SAMPLE MOMENTS

1. Population Moments

1.1. Moments about the origin (raw moments). The rth moment about the origin of a random variable X, denoted by $\mu'_r$, is the expected value of $X^r$; symbolically,

$$\mu'_r = E(X^r) = \sum_x x^r f(x)$$

for $r = 0, 1, 2, \ldots$ when X is discrete and

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) \, dx$$

when X is continuous. The rth moment about the origin is only defined if $E[X^r]$ exists. A moment about the origin is sometimes called a raw moment. Note that $\mu'_1 = E(X) = \mu_X$, the mean of the distribution of X, or simply the mean of X. The rth moment is sometimes written as function of $\theta$ where $\theta$ is a vector of parameters that characterize the distribution of X.

If there is a sequence of random variables, $X_1, X_2, \ldots, X_n$, we will call the rth population moment of the ith random variable $\mu'_{i,r}$ and define it as

$$\mu'_{i,r} = E(X_i^r)$$

1.2. Central moments. The rth moment about the mean of a random variable X, denoted by $\mu_r$, is the expected value of $(X - \mu_X)^r$ symbolically,

$$\mu_r = E[(X - \mu_X)^r] = \sum_x (x - \mu_X)^r f(x)$$

for $r = 0, 1, 2, \ldots$ when X is discrete and

$$\mu_r = E[(X - \mu_X)^r] = \int_{-\infty}^{\infty} (x - \mu_X)^r f(x) \, dx$$

when X is continuous. The rth moment about the mean is only defined if $E[(X - \mu_X)^r]$ exists. The rth central moment of X about a is defined as $E[(X - a)^r]$. If $a = \mu_X$, we have the rth central moment of X about $\mu_X$.

Note that

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\[ \mu_1 = E[X - \mu_X] = \int_{-\infty}^{\infty} (x - \mu_X) f(x) \, dx = 0 \]  
\[ \mu_2 = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x) \, dx = \text{Var}(X) = \sigma^2 \]  

Also note that all odd moments of X around its mean are zero for symmetrical distributions, provided such moments exist.

If there is a sequence of random variables, \(X_1, X_2, \ldots X_n\), we will call the \(r^{th}\) central population moment of the \(i^{th}\) random variable \(\mu_{i,r}\) and define it as

\[ \mu_{i,r} = E[X_i^r - \mu_{i,1}^r] \]  

When the variables are identically distributed, we will drop the \(i\) subscript and write \(\mu_r\) and \(\mu_r'\).

2. **Sample Moments**

2.1. **Definitions.** Assume there is a sequence of random variables, \(X_1, X_2, \ldots X_n\). The first sample moment, usually called the average is defined by

\[ \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \]  

Corresponding to this statistic is its numerical value, \(\bar{x}_n\), which is defined by

\[ \bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i \]  

where \(x_i\) represents the observed value of \(X_i\). The \(r^{th}\) sample moment for \(X_i\) is defined by

\[ \bar{X}_n^r = \frac{1}{n} \sum_{i=1}^{n} X_i^r \]  

This too has a numerical counterpart given by

\[ \bar{x}_n^r = \frac{1}{n} \sum_{i=1}^{n} x_i^r \]

2.2. **Properties of Sample Moments.**

2.2.1. **Expected value of \(\bar{X}_n^r\).** Taking the expected value of equation 10 we obtain

\[ E[\bar{X}_n^r] = E\bar{X}_n^r = \frac{1}{n} \sum_{i=1}^{n} E X_i^r = \frac{1}{n} \sum_{i=1}^{n} \mu_{i,r} \]  

If the \(X\)’s are identically distributed, then

\[ E[\bar{X}_n^r] = E\bar{X}_n^r = \frac{1}{n} \sum_{i=1}^{n} \mu_r' = \mu_r' \]
2.2.2. Variance of $\bar{X}_n^r$. First consider the case where we have a sample $X_1, X_2, \ldots, X_n$.

$$Var(\bar{X}_n^r) = Var\left(\frac{1}{n} \sum_{i=1}^{n} X_i^r\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i^r\right)$$  (14)

If the $X$'s are independent, then

$$Var(\bar{X}_n^r) = \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i^r)$$  (15)

If the $X$'s are independent and identically distributed, then

$$Var(\bar{X}_n^r) = \frac{1}{n} Var(X^r)$$  (16)

where $X$ denotes any one of the random variables (because they are all identical). In the case where $r = 1$, we obtain

$$Var(\bar{X}_n) = \frac{1}{n} Var(X) = \frac{\sigma^2}{n}$$  (17)

3. Sample Central Moments

3.1. Definitions. Assume there is a sequence of random variables, $X_1, X_2, \ldots, X_n$. We define the sample central moments as

$$C_n^r = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu'_{i,1})^r, r = 1, 2, 3, \ldots,$$

$$\Rightarrow C_n^1 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu'_{i,1})$$

$$\Rightarrow C_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu'_{i,1})^2$$  (18)

These are only defined if $\mu'_{i,1}$ is known.

3.2. Properties of Sample Central Moments.

3.2.1. Expected value of $C_n^r$. The expected value of $C_n^r$ is given by

$$E(C_n^r) = \frac{1}{n} \sum_{i=1}^{n} E(X_i - \mu'_{i,1})^r = \frac{1}{n} \sum_{i=1}^{n} \mu_{i,r}$$  (19)

The last equality follows from equation 7.

If the $X_i$ are identically distributed, then

$$E(C_n^r) = \mu_r$$

$$E(C_n^1) = 0$$  (20)
3.2.2. Variance of $C_r^n$. First consider the case where we have a sample $X_1, X_2, \ldots, X_n$.

\[
\text{Var} \left( C_r^n \right) = \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu'_i)^r \right) = \frac{1}{n^2} \text{Var} \left( \sum_{i=1}^{n} (X_i - \mu'_{i,1})^r \right) \tag{21}
\]

If the $X$'s are independently distributed, then

\[
\text{Var} \left( C_r^n \right) = \frac{1}{n^2} \sum_{i=1}^{n} \text{Var} \left[ (X_i - \mu'_{i,1})^r \right] \tag{22}
\]

If the $X$'s are independent and identically distributed, then

\[
\text{Var} \left( C_r^n \right) = \frac{1}{n} \text{Var} \left[ (X - \mu'_1)^r \right] \tag{23}
\]

where $X$ denotes any one of the random variables (because they are all identical). In the case where $r = 1$, we obtain

\[
\text{Var} \left( C_1^n \right) = \frac{1}{n} \text{Var} \left[ X - \mu'_1 \right] = \frac{1}{n} \text{Var} \left[ X - \mu \right] = \frac{1}{n} \sigma^2 - 2 \text{Cov} \left[ X, \mu \right] + \text{Var} \left[ \mu \right] = \frac{1}{n} \sigma^2 \tag{24}
\]

4. Sample About the Average

4.1. Definitions. Assume there is a sequence of random variables, $X_1, X_2, \ldots, X_n$. Define the $r$th sample moment about the average as

\[
M_r^n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^r, \quad r = 1, 2, 3, \ldots, \tag{25}
\]

This is clearly a statistic of which we can compute a numerical value. We denote the numerical value by, $m_r^n$, and define it as

\[
m_r^n = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x}_n)^r \tag{26}
\]

In the special case where $r = 1$ we have

\[
M_1^n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n) = \frac{1}{n} \sum_{i=1}^{n} X_i - \bar{X}_n = \bar{X}_n - \bar{X}_n = 0 \tag{27}
\]

4.2. Properties of Sample Moments about the Average when $r = 2$. 
4.2.1. *Alternative ways to write $M^r_n$.* We can write $M^2_n$ in an alternative useful way by expanding the squared term and then simplifying as follows

$$M^r_n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^r$$

$$\Rightarrow M^2_n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

$$= \frac{1}{n} \left( \sum_{i=1}^{n} [X_i^2 - 2 X_i \bar{X}_n + \bar{X}_n^2] \right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \frac{2 \bar{X}_n}{n} \sum_{i=1}^{n} X_i + \frac{1}{n} \sum_{i=1}^{n} \bar{X}_n^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} X_i^2 - 2 \bar{X}_n^2 + \bar{X}_n^2$$

$$= \frac{1}{n} \left( \sum_{i=1}^{n} X_i^2 \right) - \bar{X}_n^2$$

(28)

4.2.2. *Expected value of $M^2_n$.* The expected value of $M^r_n$ is then given by

$$E \left( M^2_n \right) = \frac{1}{n} E \left[ \sum_{i=1}^{n} X_i^2 \right] - E \left[ \bar{X}_n^2 \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} E \left[ X_i^2 \right] - \left( E \left[ \bar{X}_n \right] \right)^2 - Var(\bar{X}_n)$$

(29)

The second line follows from the alternative definition of variance

$$Var \left( X \right) = E \left( X^2 \right) - \left[ E \left( X \right) \right]^2$$

$$\Rightarrow E \left( X^2 \right) = \left[ E \left( X \right) \right]^2 + Var \left( X \right)$$

(30)

$$\Rightarrow E \left( \bar{X}_n^2 \right) = \left[ E \left( \bar{X}_n \right) \right]^2 + Var(\bar{X}_n)$$

and the third line follows from equation 12. If the $X_i$ are independent and identically distributed, then
\[
E \left( \frac{M_n^2}{n} \right) = \frac{1}{n} E \left[ \sum_{i=1}^{n} X_i^2 \right] - E \left[ \bar{X}_n^2 \right]
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \mu_{i,2} - \left( \frac{1}{n} \sum_{i=1}^{n} \mu_{i,1} \right)^2 - \text{Var}(\bar{X}_n)
\]

\[
= \mu_2' - (\mu_1')^2 - \frac{\sigma^2}{n}
\]

\[
= \sigma^2 - \frac{1}{n} \sigma^2
\]

\[
= \frac{n-1}{n} \sigma^2
\]

(31)

where \( \mu_1' \) and \( \mu_2' \) are the first and second population moments, and \( \mu_2 \) is the second central population moment for the identically distributed variables. Note that this obviously implies

\[
E \left[ \sum_{i=1}^{n} (X_i - \bar{X})^2 \right] = n E \left( \frac{M_n^2}{n} \right)
\]

\[
= n \left( \frac{n-1}{n} \right) \sigma^2
\]

\[
= (n-1) \sigma^2
\]

(32)

4.2.3. Variance of \( \frac{M_n^2}{n} \). By definition,

\[
\text{Var} \left( \frac{M_n^2}{n} \right) = E \left[ \left( \frac{M_n^2}{n} \right)^2 \right] - (E \left( \frac{M_n^2}{n} \right))^2
\]

(33)

The second term on the right on equation 33 is easily obtained by squaring the result in equation 31.

\[
E \left( \frac{M_n^2}{n} \right) = \frac{n-1}{n} \sigma^2
\]

\[
\Rightarrow \left( E \left( \frac{M_n^2}{n} \right) \right)^2 = (E \left( \frac{M_n^2}{n} \right))^2 = \frac{(n-1)^2}{n^2} \sigma^4
\]

(34)

Now consider the first term on the right hand side of equation 33. Write it as

\[
E \left[ \left( \frac{M_n^2}{n} \right)^2 \right] = E \left[ \left( \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \right)^2 \right]
\]

(35)

Now consider writing \( \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \) as follows
\[
\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{1}{n} \sum_{i=1}^{n} ((X_i - \mu) - (\bar{X} - \mu))^2 \\
= \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y})^2
\]  

where \( Y_i = X_i - \mu \)  
\( \bar{Y} = \bar{X} - \mu \)

Obviously,

\[
\sum_{i=1}^{n} (X_i - \bar{X})^2 = \sum_{i=1}^{n} (Y_i - \bar{Y})^2, \text{ where } Y_i = X_i - \mu, \bar{Y} = \bar{X} - \mu
\]  

Now consider the properties of the random variable \( Y_i \) which is a transformation of \( X_i \). First the expected value.

\[
E(Y_i) = E(X_i) - E(\mu) \\
= \mu - \mu \\
= 0
\]  

The variance of \( Y_i \) is

\[
Y_i = X_i - \mu \\
Var(Y_i) = Var(X_i) \\
= \sigma^2 \text{ if } X_i \text{ are independently and identically distributed}
\]  

Also consider \( E(Y_i^4) \). We can write this as

\[
E(Y^4) = \int_{-\infty}^{\infty} y^4 f(x) \, dx \\
= \int_{-\infty}^{\infty} (x - \mu)^4 f(x) \, dx \\
= \mu_4
\]  

Now write equation 35 as follows
Ignoring $\frac{1}{n^2}$ for now, expand equation 41 as follows

$$E \left[ \left( \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n) \right)^2 \right] = E \left[ \left( \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}) \right)^2 \right]$$  \hspace{1cm} (41a)

$$= E \left[ \left( \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y}) \right)^2 \right]$$  \hspace{1cm} (41b)

$$= \frac{1}{n^2} E \left[ \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right]$$  \hspace{1cm} (41c)

$$= \frac{1}{n^2} E \left[ \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right] = \frac{1}{n^2}$$  \hspace{1cm} (41d)

Ignoring $\frac{1}{n^2}$ for now, expand equation 41 as follows

$$E \left[ \left( \sum_{i=1}^{n} (Y_i - \bar{Y}) \right)^2 \right] = E \left[ \left( \sum_{i=1}^{n} (Y_i^2 - 2 Y_i \bar{Y} + \bar{Y}^2) \right)^2 \right]$$  \hspace{1cm} (42a)

$$= E \left[ \sum_{i=1}^{n} Y_i^2 - 2 \bar{Y} \sum_{i=1}^{n} Y_i + \bar{Y}^2 \right]^2$$  \hspace{1cm} (42b)

$$= E \left[ \left( \sum_{i=1}^{n} Y_i^2 \right) - 2 n \bar{Y}^2 + n \bar{Y}^2 \right]^2$$  \hspace{1cm} (42c)

$$= E \left[ \left( \sum_{i=1}^{n} Y_i^2 \right) - n \bar{Y}^2 \right]^2$$  \hspace{1cm} (42d)

$$= E \left[ \sum_{i=1}^{n} Y_i^2 \right]^2 - 2 n \bar{Y}^2 \sum_{i=1}^{n} Y_i^2 + n^2 \bar{Y}^4$$  \hspace{1cm} (42e)

$$= E \left[ \sum_{i=1}^{n} Y_i^2 \right]^2 - 2 \bar{Y}^2 \sum_{i=1}^{n} Y_i^2 + n^2 E (\bar{Y}^4)$$  \hspace{1cm} (42f)

Now consider the first term on the right of 42 which we can write as
\[ E \left[ \left( \sum_{i=1}^{n} Y_i^2 \right)^2 \right] = E \left[ \sum_{i=1}^{n} Y_i^2 \sum_{j=1}^{n} Y_j^2 \right] \]

\[ = E \left[ \sum_{i=1}^{n} Y_i^4 + \sum_{i \neq j} Y_i^2 Y_j^2 \right] \]

\[ = \sum_{i=1}^{n} E Y_i^4 + \sum_{i \neq j} E Y_i^2 E Y_j^2 \]

\[ = n \mu_4 + n (n - 1) \mu_2^2 \]

\[ = n \mu_4 + n (n - 1) \sigma^4 \]  \hspace{1cm} (43c)

Now consider the second term on the right of 42 (ignoring 2n for now) which we can write as

\[ E \left[ \sum_{i=1}^{n} Y_i^2 \sum_{j=1}^{n} Y_j \sum_{k=1}^{n} Y_k \right] = \frac{1}{n^2} E \left[ \sum_{j=1}^{n} Y_j \sum_{k=1}^{n} Y_k \sum_{i=1}^{n} Y_i^2 \right] \]

\[ = \frac{1}{n^2} \left[ \sum_{i=1}^{n} E Y_i^4 + \sum_{i \neq j} E Y_i^2 Y_j^2 + \sum_{j \neq k} E Y_j Y_k \sum_{i \neq j} Y_i^2 \right] \]

\[ = \frac{1}{n^2} \left[ \sum_{i=1}^{n} E Y_i^4 + \sum_{i \neq j} E Y_i^2 E Y_j^2 + \sum_{j \neq k} E Y_j E Y_k \sum_{i \neq j} E Y_i^2 \right] \]

\[ = \frac{1}{n^2} \left[ n \mu_4 + n (n - 1) \mu_2^2 + 0 \right] \]  \hspace{1cm} (44d)

\[ = \frac{1}{n} \left[ \mu_4 + (n - 1) \sigma^4 \right] \]  \hspace{1cm} (44e)

The last term on the penultimate line is zero because \( E(Y_j) = E(Y_k) = E(Y_i) = 0 \).
Now consider the third term on the right side of 42 (ignoring \( n^2 \) for now) which we can write as

\[
E [\bar{Y}^4] = \frac{1}{n^4} E \left[ \sum_{i=1}^{n} Y_i \sum_{j=1}^{n} Y_j \sum_{k=1}^{n} Y_k \sum_{\ell=1}^{n} Y_\ell \right]
\]

\[
= \frac{1}{n^2} E \left[ \sum_{i=1}^{n} Y_i^4 + \sum_{i \neq k} Y_i^2 Y_k^2 + \sum_{i \neq j} Y_i^2 Y_j^2 + \sum_{i \neq j} Y_i^2 Y_j^2 + \cdots \right]
\]

where for the first double sum (\( i = j \neq k = \ell \)), for the second (\( i = k \neq j = \ell \)), and for the last (\( i = \ell \neq j = k \)) and \( \cdots \) indicates that all other terms include \( Y_i \) in a non-squared form, the expected value of which will be zero. Given that the \( Y_i \) are independently and identically distributed, the expected value of each of the double sums is the same, which gives

\[
E [\bar{Y}^4] = \frac{1}{n^4} E \left[ \sum_{i=1}^{n} Y_i^4 + \sum_{i \neq k} Y_i^2 Y_k^2 + \sum_{i \neq j} Y_i^2 Y_j^2 + \sum_{i \neq j} Y_i^2 Y_j^2 + \cdots \right]
\]

\[
= \frac{1}{n^4} \left[ \sum_{i=1}^{n} E Y_i^4 + 3 \sum_{i \neq j} Y_i^2 Y_j^2 + \text{terms containing } EX_i \right]
\]

\[
= \frac{1}{n^4} \left[ \sum_{i=1}^{n} E Y_i^4 + 3 \sum_{i \neq j} Y_i^2 Y_j^2 \right]
\]

\[
= \frac{1}{n^4} \left[ n \mu_4 + 3 n (n - 1) (\mu_2)^2 \right]
\]

\[
= \frac{1}{n^3} \left[ n \mu_4 + 3 n (n - 1) \sigma^4 \right]
\]

Now combining the information in equations 44, 45, and 46 we obtain
\[
E \left[ \left( \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right)^2 \right] = E \left[ \left( \sum_{i=1}^{n} (Y_i^2 - 2Y_i\bar{Y} + \bar{Y}^2) \right)^2 \right] 
\]

\[
= E \left[ \left( \sum_{i=1}^{n} Y_i^2 \right)^2 \right] - 2n E \left[ \bar{Y}^2 \sum_{i=1}^{n} Y_i^2 \right] + n^2 E (\bar{Y}^4) 
\]

\[
= n\mu_4 + n(n-1)\mu_2^2 - 2n \left[ \frac{1}{n} \left( \mu_4 + (n-1)\mu_2^2 \right) \right] + n^2 \left[ \frac{1}{n^2} \left( \mu_4 + 3(n-1)\mu_2^2 \right) \right] 
\]

\[
= n\mu_4 + n(n-1)\mu_2^2 - 2 [\mu_4 + (n-1)\mu_2^2] + \left[ \frac{1}{n} \left( \mu_4 + 3(n-1)\mu_2^2 \right) \right] 
\]

\[
= \frac{n^2}{n} \mu_4 - \frac{2n}{n} \mu_4 + \frac{n}{n} \mu_4 + \frac{n(n-1)}{n} \mu_2^2 - \frac{2n(n-1)}{n} \mu_2^2 + \frac{3(n-1)}{n} \mu_2^2 
\]

\[
= \frac{n^2 - 2n + 1}{n} \mu_4 + \frac{(n-1)(n^2-2n+3)}{n} \mu_2^2 
\]

\[
= \frac{n^2 - 2n + 1}{n^3} \mu_4 + \frac{(n-1)(n^2-2n+3)}{n^3} \sigma_4 
\]

Now rewrite equation 41 including \( \frac{1}{n^2} \) as follows

\[
E \left[ \left( M_n^2 \right)^2 \right] = \frac{1}{n^2} E \left[ \left( \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right)^2 \right] 
\]

\[
= \frac{1}{n^2} \left( \frac{n^2 - 2n + 1}{n} \mu_4 + \frac{(n-1)(n^2-2n+3)}{n} \sigma_4 \right) 
\]

\[
= \frac{n^2 - 2n + 1}{n^3} \mu_4 + \frac{(n-1)(n^2-2n+3)}{n^3} \sigma_4 
\]

\[
= \frac{(n-1)^2}{n^3} \mu_4 + \frac{(n-1)(n^2-2n+3)}{n^3} \sigma_4 
\]

Now substitute equations 34 and 48 into equation 33 to obtain

\[
Var \left( M_n^2 \right) = E \left[ \left( M_n^2 \right)^2 \right] - \left( E M_n^2 \right)^2 
\]

\[
= \frac{(n-1)^2}{n^3} \mu_4 + \frac{(n-1)(n^2-2n+3)}{n^3} \sigma_4 - \frac{(n-1)^2}{n^2} \sigma_4 
\]

We can simplify this as
5. Sample Variance

5.1. Definition of sample variance. The sample variance is defined as

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$ (51)

We can write this in terms of moments about the mean as

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 = \frac{n}{n-1} M_n^2$$ where $M_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$ (52)

5.2. Expected value of $S^2$. We can compute the expected value of $S^2$ by substituting in from equation 31 as follows

$$E (S_n^2) = \frac{n}{n-1} E (M_n^2) = \frac{n}{n-1} \frac{n-1}{n} \sigma^2 = \sigma^2$$ (53)

5.3. Variance of $S^2$. We can compute the variance of $S^2$ by substituting in from equation 50 as follows
\[ \text{Var} \left( S_n^2 \right) = \frac{n^2}{(n - 1)^2} \text{Var} \left( M_n^2 \right) \]
\[ = \frac{n^2}{(n - 1)^2} \left( \frac{(n - 1)^2 \mu_4}{n^3} - \frac{(n - 1)(n - 3) \sigma^4}{n^3} \right) \]  
(54)
\[ = \frac{\mu_4}{n} - \frac{(n - 3) \sigma^4}{n(n - 1)} \]

5.4. Definition of \( \hat{\sigma}^2 \). One possible estimate of the population variance is \( \hat{\sigma}^2 \) which is given by
\[ \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left( X_i - \bar{X}_n \right)^2 \]
(55)
\[ = M_n^2 \]

5.5. Expected value of \( \hat{\sigma}^2 \). We can compute the expected value of \( \hat{\sigma}^2 \) by substituting in from equation 31 as follows
\[ E \left( \hat{\sigma}^2 \right) = E \left( M_n^2 \right) \]
\[ = \frac{n - 1}{n} \sigma^2 \]  
(56)

5.6. Variance of \( \hat{\sigma}^2 \). We can compute the variance of \( \hat{\sigma}^2 \) by substituting in from equation 50 as follows
\[ \text{Var} \left( \hat{\sigma}^2 \right) = \text{Var} \left( M_n^2 \right) \]
\[ = \frac{(n - 1)^2 \mu_4}{n^3} - \frac{(n - 1)(n - 3) \sigma^4}{n^3} \]  
(57)
\[ = \frac{\mu_4 - \mu_2^2}{n} - \frac{2(\mu_4 - 2\mu_2^2)}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3} \]

We can also write this in an alternative fashion
\[ \text{Var} \left( \hat{\sigma}^2 \right) = \text{Var} \left( M_n^2 \right) \]
\[ = \frac{(n - 1)^2 \mu_4}{n^3} - \frac{(n - 1)(n - 3) \sigma^4}{n^3} \]
\[ = \frac{(n - 1)^2 \mu_4}{n^3} - \frac{(n - 1)(n - 3) \mu_4^2}{n^3} \]
\[ = \frac{n^2 \mu_4 - 2n \mu_4 + \mu_4}{n^3} - \frac{n^2 \mu_2^2 - 4n \mu_2^2 + 3 \mu_2^2}{n^3} \]  
(58)
\[ = \frac{n^2 (\mu_4 - \mu_2^2)}{n^3} - \frac{2(n \mu_4 - 2 \mu_2^2) + \mu_4 - 3 \mu_2^2}{n^3} \]
\[ = \frac{\mu_4 - \mu_2^2}{n} - \frac{2(\mu_4 - 2 \mu_2^2)}{n^2} + \frac{\mu_4 - 3 \mu_2^2}{n^3} \]
6. Normal populations

6.1. Central moments of the normal distribution. For a normal population we can obtain the central moments by differentiating the moment generating function. The moment generating function for the central moments is as follows

\[ M_X(t) = e^{\frac{t^2 \sigma^2}{2}}. \]  
(59)

The moments are then as follows. The first central moment is

\[ E(X - \mu) = \frac{d}{dt} \left( e^{\frac{t^2 \sigma^2}{2}} \right) \bigg|_{t=0} = t \sigma^2 \left( e^{\frac{t^2 \sigma^2}{2}} \right) \bigg|_{t=0} = t \sigma^2 \]  
(60)

The second central moment is

\[ E(X - \mu)^2 = \frac{d^2}{dt^2} \left( e^{\frac{t^2 \sigma^2}{2}} \right) \bigg|_{t=0} = \frac{1}{2} t^2 \sigma^4 \left( e^{\frac{t^2 \sigma^2}{2}} \right) \bigg|_{t=0} = \sigma^2 \]  
(61)

The third central moment is

\[ E(X - \mu)^3 = \frac{d^3}{dt^3} \left( e^{\frac{t^2 \sigma^2}{2}} \right) \bigg|_{t=0} = \frac{1}{6} t^3 \sigma^6 \left( e^{\frac{t^2 \sigma^2}{2}} \right) + \frac{1}{2} t^2 \sigma^4 \left( e^{\frac{t^2 \sigma^2}{2}} \right) \bigg|_{t=0} = 0 \]  
(62)

The fourth central moment is
\[ E(X - \mu)^4 = \frac{d^4}{dt^4} \left( e^{\frac{t^2}{2\sigma^2}} \right) \bigg|_{t=0} \]
\[ = \frac{d}{dt} \left( t^3 \sigma^6 \left( e^{\frac{t^2}{2\sigma^2}} \right) + 3 t^4 \sigma^6 \left( e^{\frac{t^2}{2\sigma^2}} \right) \right) \bigg|_{t=0} \]
\[ = \left( t^4 \sigma^8 \left( e^{\frac{t^2}{2\sigma^2}} \right) + 3 t^2 \sigma^6 \left( e^{\frac{t^2}{2\sigma^2}} \right) + 3 \sigma^4 \left( e^{\frac{t^2}{2\sigma^2}} \right) \right) \bigg|_{t=0} \]
\[ = \left( t^4 \sigma^8 \left( e^{\frac{t^2}{2\sigma^2}} \right) + 6 t^2 \sigma^6 \left( e^{\frac{t^2}{2\sigma^2}} \right) + 3 \sigma^4 \left( e^{\frac{t^2}{2\sigma^2}} \right) \right) \bigg|_{t=0} \]
\[ = 3\sigma^4 \]

6.2. Variance of \( S^2 \). Let \( X_1, X_2, \ldots X_n \) be a random sample from a normal population with mean \( \mu \) and variance \( \sigma^2 < \infty \).

We know from equation 54 that
\[
Var \left( S_n^2 \right) = \frac{n^2}{(n-1)^2} Var \left( M_n^2 \right)
\]
\[
= \frac{n^2}{(n-1)^2} \left( \frac{(n-1)^2 \mu_4}{n^3} - \frac{(n-1)(n-3)\sigma^4}{n^3} \right) \tag{64}
\]
\[
= \frac{\mu_4}{n} - \frac{(n-3)\sigma^4}{n(n-1)}
\]

If we substitute in for \( \mu_4 \) from equation 63 we obtain
\[
Var \left( S_n^2 \right) = \frac{\mu_4}{n} \frac{(n-3)\sigma^4}{n(n-1)}
\]
\[
= \frac{3\sigma^4}{n} - \frac{(n-3)\sigma^4}{n(n-1)}
\]
\[
= \frac{(3(n-1)-(n-3))\sigma^4}{n(n-1)}
\]
\[ = \frac{(3n-3-n+3)}{n(n-1)} \sigma^4
\]
\[ = \frac{2n\sigma^4}{n(n-1)}
\]
\[ = \frac{2\sigma^4}{n-1} \]

6.3. Variance of \( \hat{\sigma}^2 \). It is easy to show that
\[
Var \left( \hat{\sigma}^2 \right) = \frac{2\sigma^4(n-1)}{n^2} \tag{66}
\]