1 Money and Storage

Consider a two-period OLG model with fiat money and a storage technology. The timing is as follows:

- The old enter period $t$ holding aggregate capital $K_t = N_{t-1}k_t$ and nominal money balances of $M_t = m_t N_{t-1}$. Each old person produces $f(k_t)$. All capital is used up in production (complete depreciation). Assume that $f(.)$ obeys Inada conditions.
- The government pays a lump-sum transfer of $x_t p_t$ units of money to each old person: $M_{t+1} = M_t + N_{t-1}x_t p_t$.
- $N_t = (1+n)^t$ young households are born. Each is endowed with $e$ units of a single good.
- The young buy money $(m_{t+1}/p_t)$ from the old, consume $c_y^t$ and save $k_{t+1}$.
- The old consume their income. The utility function is $u(c_y^t) + u(c_o^{t+1})$.

(a) State the household’s budget constraints when young and old.
(b) Derive the household’s optimality conditions. Define a solution to the household problem.
(c) The government’s only role is to grow the aggregate money supply at the constant rate $\mu$: $M_{t+1} = (1+\mu) M_t$. Define a competitive equilibrium.
(d) Define a steady state as a system of 5 equations in 5 unknowns.
(e) Find the money growth rate ($\mu$) that maximizes steady state consumption per young person, $(N_t c_y^t + N_{t-1} c_o^t)/N_t$.

2 Money in the Utility Function in an OLG Model

Consider a two-period overlapping generations model with money. Each generation is of constant size $N$. Young households supply one unit of labor when young and earn the wage $w_t$. Each consumes $c_y^t$ and saves the remainder either as capital $(s_{t+1})$ or as real money $(m^d_t/p_t)$. Old households consume their savings, including interest income $(c_o^{t+1})$. The utility function is $u(c_y^t) + \beta u(c_o^{t+1}) + v(m^d_t/p_t)$. Assume $v' > 0$.

At time 0 each member of the old generation was endowed with $m_0$ pieces of paper. No new paper is ever issued. Old persons sell their money at price $1/p_t$ to young people, in order to fund her consumption. The young then hold the paper until old at the end of period $t+1$, and sell it to fund $c_o^{t+1}$. This continues throughout time.

(a) Derive a set of 4 equations that characterize optimal household behavior. Show that the household’s first-order conditions imply rate of return dominance, i.e., the real return on money is less than the real return on capital (assuming both capital and money are held in equilibrium).
(b) Wages and interest rates are determined by competitive factor markets. Firms use capital $K_t$ and labor of young workers $L_t$ to produce output. Assume that output is $F(K_t, L_t)$, satisfying Inada conditions. $F$ has constant returns to scale. Solve the firm’s problem.
(c) Define a competitive equilibrium. Capital depreciates at rate $\delta$.

(d) Assume that the utility functions $u$ and $v$ are logarithmic. Solve in closed form for the household’s money demand function, $m_t^d/p_t = \varphi(w_t, r_{t+1}, \pi_{t+1})$, and for its saving function, $s_{t+1} = \phi(w_t, r_{t+1}, \pi_{t+1})/\pi_{t+1}$. $\pi_{t+1} = p_{t+1}/p_t$. 
3 Answers: OLG Models of Money

3.1 Answer: Money and Storage

(a) Young: \( c_t = c_t^p + k_{t+1} + m_{t+1}/p_t \). Old: \( c_{t+1}^o = f(k_{t+1}) + m_{t+1}/p_{t+1} + x_{t+1} \).

(b) The household solves

\[
\max u(e_t - k_{t+1} - m_{t+1}/p_t) + \beta u(f(k_{t+1}) + m_{t+1}/p_{t+1} + x_{t+1})
\]

First-order conditions are

\[
\begin{align*}
    u'(c_t^p) &= \beta u'(c_{t+1}^p) f'(k_{t+1}) \\
    &= \beta u'(c_{t+1}^p) p_t/p_{t+1}
\end{align*}
\]

A solution is a vector \((c_t^p, c_{t+1}^p, k_{t+1}, m_{t+1})\) that solves the 2 first-order conditions and 2 budget constraints. From the household first-order conditions, money and capital must have the same rate of return: \( f'(k_{t+1}) = p_t/p_{t+1} \).

(c) A CE consists of sequences \((c_t^p, c_{t+1}^p, k_{t+1}, m_{t+1}, M_t, p_t, x_t)\) that solve 4 household conditions, 2 government conditions, the definition \( M_t = m_t N_{t-1} \) and goods market clearing: \( N_t k_{t+1} = N_t c_t^p + N_{t-1} c_t^o \). In per capita terms: \( k_{t+1} = e_t + f(k_t)/(1+n) - c_t^p/c_t^o/(1+n) \). Transfer payments equal new money issues: \( N_{t-1} p_t x_t = \mu M_t \).

(d) A steady state consists of constants \((c^p, c^o, k, m/p, \pi, x)\) that satisfy:
- Constant \( m/p \) requires \((1+\mu) = (1+n)(1+\pi) \). This determines \( \pi \).
- \( f'(k) = (1+\pi)^{-1} \) determines \( k \).
- Both consumption levels are determined by the Euler equation and goods market clearing.
- \( m/p \) is determined from the young budget constraint.
- \( x = \mu (m/p)(1+n) \) from the law of motion for \( M \).

(e) Maximizing steady state consumption requires \( f'(k) = 1+n \). Therefore, \( \mu = 0 \). The intuition is that with a constant nominal money supply the per capita money supply shrinks at rate \( n \). For the real money supply to be constant, inflation at rate \(-n\) is needed, which satisfies the Golden Rule.

3.2 Answer: Money in the Utility Function in an OLG Model

(a) The household solves \( \max \ u(c_t^p) + \beta u(c_{t+1}^p) + v(m_t^d/p_t) \) subject to the budget constraints \( w_t = c_t^p + m_t^d/p_t + s_{t+1} \) and \( c_{t+1}^o = s_{t+1} R_{t+1} + m_t^d/p_{t+1} \). This is most easily set up using a lifetime budget constraint:

\[
w_t - c_t^p - m_t^d/p_t = s_{t+1} = \frac{c_{t+1}^o - m_t^d/p_{t+1}}{R_{t+1}} \tag{1}
\]

The first-order conditions are:

\[
\begin{align*}
    u'(c_t^p) &= \beta R_{t+1} u'(c_{t+1}^p) \\
    v'(m_t^d/p_t) &= \beta u'(c_{t+1}^p) [R_{t+1} - p_t/p_{t+1}]
\end{align*}
\]

Interpretation: If the household saves one unit as money rather than capital, he gains \( u' \), but loses some old consumption because the rate of return of money is lower than that of capital. Note that this implies rate of return dominance because positive marginal utility requires \( R_{t+1} > p_t/p_{t+1} \).
A solution to the household problem is a vector \((c_t', c_{t+1}', m_t', s_{t+1})\) which solves the 2 first-order conditions and the 2 budget constraints.

(b) The firm’s problem is standard with first order conditions \(r_t = F_K(K_t, L_t)\) and \(w_t = F_L(K_t, L_t)\).

(c) A competitive equilibrium is an allocation \((c_t', c_{t+1}', m_t', s_{t+1}, K_t, L_t)\) and a price system \((R_t, r_t, w_t, p_t)\) that satisfies 4 household FOCs, 2 firm FOCs, the identity \(R_t = 1 + r_t - \delta\), and market clearing. The capital market clears if \(N_{s_{t+1}} = K_{t+1}\). The money market clears if \(m_t' = m_0\). Goods market clearing requires that \(F(K_t, L_t) + (1 - \delta)K_t = Nc_t' + Nc_{t+1}' + K_{t+1}\). We have 10 variables and 11 equations, one of which is redundant by Walras’ law.

(d) With log utility the household first-order conditions become \(c_{t+1}' = R_{t+1} c_t' = \beta m_t/p_t (R_t + 1 - p_t/p_{t+1})\). Define the inflation rate \(\pi_{t+1} = p_{t+1}/p_t\). Then the budget constraint 1 together with the first-order condition for \(c_t'\) imply

\[
\frac{c_{t+1}'}{R_{t+1} (1 + 1/\beta)} = w_t - \frac{m_t}{p_t} (1 - 1/\pi_{t+1} R_{t+1})
\]

Substituting out \(c_{t+1}'\) and simplifying yields the money demand function

\[
w_t = \frac{m_t}{p_t} (1 + 2\beta) (1 - 1/\pi_{t+1} R_{t+1})
\]

This has sensible properties. A higher nominal interest rate reduces money demand. If the household is more patient or richer, more money is held. To solve for the saving function:

\[
s_{t+1} = \frac{c_{t+1}' - m_t/(p_t \pi_{t+1})}{R_{t+1}}
\]

\[
= \beta m_t/p_t \left( R_{t+1} - 1/\pi_{t+1} \right) - m_t/(p_t \pi_{t+1})
\]

\[
= \frac{m_t}{p_t} \left( \beta (1 - 1/\pi_{t+1} R_{t+1}) - 1/\pi_{t+1} R_{t+1} \right)
\]

Substitute out \(m_t/p_t\) using the money demand function.

\[
s_{t+1} = \frac{\beta (\pi_{t+1} R_{t+1} - 1)}{(2 + \beta) (\pi_{t+1} R_{t+1} - 1)}
\]

From the saving function one could derive a single equation that characterizes the steady state capital stock. In steady state, \(\pi = 1\); otherwise real money balances would not be constant. Apply this to the saving function and equate \(s = K\). Then note that \(w\) and \(R\) are functions of \(K\).