Econ602: The Two-Sector Growth Model

Spring 2005. Prof. Lutz Hendricks

So far we assumed that there is only one good at each point in time. We now relax this assumption by assuming that capital and consumption goods are produced in different sectors. This setup is important for investigating, for example, the consequences of technical change that is "embodied" in capital goods (Hercovitz 1998; Greenwood, Hercovitz, and Krusell 1997), or for models of human capital.

1. Planning Problem

There is a unit mass of **households** who live forever. Their preferences over consumption and leisure are $\sum_{t=0}^{\infty} \beta^t u(c_t, 1-v_t)$. The household has one unit of time available in each period, which can be allocated to work (v) or leisure ($1-v$).

Consumption goods are produced according $Y_1 = F(K_1, L_1)$ and capital goods according to $Y_2 = G(K_2, L_2)$. The resource constraints are

$$L_1 + L_2 = v, \ K_1 + K_2 = K, \ Y_1 = c, \ Y_{2t} = K_{t+1} - (1-\delta)K_t.$$

The **planner** maximizes $\sum_{t=0}^{\infty} \beta^t u(c_t, 1-v_t)$ subject to the resource constraints. We may think of the planner as choosing investment, consumption, and the fractions of capital and labor allocated to each sector. Let's call φ the fraction of capital employed in sector 1:

$$K = K_1 + K_2 = \varphi K + (1-\varphi)K.$$

The planner's **state** variable is then simply K and there is no need to separately keep K_1 and K_2 around because the two types of capital can be transformed into each other one-for-one. Note: this would not be admissible if K_1 and K_2 were produced with different technologies. The Bellman equation is

$$V(K) = \max \ u\big(F(\varphi K, L_1), 1-L_1-L_2\big) + \beta V\big(K(1-\delta) + G([1-\varphi]K, L_2)\big)$$

where the choice variables are L_1, L_2, and φ.

FOCs: $u_l = \beta V'(K') G_L = u_c F_L, \ u_c F_K = \beta V'(K') G_K$

Envelope: $V'(K) = \varphi F_K u_c + \beta V'(K')\{1-\delta+(1-\varphi)G_K\}$

Combining these yields the **Euler** equation

$$u_c \frac{F_K}{G_K} = \beta u_c(.') \frac{F_K(.')}{G_K(.')} \{1 - \delta + G_K(.')\}$$

and the static condition

(1) $$F_K / F_L = G_K / G_L \,.$$

Intuition: The static condition simply equates marginal rates of substitution in the two sectors, which is necessary for maximizing output for given inputs. The Euler equation refers to the consequences of allocating a bit more capital towards producing capital today. This looks more familiar if we impose the condition from the one-sector case: $F_K = G_K$. The Euler equation then means: Giving up one unit of consumption today allows to consume MPK + 1 − δ more tomorrow.

To interpret the general condition, consider a feasible perturbation that only affects variables today and tommorrow, but not beyond. The **short version** goes like this: At any point in time, consumption can be converted into next period capital at a marginal rate of transformation of G_K / F_K. In period t we convert one unit of consumption into G_K / F_K units of capital. Next period this additional capital could produce $\{(1-\delta) + G_K(.')\} G_K / F_K$ units of date $t+2$ capital. But we want to hold date $t+2$ variables constant. So we convert the additional date $t+2$ capital into date $t+1$ consumption at the rate of transformation $F_K(.') / G_K(.')$. Why is the MRT between c and k equal to G_K / F_K? Suppose we give up ε units of consumption today (cost: $\varepsilon\, u_c$). This frees up ε / F_K units of K_1 which can now be moved to sector 2: $dK_2 = \varepsilon / F_K$. Output of sector 2 therefore increases by $dK' = dY_2 = G_K / F_K\, \varepsilon$.

The **long version** is more complicated. We give up ε units of c and gain $dK' = dY_2 = G_K / F_K\, \varepsilon$. We now want to increase c' so as to hold K'' unchanged. In a one-sector model we would simply eat the additional K'. Here, we can't eat it directly, but we can eat more if we reduce investment next period. The question is now: how much capital can we take away from sector 2 next period without affecting K''?

First, let's assume we use all the additional K' in sector 1. That produces $F_K(.')\, dK' = F_K(.')\, G_K / F_K\, \varepsilon$ which we can eat. But that would still leave us with additional K in period $t+2$ because we'd retain the undepreciated part of dK'. For each unit of capital that we want less in $t+2$ we can move an additional $1 / G_K(.')$ into sector 1 because $dK'' = G_K(.')\, dK_1'$. This is where the term $(1-\delta) / G_K(.')$ comes from.

In other words, we come into period $t+1$ with an additional dK' units of capital, but we want K'' to be unchanged. First, we move all the dK' into sector 1 and produce consumption goods from that. But we can move an additional $(1-\delta) / G_K(.')\, dK'$ into sector 1, which amounts to reducing investment to "eat" the undepreciated capital that would otherwise be carried over into period $t+2$.

Steady State

In steady state, the Euler equation simplifies to $\beta\{1-\delta+G_K\}=1$. Since the conversion rate between consumption and capital is constant, the rate of return to investment (the interest rate in competitive equilibrium) is $G_K-\delta$ which shows that the Euler equation is the same as in the one-sector model.

2. Competitive Equilibrium

Let P_1 and P_2 denote the "nominal" prices of consumption and investment goods, respectively (by "nominal" I mean in terms of units of account). The relative price of good 2 is p_2. The rental prices for capital and labor are $R\,P_1$ and $w\,P_1$, respectively.

Consumption sector **firms** maximize period profits: $\max\ Y_1 - R\,K_1 - w\,L_1$. The FOCs are as usual: $R = F_K$, $w = F_L$.

Capital sector firms have a more complicated problem because their output price does not equal one. Start by writing out the problem in *nominal* terms: $\max\ P_2\,Y_2 - P_1\,R\,K_2 - P_1\,w\,L_2$. Divide through by P_1 to obtain

$$\max\ p_2\,Y_2 - R\,K_2 - w\,L_2.$$

The FOCs are $\qquad\qquad R / p_2 = G_K,\ w / p_2 = G_L.$

Households: To understand the budget constraint, write it out in nominal terms:

(2) $$P_{2t}\,k_{t+1} = P_{2t}\,(1-\delta)k_t + P_{1t}\,R_t\,k_t + P_{1t}\,(w_t\,v_t - c_t).$$

The household comes into the period with k_t, of which fraction δ depreciates in production. Firms pay a nominal rental price (not rental rate!) $P_1\,R$. Therefore, total capital income is $[P_2\,(1-\delta) + P_1\,R]k$. In addition, the household receives nominal earnings of $P_1\,wv$ and spends $P_1\,c$ on consumption. The total value saved is therefore the right-hand side of (2), which the household invests in k_{t+1}. Now divide through by P_1 to obtain the budget constraint in real terms:

$$p_{2t}\,k_{t+1} = (1-\delta)\,p_{2t}\,k_t + R_t\,k_t + w_t\,v_t - c_t.$$

What then is the rate of return on investments in capital? The value of assets at $t+1$ is $a_{t+1} = p_{2t+1}\,k_{t+1}$, so that the budget constraint becomes

$$\begin{aligned} a_{t+1} = p_{2,t+1}\,k_{t+1} &= (p_{2,t+1} / p_{2,t})\,p_{2,t}\,k_{t+1} \\ &= \pi_{t+1}\,\{(1-\delta)\,a_t + R_t / p_{2,t}\,a_t + w_t\,v_t - c_t\} \end{aligned}$$

where π is the rate of growth of p_2. Intuitively, buying capital goods that appreciate leads to a capital gain. To find the real rate of return consider reducing c_t by 1 and ask: "by how much can c_{t+1} be increased?" Clearly, $d\,a_{t+1} = -\pi_{t+1}\,d\,c_t$. Holding a_{t+2} constant, this allows to increase c_{t+1} by

$$dc_{t+1} = -\pi_{t+1}\,(1-\delta + R_{t+1} / p_{2,t})\,dc_t.$$

The **rate of return** is therefore $1 + r_{t+1} = R_{t+1} / p_{2,t} + (1-\delta)\pi_{t+1}$. The first term is the real rental price received per unit of c spent on capital. In a one-sector model this is simply the MPK because $p_2 = 1$. The second term reflects the capital gain/loss from holding one unit of capital: it may experience price appreciation, but it also depreciates.

The Lagrangean for this problem is:

$$\sum_{t=0}^{\infty} \beta^t u\big((1-\delta)a_t + R_t / p_{2,t}\, a_t + w_t\, v_t - a_{t+1} / \pi_{t+1},\, 1 - v_t\big).$$

FOCs:
$$\beta u_c(t)\{1 - \delta + R_t / p_{2,t}\} = u_c(t-1) / \pi_t\ ,\quad u_l / u_c = w$$

As in the one-sector model, the marginal rate of substitution between consumption and leisure equals the real wage. The Euler equation is also the same as in the one-sector model once we substitute the rate of return for the expression in $\{\}$: $u_c(t) = \beta(1 + r_{t+1})u_c(t+1)$.

2.1 Characterization of competitive equilibrium

Market clearing requires:

$$L_1 + L_2 = v,\quad K_1 + K_2 = K = a / p_2,\quad Y_1 = c,\quad Y_{2t} = K_{t+1} - (1-\delta)K_t.$$

A CE is a sequence of prices (w_t, R_t, p_{2t}) and quantities $(L_{1t}, L_{2t}, K_{1t}, K_{2t}, c_t, v_t, a_t)$ which satisfy (11 equations in 10 unknowns):

- 2 FOCs for each type of firm
- 2 household FOCs and 1 budget constraint
- 4 market clearing conditions

The firms' FOCs imply that $R = p_2 G_K = F_K$ and therefore $p_2 = F_K / G_K$. In words, the relative price equals the marginal rate of transformation. It is now easy to see that the solutions of the planning problem and the CE coincide by substituting prices for derivatives of F and G in the planner's FOCs.

2.2 A One-Sector Reduced Form

Two-sector models are usually much less convenient to analyze than are one-sector models. It is also much harder to guarantee existence and uniqueness of a solution. However, for some purposes it is possible to construct a reduced form that looks almost like a one-sector model. This requires the assumption

$$G(K, L) = A F(K, L)$$

for some constant A. Then static optimality, $F_K / F_L = G_K / G_L$, implies $k_1 = k_2$, where $k = K/L$. Moreover, The relative price of capital is constant, $p_2 = 1/A$. It is then always the case that we can transform one unit of into $1/A$ units of good 2. To see this aggregate the resource constraints for the two sectors into a single one. Define aggregate real output as

$$\begin{aligned}
Y = Y_1 + Y_2 / A &= F(K_1, L_1) + F(K_2, L_2) \\
&= (L_1 + L_2) f(k) \\
&= F(K, L) \\
&= c + (K_{t+1} - [1-\delta]K_t)/A
\end{aligned}$$

The last equality comes from adding market clearing (or feasibility) for both goods. Moreover, we can choose units of capital such that $Y_t = c_t + \widetilde{K}_{t+1} - (1-\delta)\widetilde{K}_t = F(\widetilde{K}_t, L_t)$. Effectively, the model becomes a one-sector model.

Why is this useful? First, it helps study issues such as cross-country (or cross-industry) productivity differentials without having to construct a full-blown multi-sector model with endogenous prices. Secondly, the idea can be generalized slightly. Instead of assuming that A is a constant, it can be assumed that A grows at some rate. This allows to construct models in which relative prices and output shares behave the way they do in the data in a simple way. For example, the relative price of capital and the aggregate share of output due to agriculture are both falling in the data.

A recent application of such models is **investment specific technical change** where technical change takes the form of improved productivity of capital goods. Greenwood, Hercowitz, and Krusell (1997) claim that such technical change accounts for 60 percent of overall productivity growth. To capture this, assume that the technology for producing capital is $A_t G(K_{2,t}, L_{2,t})$ with $A_t = \gamma^t$, $\gamma > 1$. There are two equivalent interpretations. Either there is "disembodied" productivity growth in the production of new capital. Or more recent vintages of capital are more productive. In either case, a given addition to the capital stock takes fewer and fewer resources as time goes by. The previous analysis is unchanged, except that the price of investment now falls over time (at rate γ in steady state): $p_{2,t} = F_K(K_{1,t}, L_{1,t})/\{\gamma^t G_K(K_{2,t}, L_{2,t})\}$. This slight modification of the standard growth model has important implications for growth accounting.

2.3 Summary

Nothing fundamental changes when there are multiple sectors. The main additional complexity is in the household budget constraint because there may be capital gains terms. The dynamics of such models is, by the way, remarkably complex. The final example covers a variation of a two-sector model that is important for studying economic growth.

3. References

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