In this section we study a second model of money. Recall the central questions of monetary theory:

1. Why do people hold money, an asset that does not pay interest (rate of return dominance)?
2. Why is money valued in equilibrium?
3. What are the effects of monetary policy: one time increases in the money supply or changes in the money growth rate?

The OLG models we studied provides a satisfactory answer why money is valued: it is a bubble. However, models of this kind fall short because money is never held if its rate of return is dominated by another asset. The idea of CIA models is to explicitly model the frictions that make money valuable, but the modeling has very much the character of a reduced form: it is simply assumed that certain goods can only be purchased with bits of green paper.

Other variations of monetary models (that we won’t cover) have much cleaner micro foundation. In search models a household cannot eat his own production and must therefore trade. But finding a trading partner is costly. Money is valued because of the expectation that others will value it in the future as well. If I meet an individual who would like to consume my good, I may sell it for money, because I don’t like to eat his good. But I may expect to meet someone in the future who has something edible to offer in exchange for my money.

Limited participation models are similar to CIA models, but spell out the institutional structure that gives rise to market frictions.

1. Timing

The overall model structure resembles that of the neoclassical growth model. But it is now assumed that some goods can only be purchased for money, not for other goods. The timing of the model is crucial.

1. The household enters the period with capital $k_t$ and a stock of money $m_{t-1}^d$.
2. He then receives a transfer of money from the government so that his period $t$ money holdings are $m_t = m_{t-1}^d + \tau_t$.
3. Next the household produces and sells his output for money to be received at the “end of the period.” At the same time he uses $m_t$ to buy goods from other households ($c_t$ and $k_{t+1}$).

4. The household is paid for the goods he sold in step 3 so that his end of period money stock is $m_t^d$.

Note that money is “in limbo” for a while. The household needs money to pay others, but does not actually receive payments for his own sales until the end of the period. One way of thinking about this is that the money is “in the mail” or in escrow until the end of the period. It cannot be used for anything during that time. As a consequence, households must hold money at $t$ if they want to buy consumption goods at $t+1$.

2. Household Problem

While this sounds like a complicated setup, the modification to the household problem is actually quite minor. There is simply an additional constraint (the CIA constraint) which says that expenditures on goods that need to be paid for with cash cannot exceed $m_t$.

The household solves

$$\max \sum_{t=1}^{\infty} \beta^t u(c_t) \quad \text{subject to}$$

(BC) $$k_{t+1} + c_t + m_t^d / p_t = f(k_t) + (1-\delta)k_t + m_t / p_t$$

(CIA) $$m_t / p_t \geq c_t + k_{t+1} - (1-\delta)k_t$$

$$m_{t+1} = m_t^d + \tau_{t+1}$$

The budget constraint is fairly standard. After production the household receives goods $f(k)$ and undepreciated capital. In addition, the household carries money balances $m_t$ into the period, which must be divided by the price level because they are in nominal terms. There are three uses for the household’s resources: consumption, investment in capital, and money demand, $m_t^d$. The law of motion for money states that $m_{t+1}$ equals today’s money demand plus a money transfer from the government, $\tau_{t+1}$.

Exactly what kinds of goods have to be bought with cash is arbitrary. Some models have cash and credit goods. Here we assume that all goods (including replacement investment) are bought with cash. It will turn out that the CIA constraint holds with equality if the rate of return on money is less than that on capital (the nominal interest rate is positive).
2.1 DP Formulation

Note that the household now has two (individual) state variables \((m, k)\). The Bellman equation is therefore:

\[
V(m, k) = \max u(c) + \beta V(m', k') + \lambda (BC) + \gamma (CIA). 
\]

To write this properly in recursive form, we need to impose \(m_t = m_{t-1}^d + \tau_t\). We can then use \(m_{t+1}\) as a choice variable (this would not work under uncertainty).

\[
V(m, k) = \max u(c) + \beta V(m', k') + \\
\quad + \lambda [f(k) + (1-\delta)k/m - c - k' - (m' - \tau')/p] \\
\quad + \gamma [m/p - c - k' + (1-\delta)k] \\
\]

FOC: \(u'(c) = \lambda + \gamma\), \(\beta V_m(\cdot') = \lambda/p\), \(\beta V_k(\cdot') = \lambda + \gamma\)

KT: \(\gamma [m/p - c - k' + (1-\delta)k] = 0\), \(\gamma \geq 0\)

Thus: \(u'(c) = \beta V_k(\cdot')\)

Envelope conditions: \(V_m = (\lambda + \gamma)/p\), \(V_k = \lambda [f'(k) + 1 - \delta] + \gamma [1 - \delta]\)

Eliminate derivatives of \(V\): \(\beta [\lambda' + \gamma']/p' = \lambda/p\), \((\lambda + \gamma)/\beta = \lambda'. f'(k') + [1 - \delta][\lambda' + \gamma']\)

(1) \(\beta u'(c') p'/p' = \lambda\)

and obtain the Euler equation for \(c\):

(2) \(u'(c) = \beta^2 u'(c')(p'/p^*) f'(k') + (1-\delta)\beta u'(c')\)

To summarize, the solution of the household problem takes prices \((p)\) as given and consists of a value function and policy functions \([c(k, m), k'(k, m), m'(k, m)]\) that satisfy the Bellman equation and the policy functions solve the max problem and satisfy the budget constraint. In sequence language a solution consists of sequences \((c_t, k_t, m_t)\) that satisfy the Euler equation and the budget constraint. If the CIA constraint binds, it provides the third equation. What happens if the CIA constraint does not bind is discussed below. We also need two transversality conditions that prevent the present value of wealth from blowing up. Note that we need four boundary conditions because we have two first-order and one second-order difference equation.

The interpretation of the Euler equation is as follows. Suppose the household gives up a unit of \(c\) and buys more capital instead \((k'\) rises). He gains two benefits: Next period, he can consume the undepreciated capital. Plus he produces additional output \((f'(k'))\), which he can invest in money that yields \((p'/p^*)\) until period \(t+2\), when it can be consumed. The complication is that any reduction in consumption today can only be consumed in \(t+2\). Of course, it could be
consumed in $t+1$, if the household held money instead of additional capital, but that would not be optimal as long as money pays a lower interest rate than $k$.

### 2.2 When does the CIA constraint bind?

We next show that the CIA constraint binds unless the return on money equals that on capital, i.e. the nominal interest rate is zero. If the CIA constraint does not bind, then $\gamma = 0$ and therefore $\lambda^t/\beta = \lambda'[f'(k') + 1 - \delta]$, so we get a standard Euler equation – the same as in a non-monetary economy. What happens is that holding money is free; it has no opportunity cost. Money then does not distort the real allocation in any way.

What does monetary policy have to do in order to achieve outcome? With $\gamma = 0$, we have $\beta \lambda'/p' = \lambda/p$ and thus $\lambda/(\beta \lambda') = p/p' = [f'(k) + 1 - \delta]$. The real return on money is the negative of the inflation rate and the RHS of the equation is the real return on capital.

What happens to the solution of the household problem in that case? We still need to determine $c, k,$ and $m$ sequences using the Euler equation, the budget constraint and a third condition that is no longer the CIA constraint. Now the capital stock is determined by $p/p' = f'(k) + 1 - \delta$, while money demand follows from the budget constraint. The intuition is illustrated in the next figure. The household can hold two assets, $k$ and $m$. The rate of return to $m$ is given $(p/p')$. The rate of return to $k$ is a decreasing function. If the CIA constraint does not bind, the household accumulates $k$ until its return falls to that of money. Any additional wealth is accumulated in the form of money.
3. Equilibrium

Government: The government’s only role is to hand out lump-sum transfers of money. The money growth rule is \( \tau_t = g m_{t-1}^d \) so that money holdings in period \( t \) are

\[
m_t = m_{t-1}^d + \tau_t = (1 + g)m_{t-1}^d.
\]

Market clearing: Goods market clearing requires (the same as feasibility):
\( c + k' = f(k) + (1 - \delta)k \). Money market clearing is implicit in the notation.

Definition: An equilibrium is a sequence \((k_t, m_t, c_t, \tau_t, p_t)\) that satisfies

1. the money growth rule and definition of \( \tau \) (sort of a government budget constraint);
2. the household optimality conditions (see above) (3 equations)
3. the goods market clearing condition.

4. Steady State

In steady state, all real variables are constant \((c, k, m/ p)\). This requires \( \pi = g \) to hold real money balances constant. Since \( u'(c) \) is constant, the Euler equation (2) implies

\[
1 = \beta^2 (1 + \pi)^{-1} f'(k') + (1 - \delta)\beta.
\]

Using \( 1 + \pi = 1 + g \) this can be solved for the capital stock:

\[
f'(k) = (1 + g)[1 - \beta(1 - \delta)]/\beta^2
\]

Assuming that the CIA constraint binds: \( f(k) = m/ p \). Goods market clearing with constant \( k \) implies \( c = f(k) - \delta k \).

4.1 Properties of the Steady State

1. Money is not superneutral. Higher inflation \((g \uparrow)\) implies a lower capital stock and lower consumption. The reason is that inflation increases the cost of holding money, which is required for investment. There is an inflation tax on any activity that requires money holdings. Models of this kind are, in fact, commonly used to study the welfare costs of inflation (e.g. Abel 1997).

Definition: Money is called neutral if changing the level of \( M \) does not affect the real allocation. It is called superneutral if changing the growth rate of \( M \) does not affect the real allocation.

Exercise: Show that super-neutrality would be restored, if the CIA constraint applied only to consumption \((m/ p \geq c)\). What is the intuition for this finding?

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2. The velocity of money is one. Higher inflation reduces money demand only by reducing output. This is a direct consequence of the rigid CIA constraint and probably an undesirable result. Obviously, this would not be a good model of hyperinflation.

3. The Friedman rule ensures that the CIA constraint does not bind. It can be shown that this maximizes welfare.

The Friedman rule requires the nominal interest rate to be zero. The idea is to eliminate the opportunity cost of holding money, which induces agents to inefficiently economize on holding a good that can be produced costlessly. The opportunity cost of money is the gap between its rate of return and that of other assets (here \( k \)). Setting the real return on money (the negative inflation rate) equal to that of capital, in which case the nominal interest rate is zero, eliminates this inefficiency. This policy rule turns out to be optimal (in steady state) in a number of models.

To see that the Friedman rule is optimal here, it is easy to show that the Friedman rule induces the first-best allocation. The planner maximizes discounted utility subject to the feasibility constraint only. There is no money in the planner’s problem. Even if the planner faces something akin to a transactions technology that requires pieces of paper as input, the fact that paper can be produced costlessly means that such a technology cannot affect the allocation. The planner’s problem thus reduces to a standard non-monetary growth model. The Pareto optimal allocation is characterized by an Euler equation and a feasibility constraint. The Euler equation is the same as the household Euler equation when the CIA constraint does not bind. The feasibility constraint is the same as the goods market clearing condition. Hence, both allocations must be the same and the Friedman rule is first best.

Is this a good theory of money? For many purposes the answer is “yes” (although Neil Wallace disagrees). The model is consistent with rate of return dominance and “explains” why money is valued: it provides liquidity services. Of course, this result is essentially assumed. Why does one need money instead of t-bills to pay bills? Clearly, one should think of the CIA constraint as a reduced form for a fully specified model with trading frictions (a search model?). Still, the model does allow to think about questions for which endogeneity of the institutional structure (that gives rise to the CIA constraint) is not essential. There is one caveat, though: the model’s properties depend on the exact nature of the CIA constraint (see the homework), which is arbitrary.

5. Reading

BF chapter 4 (CIA model in continuous time).


[A critique of reduced form models of money (such as CIA or money in the utility function).]

The extension to a shopping time model in which the CIA constraint is less rigid is discussed in Ljundquist and Sargent (2000), ch. 17. They also show that the implications of monetary policy can be very different, if money is used to finance government spending.