What this section is about:

We study a second model of money.

Recall the central questions of monetary theory:

1. Why do people hold money, an asset that does not pay interest (rate of return dominance)?
2. Why is money valued in equilibrium?
3. What are the effects of monetary policy: one time increases in the money supply or changes in the money growth rate?
OLG models have 2 shortcomings:

1. They fail to explain rate of return dominance.
2. Money has no transaction value.

CIA models focus on transactions demand for money.

Their main shortcoming: the fact that money must be used in transactions is assumed, not explained.
1 Timing

The overall model structure is that of the standard growth model.

The timing within each period is crucial.

1. The household enters the period with capital $k_t$ and a stock of money $m_{t-1}^d$.
2. He then receives a transfer of money from the government. His period $t$ money holdings are

$$m_t = m_{t-1}^d + \tau_t$$

3. The household produces and sells his output for money to be received at the “end of the period.”
4. He uses $m_t$ to buy goods from other households ($c_t$ and $k_{t+1}$).
5. The household is paid for the goods he sold in step 3 so that his end of period money stock is $m_t^d$.

Note that money earned in period $t$ cannot be used until $t + 1$. 
2 Household Problem

We simply add one constraint to the household problem: the CIA constraint.

The household solves

$$\max \sum_{t=1}^{\infty} \beta^t u(c_t)$$

subject to the budget constraint

$$k_{t+1} + c_t + m_t^d/p_t = f(k_t) + (1 - \delta)k_t + m_t/p_t$$

and the CIA constraint

$$m_t/p_t \geq c_t + k_{t+1} - (1 - \delta)k_t$$

and the law of motion

$$m_{t+1} = m_t^d + \tau_{t+1}$$

Exactly what kinds of goods have to be bought with cash is arbitrary.

The CIA constraint holds with equality if the rate of return on money is less than that on capital (the nominal interest rate is positive).
2.1 DP Formulation

Individual state variables: \( m, k \).

Bellman equation:

\[
V(m, k) = \max u(c) + \beta V(m', k') + \lambda(BC') + \gamma(CIA)
\]

We need to impose

\[
m_t = m_{d}^{t} + \tau_t
\]

Then we can use \( m_{t+1} \) as a control (this would not work under uncertainty).
\[ V(m, k) = \max u(c) + \beta V(m', k') \]
\[ + \lambda [f(k) + (1 - \delta)k + m/p \]
\[ - c - k' - (m' - \tau')/p] \]
\[ + \gamma [m/p - c - k' + (1 - \delta)k] \]

**FOC:**

\[ u'(c) = \lambda + \gamma \]
\[ \beta V_m(c') = \lambda/p \]
\[ \beta V_k(c') = \lambda + \gamma \]

**Kuhn Tucker:**

\[ \gamma [m/p - c - k' + (1 - \delta)k] = 0 \]
\[ \gamma \geq 0 \]
Thus:

\[ u'(c) = \beta V_k(\cdot') \]

Envelope conditions:

\[
\begin{align*}
V_m &= (\lambda + \gamma)/p \\
V_k &= \lambda[f'(k) + 1 - \delta] + \gamma[1 - \delta]
\end{align*}
\]

Eliminate derivatives of \( V \):

\[
\beta[\lambda' + \gamma']/p' = \lambda/p \\
(\lambda + \gamma)/\beta = \lambda' f'(k') + [1 - \delta][\lambda' + \gamma'] \\
\beta u'(c') p/p' = \lambda
\]

and obtain the Euler equation

\[ u'(c) = \beta^2 u''(c'')(p'/p'') f'(k') + (1 - \delta)\beta u'(c') \]
A solution to the household problem is:

A value function \( (V) \) and policy functions \( (m' (m, k), k' (m, k)) \) that "solve" the Bellman equation.

Or: Sequences \( \{c_t, m_{t+1}, k_{t+1}\} \) that satisfy:

1. the Euler equation;
2. the budget constraint;
3. the CIA constraint (with the law of motion).
4. transversality conditions:

\[
\lim_{t \to \infty} \beta^t u'(c_t) k_t = 0
\]
\[
\lim_{t \to \infty} \beta^t u'(c_t) m_t/p_t = 0
\]
2.1.1 Interpretation of the Euler equation

Today:

- Give up $dc = -\varepsilon$.

Tomorrow:

- $dk' = \varepsilon$.
- Eat the undepreciated capital: $dc' = (1 - \delta) \varepsilon$.
- Produce additional output $f'(k') \varepsilon$.
- Save it as money: $dm'' = f'(k') \varepsilon p'$.

The day after:

- Eat an additional $dm''/p''$. 
2.2 When does the CIA constraint bind?

We show: The CIA constraint binds unless the return on money equals that on capital, i.e. the nominal interest rate is zero.

If the CIA constraint does not bind, then $\gamma = 0$ and therefore we get the standard Euler equation

$$\frac{\lambda}{\beta} = \lambda' [f'(k') + 1 - \delta]$$

Holding money has no opportunity cost. Money then does not distort the real allocation in any way.
2.2.1 How would monetary policy achieve this outcome?

With $\gamma = 0$ we have

$$\beta \lambda'/p' = \lambda/p$$

and thus

$$\lambda/(\beta \lambda') = p/p' = [f'(k) + 1 - \delta]$$

The real return on money is the negative of the inflation rate.

This equals the real return on capital.

The nominal interest rate is 0.
What happens to the solution of the household problem?

The Euler equation and the budget constraint still hold.

The CIA constraint is replaced by

$$\frac{p}{p'} = f'(k) + 1 - \delta$$

**Intuition:** If the CIA constraint does not bind, the household only accumulates $k$ until its rate of return falls to that of money.
3 Equilibrium

Government:

The government’s only role is to hand out lump-sum transfers of money.

The money growth rule is

$$\tau_t = gm_{t-1}^d$$

Money holdings in period $t$ are

$$m_t = m_{t-1}^d + \tau_t$$
$$= (1 + g)m_{t-1}^d$$

Market clearing:

Goods: $c + k' = f(k) + (1 - \delta)k$.

Money market: implicit in the notation.

An equilibrium is a sequence $(k_t, m_t, c_t, \tau_t, p_t)$ that satisfies

1. the money growth rule and definition of $\tau$ (sort of a government budget constraint);
2. the household optimality conditions (see above) (3 equations)
3. the goods market clearing condition.
4 Steady State

In steady state all real, per capita variables are constant $(c, k, m/p)$.

This requires $\pi = g$ to hold real money balances constant.

The Euler equation implies

$$1 = \beta^2 (1 + \pi)^{-1} f'(k') + (1 - \delta)\beta$$

Using $1 + \pi = 1 + g$ this can be solved for the capital stock:

$$f'(k) = (1 + g)[1 - \beta(1 - \delta)] / \beta^2$$

Assuming that the CIA constraint binds:

$$f(k) = m/p$$

Goods market clearing with constant $k$ implies

$$c = f(k) - \delta k$$

A steady state is a vector $(c, k, m/p)$ that satisfies (1) through (3).
4.1 Properties of the Steady State

**Definition:** Money is called neutral if changing the level of $M$ does not affect the real allocation.

It is called super neutral if changing the growth rate of $M$ does not affect the real allocation.

**Exercise:** Show that super-neutrality would be restored, if the CIA constraint applied only to consumption ($m/p \geq c$). What is the intuition for this finding?

**Money is not super neutral.**

- Higher inflation ($g$) implies a lower $k$.
- Inflation increases the cost of holding money, which is required for investment (inflation tax).
The velocity of money is one.

Higher inflation reduces money demand only be reducing output. This is a direct consequence of the rigid CIA constraint and probably an undesirable result. Obviously, this would not be a good model of hyperinflation.

The Friedman rule ensures that the CIA constraint does not bind.

It can be shown that this maximizes welfare.

The Friedman rule requires the nominal interest rate to be zero.

It is optimal to make holding money costless b/c money can be costlessly produced.

This requires that the rate of return on money $\frac{1}{1+\pi}$ equal that on capital.

Exercise: Show that the Friedman rule attains the planner’s allocation.
5 Is this a good theory of money?

Positive features:

1. Rate of return dominance.
2. Money plays a liquidity role.

Drawbacks:

1. The reason why money is needed for transactions is not modeled.
2. The form of the CIA constraint is arbitrary (and important for the results).
3. The velocity of money is fixed.