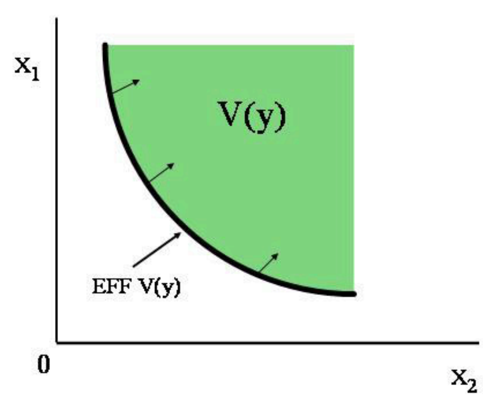
5.3. **The efficient subsets of V(y).** While the input correspondence maps a given output vector into the set of all input vectors capable of producing it, economic efficiency is concerned with minimizing the use of inputs necessary to produce a given output level. Different ways of defining this minimal set of inputs gives rise to different notions of efficiency.

5.3.1. *The efficient subset of V(y).* The efficient subset of V(y) is defined as follows:

$$Eff\ V(y) = \{x : x\epsilon V(y), x' \leq x \Rightarrow x' \notin V(y),\ Eff\ V(0) = \{0\}\, \}$$

An efficient element of V(y) is an input level that cannot be reduced in any component and still produce the set of outputs y. In essence, the efficient set is elements of V(y) such that any reduction in any element in x removes the vector from V(y). Production is efficient only along this lower boundary of V(y), or alternatively Eff V(y) is the lower boundary of V(y). The efficient set is that portion of the boundary of V(y) that is negatively sloped as shown in figure 15.

FIGURE 15. Efficient Subset of V(y)



5.3.2. *The weak efficient subset of V(y).* The weak efficient subset of V(y) is defined as follows:

$$WEff\ V(y) = \{x : x\epsilon V(y), x' <^{*} x \Rightarrow x' \notin V(y),\ WEff\ V(0) = \{0\}\, \}$$

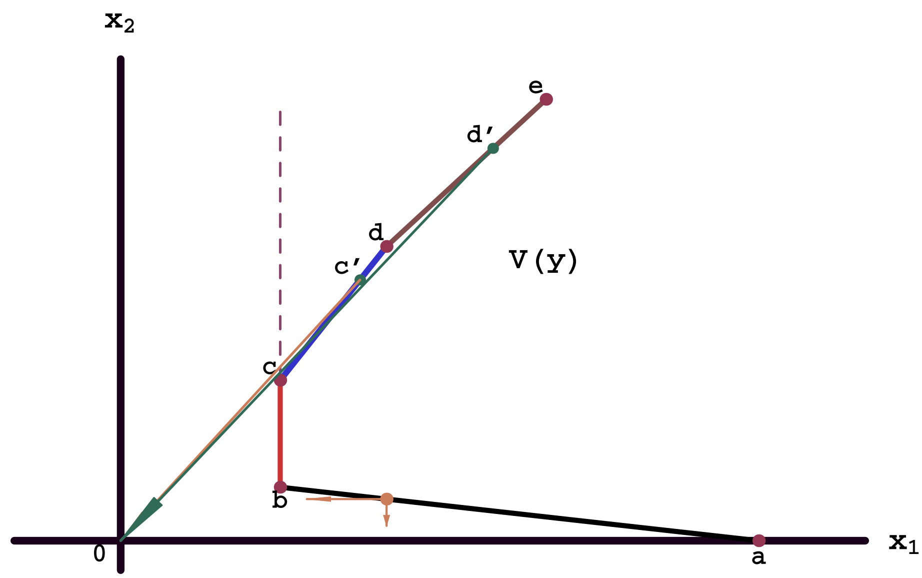
In essence, the weak efficient set is elements of V(y) such that any reduction in some elements in x will remove the vector from V(y). But in this set, the levels of some components of x may be reduced without making the vector x' $\notin$ V(y).

5.3.3. *The input isoquant of V(y).* An isoquant is in some sense the effective boundary of the input requirement set. Positively sloped sections and those with infinite slope are allowed, but radial contractions of x in this set must make the resulting x′ $\notin$ V(y), It is defined as follows:

$$IsoqV(y) = \{x : x \epsilon V(y), \lambda x \notin V(y), \lambda \epsilon [0,1),\ IsoqV(0) = \{0\}\}$$

An isoquant is elements of V(y) such that any radial contraction removes them from V(y). This is made more precise by considering figure 16.

FIGURE 16. Efficient Subsets of V(y)



The isoquant is given by abcd. A radial contraction from d′ is still in V(y), while a radial contraction from c′ is outside of V(y). The weak efficient subset is given by abc. In this portion of the V(y), a reduction in x_2 will remove a point from V(y) but a reduction in x_1 above the point b will not. The strong efficient subset is given by ab. A reduction in either input will remove a point from the set.

5.3.4. *Relationships among various notions of efficiency.* The following relationships hold between the various efficiency concepts.

$$Eff\ V(y) \subseteq WEff\ V(y) \subseteq Isoq\ V(y)$$

5.4. **Properties of V(y).** The following are a set of axioms proposed for the input correspondence.

5.4.1. *V.1 No Free Lunch.*

- **a:** V(0) = R^n_+
- **b:** 0 $\notin$ V(y), y $>$ 0.

5.4.2. *V.2 Weak Input Disposability.* $\forall\, y \,\epsilon\, R^m_+,\ x \,\epsilon\, V(y)$ and $\lambda \geq 1 \Rightarrow \lambda x \,\epsilon\, V(y)$

5.4.3. *V.2.S Strong Input Disposability.* $\forall\, y \,\epsilon R^m_+\ x \,\epsilon\, V(y)$ and $x' \geq x \Rightarrow x' \,\epsilon\, V(y)$

5.4.4. *V.3 Weak Output Disposability.* $\forall\, y\, \epsilon\, R^m_+\; V(y)\; \subseteq V(\theta y),\; 0 \leq\; \theta\; \leq\; 1.$

5.4.5. *V.3.S Strong Output Disposability.* $\forall\, y,\; y'\; \epsilon R^m_+\;,\; y'\; \geq\; y\; \Rightarrow\; V(y')\; \subseteq\; V(y)$

5.4.6. *V.4 Boundedness for vector y.* If $\|y^\ell\| \to +\infty$ as $\ell \to +\infty$,

$$\cap_{\ell=1}^{+\infty}\; V(y^\ell)\;=\;\emptyset$$

If y is a scalar,

$$\bigcap_{y\epsilon(0,+\infty)} V(y)\;=\;\emptyset$$

5.4.7. *V.5 T(x) is a closed set.* V: $R^m_+ \to 2^{R}_{+^n}$ is a closed correspondence.

5.4.8. *V.6 Attainability.* If x ϵ V(y), y $\geq$ 0 and x $\geq$ 0, the ray $\{\lambda$x: $\lambda \geq 0\}$ intersects all V(θy), $\theta \geq 0$.

5.4.9. *V.7 Quasi-concavity.* V is quasi-concave on R^m_+ which means $\forall$ y, y′ ϵ R^m_+ , 0 $\leq \theta \leq 1$, V(y) $\cap$ V(y′) $\subseteq$ V(θy + (1-θ)y′).

5.4.10. *V.8 Convexity of V(y).* V(y) is a convex set for all y ϵ R^m_+

5.4.11. *V.9 Convexity of T(x).* V is convex on R^m_+ which means $\forall$ y, y′ ϵ R^m_+ , 0 $\leq \theta$ ≤ 1, θV(y)+(1-θ)V(y′) $\subseteq$ V(θy+ (1-θ)y′)

5.5. **Discussion of properties of V(y).**

5.5.1. *V.1 Near Inaction and No Free Lunch.*

a: V(0) = R^n_+
b: 0 $\notin$ V(y), y $>$ 0.

The first part says that any nonnegative input is sufficient to produce at least zero output. The second part says that if any element of y is positive, that at least some input in needed for production. This part of the axiom is often called "no free lunch".

5.5.2. *V.2 Weak Input Disposability.*

$$\forall\, y\, \epsilon\, R^m_+,\;\; x\, \epsilon\, V(y) \text{ and } \lambda \geq 1\; \Rightarrow\; \lambda x\, \epsilon\, V(y)$$

Weak disposability of inputs says that if inputs are proportionally increased, outputs do not decrease.

5.5.3. *V.2.S Strong Input Disposability.*

$$\forall\, y\, \epsilon R^m_+\;\, x\, \epsilon\, V(y) \text{ and } x'\; \geq\; x\; \Rightarrow x'\, \epsilon\, V(y)$$

Strong disposability says that if any element of x is increased, outputs will not decrease. Strong disposability implies weak disposability. Weak disposability allows for backward bending isoquants while strong disposability requires isoquants that are parallel to the axes or have negative slope. Strong disposability prevents uneconomic regions and any type of input congestion. COnsider the relationship between disposability and efficiency in figure 16 A strongly disposable input set has only negatively sloped sections

5.5.4. *V.3 Weak Output Disposability.*

$$\forall\, y \,\epsilon\, R^m_+\, V(y) \,\subseteq V(\theta y),\ 0 \le\ \theta\ \le\ 1.$$

Weak output disposability says that proportional reductions in output are possible for a given set of inputs, x.

5.5.5. *V.3.S Strong Output Disposability.*

$$\forall\, y,\, y' \,\epsilon R^m_+\,,\ y'\ \ge\ y\ \Rightarrow\ V(y')\ \subseteq\ V(y)$$

Strong output disposability states that any output can be disposed of without affecting the inputs. This may not be reasonable if some of the outputs are viewed as bads and must be disposed of by the producer.

5.5.6. *V.4 Boundedness for vector y.* If $\|y^\ell\| \to +\infty$ as $\ell \to +\infty$,

$$\cap_{\ell=1}^{+\infty}\, V(y^\ell)\ =\ \emptyset$$

If y is a scalar,

$$\bigcap_{y \epsilon (0,+\infty)} V(y)\ =\ \emptyset$$

This axiom ensures that the technology is bounded. It is a precise way to saying that an unbounded output rate cannot arise from a bounded input vector. In the scalar case it is obvious an output cannot suddenly become unbounded as produced by a sequence of bounded input vectors. In the case of multiple outputs, we use the norm of the vectors to represent the idea that it is getting larger. The fact that the intersection of the input sets is $\emptyset$ means that the input set (and thus the intersections) becomes smaller and smaller as y is increased and in the limit vanishes.

5.5.7. *V.5 T(x) is a closed set.* V: $R^m_+ \to 2^R_{+^n}$ is a closed correspondence.

This axiom is equivalent to saying that the production possibility set or the graph of the technology is a closed set. It further implies that V(y) is a closed set. This property is used to define the isoquant and the efficient input set as subsets of the boundary of V(y). V.4 and V.5 together imply that V(y) is compact.

5.5.8. *V.6 Attainability.* If x ϵ V(y), y $\ge$ 0 and x $\ge$ 0, the ray $\{\lambda$x: $\lambda \ge 0\}$ intersects all V(θy), $\theta \ge 0$.

This axiom is often referred to as the attainability axiom. It states that if a given output vector is attainable, any scalar multiple of it is attainable by proportional scaling of inputs. This, of course, assumes no constraints on input use.

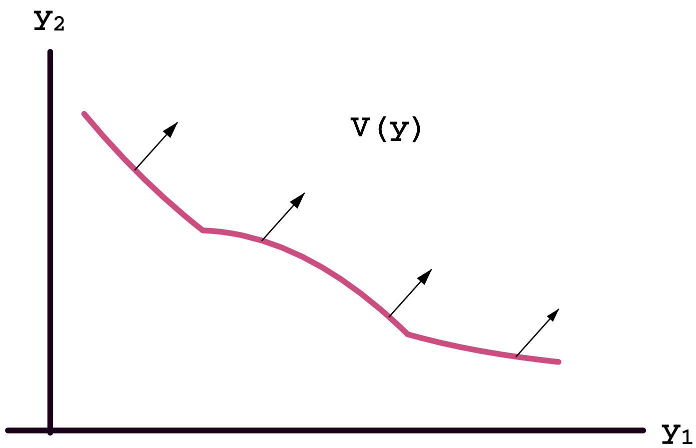
5.5.9. *V.7 Quasi-concavity.* V is quasi-concave on R^m_+ which means $\forall$ y, y′ ϵ R^m_+ , 0 $\le \theta \le 1$, V(y) $\cap$ V(y′) $\subseteq$ V(θy + (1-θ)y′)

This axiom implies that the output set P(x) is a convex set. Specifically it says that if x will produce both y and y′, i.e. y $\in$ P(x) and y′ $\in$ P(x), then θy + (1-θ)y′ $\in$ P(x). Similarly P.7 implies that V is quasiconcave so that V.7 $\Longleftrightarrow$ P.7.

5.5.10. *V.8 Convexity of V(y).* V(y) is a convex set for all y ϵ R^m_+

If V(y) is a convex set, then convex combinations of elements in V(y) are also in V(y), i.e. if x $\in$ V(y) and x′ $\in$ V(y) then θx + (1-θ)x′ $\in$ V(y) for $\theta \in$ [0,1]. This implies that P is quasiconcave on Re^n_+, which is property P.8 about quasiconcavity of P. And P.8 implies that V.8 so V.8 $\Longleftrightarrow$ P.8. The input requirement set in figure 17 is not convex.

FIGURE 17. V(y) is not Convex



5.5.11. *V.9 Convexity of T(x).* V is convex on R^m_+ which means $\forall$ y, y′ ϵ R^m_+ , $0 \leq \theta \leq 1$, θV(y)+(1-θ)V(y′) $\subseteq$ V(θy+ (1-θ)y′)

This simply states that V is a convex function and that the graph or technology set will be a convex set. Together with V.1 this eliminates increasing returns to scale. This axiom implies both V.7 and V.8 but not vice versa because convex functions have convex level sets but not necessarily vice versa.

6. RELATIONSHIPS BETWEEN VARIOUS REPRESENTATIONS OF TECHNOLOGY

6.1. **Relationships between representations: V(y), P(x) and T(x,y).** The technology set can be written in terms of either the input or output correspondence.

$$T \ = \{(x,\ y) : x \ \epsilon \ R^n_+,\ y \ \epsilon \ R^m_+, \text{such that x will produce y}\} \tag{1a}$$

$$T \ = \{(x,\ y) \ \epsilon \ R^{n+m}_+ \ : y \ \epsilon P(x),\ x\epsilon \ R^n_+\} \tag{1b}$$

$$T \ = \{(x,\ y) \ \epsilon \ R^{n+m}_+ \ : x \ \epsilon V(y),\ y\epsilon \ R^m_+\} \tag{1c}$$

The output and input correspondences can be determined from the technology set

$$P(x) \ = \{y : \ (x,\ y) \ \epsilon \ T\} \tag{2a}$$

$$V(y) \ = \{x : \ (x,\ y) \ \epsilon \ T\} \tag{2b}$$

We can summarize the relationships between the input correspondence, the output correspondence, and the production possibilities set in the following proposition.

Proposition 1. *$y \epsilon P(x) \Leftrightarrow x \epsilon V(y) \Leftrightarrow (x,y) \epsilon T$*

The three representations present alternative aspects of the technology.

1: The input correspondence (V) emphasizes the substitution of inputs.
2: The output correspondence (P) emphasizes the substitution of outputs.
3: The technology set (T) emphasizes input-output transformations.

6.2. **Relationships between axioms for V(y), P(x) and T(x,y).** The set of axioms V.1 - V.9 on V can be shown to be equivalent to the set of properties P.1 - P.9 on P. For a complete set of proofs see Fare [4, p. 9-10] and Shephard [8, p. 178-192]. The key element in the proofs is the use of proposition 1 from section 6.1. For convenience it is repeated here.

$$y \epsilon P(x) \Leftrightarrow x \epsilon V(y) \Leftrightarrow (x, y) \epsilon T \tag{3}$$

6.2.1. *Proof that P.2 implies V.2.* Let y$\in$ P(x)$\subseteq$P(λx) for $\lambda \geq 1$, then by Proposition 1, λx$\in$P(x) for $\lambda \geq 1$.

6.2.2. *Proof that P.4 implies V.4.* Suppose $\exists$ x such that $x \, \epsilon \, \cap_{\ell=1}^{\infty} \, y^{\ell}$ as $\ell \to \infty$, then by Proposition 1, y^{ℓ} ϵ P(x) $\forall$ ℓ, contradicting P.4.

6.2.3. *Proof that P.8 implies V.8.* Suppose that P.8 holds and that y ϵ $R^m{}_+$. If y $\notin$ P(x) $\cap$ P(x′) for any x and x′ in $R^n{}_+$, then V(y) is empty and convex by definition. So assume that y ϵ P(x) $\cap$ P(x′) for some x and x′. By Proposition 1, x, x′ ϵ V(y). Furthermore by P.8 and Proposition 1, y ϵ P(θx + (1-θ)x′) and (θx + (1-θ)x′) ϵ V(y) which proves convexity. For a reverse proof see Shephard [8, p. 182,191].

6.2.4. *Proof that P.9 and V.9 imply that T(x,y) is convex.* We need to show that a convex combination of two points in the graph is also in the graph. Consider two elements of the graph (x,y) ϵ T and (x′,y′) ϵ T. By Proposition 1, x ϵ V(y) and x′ ϵ V(y′). Thus V.9 and the proposition imply that λx + (1-λ)x′ ϵ V(λy + (1-λ)y′). Now apply the proposition again to obtain that (λx + (1-λx′, λy + (1-λ)y′) ϵ T.

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