Dual Bargaining and the Talmud Bankruptcy Problem

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Abstract: In dual bargaining players minimize their losses and will incur the highest loss at the disagreement point if no agreement is reached. This paper extends the solutions and axioms of utility bargaining to dual bargaining, and it solves four classes of linear dual bargaining problems. The dual approach solves a bankruptcy problem directly rather than converting it to a utility bargaining problem, and it provides a new understanding of the problem. For example, the ancient Talmud solutions can be understood as the Kalai-Smorodinsky solution of the related dual bargaining problems. Another important finding is that the set of dual bargaining problems is larger than the set of utility bargaining problems.

Keywords: Bargaining, core selection, dual bargaining, cost sharing, bankruptcy problem

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1. Introduction

This paper studies axiomatic bargaining in loss-loss situations like carpooling (or equivalently, municipal cost sharing [Young, 1985]) and bankruptcy problems. In a carpooling problem, it costs a commuter $D_j$ to drive alone, and costs only $L (\leq \sum D_j)$ in total if all commuters carpool in one vehicle. In a bankruptcy problem, a bankrupted firm owes $D_j$ to creditor $j$ and has assets of $E (\leq \sum D_j)$; a creditor $j$ loses $D_j$ if no agreement is reached, and cooperation reduces the total loss to $L = (\sum D_j - E) < \sum D_j$. Splitting $E$ in a bankruptcy problem is equivalent to sharing the total loss $(\sum D_j - E)$, hence the bankruptcy and carpooling problems are equivalent.

In general, a dual bargaining problem is defined by a pair, a feasible set $Y$ and a disagreement point $D$ (d-point, henceforth) both in $\mathbb{R}^n_+$, where $n$ is the number of players, the feasible set is bounded from above by the d-point, and components of a payoff vector represent players' losses or costs. Players' objectives are to minimize their losses, and they incur the highest loss at the d-point if no agreement is reached. This paper first extends egalitarian, Kalai-Smorodinsky (KS), Nash and proportional solutions and their axioms to dual bargaining. It then solves four classes of linear dual bargaining problems.

It is worth noting three features of dual bargaining. First, existing literature solves a bankruptcy problem by converting it to a utility bargaining problem (or a coalitional utility game) and then using the utility bargaining solutions [see O'Neil, 1982; Aumann and Maschler, 1985]; while dual bargaining solves it directly and brings new insights that are lost in the conversion to utility bargaining. For example, the ancient Talmud bankruptcy solutions can be understood as the KS solutions of the related dual bargaining problems.
Second, a dual bargaining problem can not be solved by replacing its feasible set and d-point by “-Y” and “-D” and then solving it as a utility bargaining problem (this would violate the non-negative requirement). Third, the relation between utility and dual bargaining problems is similar to the relation between a primary programming and its dual problems: they are dual to each other in economic interpretation. In utility bargaining (similar to vector-max), players try to maximize their returns and no player will accept a return less than his d-point; while in dual bargaining (similar to vector-min): players try to minimize their costs and no one will pay a cost greater than his d-point.

However, because the set of dual bargaining problems is larger and the existing literature focuses heavily on utility bargaining, it would be useful to devote more future works on dual bargaining, to check if the known conclusions of utility bargaining hold in dual bargaining, and to examine the duality between utility and dual bargaining problems.

The rest of the paper is organized as follows: Section 2 studies solution concepts and axioms, Section 3 revisits the Talmud bankruptcy problem, Section 4 studies four classes of linear dual bargaining problems, and Section 5 concludes. The Appendix provides all proofs.

2. Description of the problem

Let \( N = \{1, 2, ..., n\} \) be a finite set of players. A dual bargaining problem is a pair \((D; Y)\) of a d-point and a feasible set satisfying: (i) \( D \in \mathbb{R}^n_+, Y \subset \mathbb{R}^n_+ \), \( Y \neq \emptyset \), and \( Y \) is compact; (ii) \( y \leq D \) for all \( y \in Y \); and (iii) there is \( y \in Y \) such that \( y \ll D \). This can alternatively be understood as a Vector-Min problem\(^3\) with an opportunity cost \( D: \text{Min} \{ F(y) = y \mid y \in Y \} \), with the objective function taking the high value \( F(D) = D \) if no solution is chosen.
Let $U$ be the set of all dual bargaining problems $(D; Y)$. A dual bargaining solution is a function $\theta: U \rightarrow \mathbb{R}_n^+$ such that $\theta(D; Y) \in Y$. Let $e = (1, \ldots, 1)$ be a vector of ones in $\mathbb{R}_n^+$, $Y^*_E = \{ y \in Y \mid \{ z \in Y \mid z > y \} = \emptyset \}$ denote the efficient frontier of $Y$, and $\text{Arg-Max} \{ f(x) \mid x \in X \} = \{ z \in X \mid f(z) \geq f(x) \ \forall \ x \in X \}$ denote the set of optimal solutions for $\text{Max} \{ f(x) \mid x \in X \}$.

The existing literature focuses on the equal surplus (ES), KS, Nash (NA), and proportional (P) solutions for utility bargaining $(d, X)$ (i.e., primary bargaining), which can be viewed as a Vector-Max problem with an opportunity cost $d$, $\text{Max} \{ F(x) = x \mid x \in X \}$, and the agents receive the low value $F(d) = d$ if they fail to reach an agreement. Definition 1 below extends these solutions to a dual bargaining problem $(D; Y)$.

**Definition 1**: Given $(D; Y)$. (i) Its ES solution is $\text{ES} = \text{ES}(D; Y) = D - \lambda_{\text{ES}}e$, where $\lambda_{\text{ES}} = \text{Max} \{ \lambda \mid D - \lambda e \in Y \}$; (ii) its KS solution is $\text{KS} = D + \lambda_{\text{KS}} t_{\text{KS}}$, where $t_{\text{KS}}$ is the optimal direction, and $\lambda_{\text{KS}} = \text{Max} \{ \lambda \mid D + \lambda t_{\text{KS}} \in Y \}$; (iii) its Nash solution is $\text{NA} = \text{Arg-Max} \{ \Pi(D - y^j) \mid y \in Y \}$; and (iv) its P solution is $P = (1 - \lambda_P)D$, where $\lambda_P = \text{Max} \{ \lambda \mid (1 - \lambda)D \in Y \}$.

The "optimal direction" $t_{\text{KS}}$ is defined by

$$t_{\text{KS}}^i = t_{\text{KS}}^i(Y) = \text{Min} \{ (y^i - D^j) \mid y \in Y \}, \ i = 1, \ldots, n.$$  

In other words, the $i^{th}$ component of $t_{\text{KS}}$ is the difference between $i$'s lowest feasible cost and his d-value $D^j$. The KS solution is obtained geometrically: One starts at the d-point, walks towards the efficient frontier $Y^*_E$ along the optimal direction $t_{\text{KS}}$, and stops at a solution on the frontier. The above NA solution is defined by a max-problem because minimizing cost $(y^i - D^j)$ is equivalent to maximizing the savings $(D^j - y^i)$. 

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A dual bargaining problem \((D; Y)\) is called a convex dual bargaining problem if \(Y\) is a convex set. As shown in Proposition 1, KS and NA solutions always exist in convex dual bargaining and are both efficient. However, efficient ES and P solutions might not exist in convex dual bargaining (see Figure 1). The uniform solution \(U = \lambda_U e \ (\lambda_U > 0)\) is omitted because it generally does not exist.

(Figure 1 about here)

**Definition 2** [Nash, 1950]: Given \((D; Y)\), efficiency, symmetry, scale invariance, and independence of irrelevant alternatives (IIA) of a solution \(\theta(D; Y)\) are defined below.

(i) **Efficiency**: \(\theta \in Y_E^*\); (ii) **Symmetry**: If \((D; Y)\) is invariant under all permutations of agents, then \(\theta^i(D; Y) = \theta^j(D; Y)\) for all \(i\) and \(j\); (iii) **Scale invariance**: If \(T(y) = \{a^1y^1, \ldots, a^ny^n\}\) is a scale transformation for some \(a \gg 0\), \(T(Y) = \{T(y) \mid y \in Y\}\), then \(\theta(T(D); T(Y)) = T(\theta(D; Y))\);

(iv) **IIA**: If \(Y' \subseteq Y\) and \(\theta(D; Y) \in Y'\), then \(\theta(D; Y) = \theta(D; Y')\).

We now extend monotonicity of the KS solution to an n-person dual bargaining problem \((D; Y)\). A set \(Y' (\supseteq Y)\) is an enlargement of \(Y\) in agent \(i\)’s favor if

\[
(2) \quad t_{KS}^i(Y') = \min \{(y^i-D^i) \mid y \in Y'\} < t_{KS}^i(Y), \text{ and } t_{KS}^j(Y') = t_{KS}^j(Y) \text{ for all } j \neq i.
\]

**Definition 3** [Kalai and Smorodinsky, 1975]: A solution \(\theta(D; Y)\) satisfies monotonicity if for each \(i\), \(\theta^i(D; Y') \leq \theta^i(D; Y)\) for any enlargement \(Y'\) of \(Y\) in \(i\)’s favor.

**Proposition 1**: Consider a convex dual bargaining problem \((D; Y)\). (i) The KS and NA solutions always exist and are both efficient; (ii) the NA solution is the only solution satisfying efficiency, symmetry, scale invariance, and IIA; and (iii) the KS solution is the only solution satisfying efficiency, symmetry, scale invariance, and monotonicity.
These axiomatic properties are parallel to those in utility bargaining. The next two sections solve the bankruptcy and cost-sharing problems directly rather than converting them into utility bargaining. As shown in the next section, the dual approach provides a new understanding of the bankruptcy problem that had not been reported in the literature.

3. The Talmud bankruptcy problem revisited

The Babylonian Talmud is a collection of Jewish religious and legal decisions copied from the oral heritage of the first five centuries A.D. It recorded an intriguing three person bankruptcy problem [see page 370 in O'Neill, 1982; or Epstein, 1936]: The asset of a deceased husband is $E$ units ($0 < E < 600$), and his will leaves 100, 200 and 300 units respectively to his first, second and third wives. The question is how to split the $E$ units among the three parties.

(Table 1 about here)

For $E = 100, 200, 300$, the Mishna\textsuperscript{6} allocations of assets in the Talmud are underlined in the right column of Table 1. These ancient numbers puzzled scholars over 2000 years. In a fascinating recent study [Aumann and Maschler, 1985], these Mishna solutions are shown to be the nucleolus [Schmeidler, 1969] of the corresponding coalitional utility games. The right column of Table 1 also provides the KS, NA and P solutions of the corresponding bargaining problems. One interesting feature of these dual solutions is summarized in the next proposition.

**Proposition 2:** For $E = 100$ and 300, the Mishna solutions in the Talmud bankruptcy problem are the KS solutions of the corresponding dual bargaining problems.
As shown in the table, the KS solution coincides with the Mishna solution for \( E = 100 \) and \( 300 \), and it coincides with the NA solution for \( E = 100 \), with the P solution for \( E = 300 \). It is straightforward to see that the NA solution is identical to the egalitarian (i.e., equal savings) solution in all three cases.

4. Properties of linear dual bargaining

A dual bargaining problem is a linear dual bargaining problem if its feasible set is defined by linear inequalities.

**Definition 4:** A linear dual bargaining problem is defined by \( (D; G, h) \in \mathbb{R}^{n+mn+m} \), which is a dual bargaining problem \( (D; Y) \) whose feasible set is

\[
Y = \{y \in \mathbb{R}^n | Gy \geq h\},
\]

where \( D \) is the earlier d-point, \( G = G_{m \times n} \) is an \( m \times n \) matrix, and \( h = h_{m \times 1} \in \mathbb{R}^m \).

**Remark 1:** Computation in a general linear bargaining problem is non-trivial. Because bargaining theory has grown without considering computation, effective methods in finding the KS and other solutions (the NA solution can be computed using the known optimization methods) in linear bargaining could help push the theory towards applications to the same extent that the computation of fixed point has helped the applications of computational general equilibrium models.

Another advantage of linear dual bargaining is that monotonicity can be defined more intuitively. Given \( (D; G, h) \in \mathbb{R}^{n+mn+m} \), let \( G = (G^1, \ldots, G^n) \) be the \( n \) columns of \( G \). An enlargement of \( Y \) in agent \( i \)'s favor can be represented by a change from \( G \) to \( \bar{G} = (G^1, \ldots, G^{i-1}, G^i+t, G^{i+1}, \ldots, G^n) \) for some \( t \in \mathbb{R}^m \), leading to \( Y \subseteq Y' = \{y \in \mathbb{R}^n | \bar{G}y \geq h\} \). The
effects of such changes are similar to sensitivity analysis in linear programming, and their characterizations remain a challenging topic for future research.

Linear dual bargaining is clearly convex bargaining. The rest of this section studies four classes of linear dual bargaining problems: (a) the base polyhedron selection problem; (b) the rectangular dual bargaining problem; (c) the cost sharing problem; and (d) the bankruptcy problem.

4.1 The base polyhedron selection problem:

Given a set function $v: \mathcal{N} \rightarrow \mathbb{R}_+$ with $v(\emptyset) = 0$, where $\mathcal{N} = 2^\mathcal{N}$ is the set of subsets of $\mathcal{N}$, its base polyhedron [see Fujishige, 1991] is dual to the core and is defined by

$$(4) \quad B(v) = \{y \in \mathbb{R}_n^\mathcal{N} \mid \sum_{i \in T} y_i \leq v(T) \text{ for all } T \in \mathcal{N}, \text{ and } \sum_{i \in \mathcal{N}} y_i = v(\mathcal{N})\}.$$  

**Remark 2:** A base polyhedron is often defined for submodular functions satisfying

$$(5) \quad v(S) + v(T) \geq v(S \cup T) + v(S \cap T) \text{ for every } S, T \in \mathcal{N}.$$  

Such submodular functions arise from situations, like carpooling, where costs exhibit scale economies: For any partition $\{S_1, ..., S_k\}$ of $S$, it costs $v(S_i)$ for travelers in $S_i$ to take a taxicab alone, while it costs only $v(S) \leq \sum_{i=1}^k v(S_i)$ if all travelers in $S$ share one taxicab. This situation is clearly dual to a convex coalitional game or a supermodular function (i.e., (5) is reversed), which arises from situations (like networks) where returns exhibit scale economies. It is straightforward to see that the solution of dual bargaining can be used to select an allocation from the base polyhedron.
**Remark 3:** Given a set function $v: \mathcal{N} \to \mathbb{R}_+$, assume $B(v) \neq \emptyset$, and let a dual bargaining problem $(D; Y)$ be defined by $D^i = v(i)$ for all $i$, and $Y = B(v)$. Then any non-empty solution $\theta(D; Y)$ is an answer to the base polyhedron selection problem.

An allocation $y$ in the base polyhedron splits the total costs $v(N)$ in such a way that no coalition $T$ pays more than its own cost $v(T)$, while a core allocation in a coalitional game $\Gamma = \{N, u(S)\}$ splits the total return $u(N)$ such that no coalition $S$ receives less than its own payoff $u(S)$ (i.e., $\Sigma_{i \in S} x^i \geq u(S)$ for all $S$ and $\Sigma x^i = u(N)$). Hence, the base polyhedron and the core are dual to each other, and their selection problems are solved by the solutions of the corresponding dual or utility bargaining problems.

**4.2 The rectangular dual bargaining problem:**

The feasible set of a rectangular dual bargaining problem is defined as

$$Y = \{ y \in \mathbb{R}_n^+ \mid g \leq y \leq D \}.$$  

Such a rectangular bargaining problem is equivalent to a vector $(D, g) \in \mathbb{R}_n^{2n}$ such that $g << D$, and its solutions are provided in the following Proposition 3.

**Proposition 3:** The solutions for a rectangular dual bargaining problem $(D, g)$ are: (i) $\text{KS} = \text{NA} = g$; (ii) $\text{ES}$ solution is efficient $\iff D - g = \lambda e$ for some $\lambda > 0 \iff \text{ES} = \text{KS} = \text{NA} = g$; (iii) the $P$ solution is efficient $\iff g = \lambda D$ for some $\lambda > 0 \iff P = \text{KS} = \text{NA} = g$; and (iv) the $U$ solution exists and is efficient $\iff g = \lambda e$ for some $\lambda > 0 \iff U = \text{KS} = \text{NA} = g$.

**4.3 The cost sharing and the bankruptcy problems:**
A cost sharing problem is defined by an (n+1) vector such that the sum of d-point values is greater than the total cost, which has been well studied in the literature [see Young (1985) for surveys]. As will be seen in Remark 4, such (n+1) vectors imply three classes of economic problems, which are solved directly using the dual approach.

**Definition 5**: A *cost-sharing* (CS) problem is a vector \((D; L) \in \mathbb{R}^{n+1}_+\) such that \(\Sigma D^j > L\), or a linear dual bargaining problem* \(^8\) \((D; Y)\) whose feasible set is

\[
Y = \{ y \in \mathbb{R}^n_+ \mid y \leq D \text{ and } \Sigma y^j \geq L \} = \{ y \in \mathbb{R}^n_+ \mid Gy \geq h \},
\]

where \(L\) is called the total costs, and \(\omega = (\Sigma D^j - L) > 0\) is the cost saving.

Such CS problems represent situations like carpooling or municipal cost sharing where costs exhibit scale or scope economies: It costs a commuter \(\$ D^j\) to drive alone, while it costs only \(\$ L < \Sigma D^j\) if they carpool together.

The above (n+1) vector can also be used to define a bankruptcy problem:

**Definition 6**: A *bankruptcy problem* is an (n+1) vector \((D; E) \in \mathbb{R}^{n+1}_+\) satisfying \(\Sigma D^j > E\):

How to split the asset of \$E in a bankrupted firm that owes \(\$ D^j\) to creditor \(j\).

If no agreement is reached, creditor \(j\) loses the entire bad loan \(\$ D^j\), while cooperation by all creditors could reduce the total loss from \(\Sigma D^j\) to \((\Sigma D^j - E)\) [see O'Neil, 1982; Aumann and Maschler, 1985]. Since splitting the asset \$E is equivalent to sharing the loss \((\Sigma D^j - E)\), a bankruptcy problem \((D; E) \in \mathbb{R}^{n+1}_+\) is equivalent to a carpooling problem \((D; \Sigma D^j - E) \in \mathbb{R}^{n+1}_+\) or a dual bargaining problem \((D; Z)\) with

\[
Z = \{ y \in \mathbb{R}^n_+ \mid y \leq D \text{ and } \Sigma y^j \geq \Sigma D^j - E \},
\]
Remark 4: By above discussions, a vector \((D; L)\)\(\in\mathbb{R}^{n+1}\) (with \(\Sigma D^j > L\)) defines three classes of economic problems: (i) Carpooling: It costs \(D^j\) for \(j\) to travel alone and only \(L\) if all \(n\) travelers share one taxicab; (ii) bankruptcy problem: \(L\) is the asset left in a bankrupted firm who owes \(D^j\) to creditor \(j\); and (iii) public good provision: \(L\) is the cost of a public good that provides a benefit \(D^j\) to \(j\).

Since carpooling and bankruptcy problems are equivalent, one only needs to solve the carpooling problem. Propositions 4 and 5 below provide the solutions.

**Proposition 4:** Given a CS problem \((D; L)\), let \(\omega = (\Sigma D^j - L) > 0\) be the cost savings, and assume \(D^1 \leq D^2 \leq \ldots \leq D^n\). Then, its KS solution has the following three forms.

(i) If \(\omega \leq D^1\), then \(KS = NA = ES = D - (\omega/n) e\);

(ii) If \(\omega \geq D^n\), then \(KS = P = (L/\Sigma D^j) D\);

(iii) Let \(k\) be the agent such that \(D^k < \omega \leq D^{k+1}\). Then, \(KS = D + \lambda_{KS} t_{KS}\), where

\begin{equation}
\lambda_{KS} = \frac{\omega}{\sum_{j=1}^{k} D^j + \omega (n-k)}, \quad t_{KS}^i = -D^i \text{ for } i \leq k, = -\omega \text{ for } i \geq k+1.
\end{equation}

**Proposition 5:** Assume \(D^1 \leq \ldots \leq D^n\) in a CS problem \((D; L)\), let \(\omega = (\Sigma D^j - L) > 0\) be the cost savings. Its NA solution is obtained in the following \(n\) steps:\(^9\):

**Step 1.** If \(\omega/n \leq D^1\), let \(NA^i = D^i - \omega/n\) for all \(i\). Otherwise, let \(NA^1 = 0\), and go to the next step;

**Step 2.** Now the available savings are \(\delta(2) = (\Sigma_{j=2}^{n} D^j - L)\) because agent 1 already saved \((D^1 - NA^1) = D^1\). If \(\delta(2)/(n-1) \leq D^2\), then \(NA^i = D^i - \delta(2)/(n-1)\) for all \(i \geq 2\). Otherwise, let \(NA^2 = 0\), and go to next step;
Step k. Let $\delta(\kappa) = (\sum_{j=k}^{n} D_j - L)$ denote available savings ($k = 3, \ldots, n$). The procedure ends with $NA^k = D^k - \delta(\kappa)$ when $k = n$, and continues when $k < n$: If $\delta(k)/(n-k+1) \leq D^k$, then $NA^i = D^i - \delta(\kappa)/(n-k+1)$ for all $i \geq k$. Otherwise, let $NA^k = 0$, and continue the same computation for agents $i = k+1, \ldots, n$.

Note that there is an economic duality\textsuperscript{10} between a CS problem and a surplus-sharing (SS) problem, which is an $(n+1)$ vector $(d; r) \in R_{n+1}^{+}$ ($r > \sum d^i)$. SS problems arise from situations like networks in which returns exhibit scale economies: Provider $i$ alone generates a revenue $S d^i$, while the grand coalition generates $S r > \sum d^i$. In an SS problem, players try to maximize returns and no player $i$ will accept a return less than $d^i$; while in a CS problem, players try to minimize costs and no player $j$ will pay a cost greater than $D^j$.

(Figure 2 about here)

Remark 5: In spite of the economic duality between the CS and SS problems, the set of CS problems is larger than the set of SS problems in two aspects: (i) SS represents one economic problem (i.e., networks), while CS represents three problems (i.e., public good, carpool and bankruptcy); (ii) In SS problems, ES, KS, and NA solutions are always identical; while in CS problems, the three are identical only if cost savings are sufficiently small (see Figure 2-b), so SS problems correspond to a subset of CS problems with small savings. Such asymmetry is caused by the non-negative restriction. Because existing literature focuses heavily on SS related problems, it would be useful in future works to check if the known conclusions of utility bargaining hold in dual bargaining and to examine the precise duality between utility and dual bargaining problems\textsuperscript{11}.
5. Concluding Remarks

This paper extends Nash and KS solutions and their axioms of utility bargaining to dual bargaining. It is a non-trivial future task to compute the KS and other solutions (except NA solution) in linear (or non-linear) dual (or utility) bargaining. The paper solves CS problems like carpooling problems directly using the dual approach rather than converting them to utility bargaining. The dual approach brings new insights into the bankruptcy problem: the ancient Talmud solutions can be understood as the KS solution of the related dual bargaining problems.

Since dual bargaining represents a larger set of economic problems than that represented by utility bargaining, it obviously would be useful in future works to check whether the known properties of utility bargaining can be extended to dual bargaining. Three such future topics are discussed below. (a) A linear dual bargaining problem \((D; Y)\) leads to a Linear Multiobjective Programming (LMP) problem with opportunity cost \(D\):

\[
\text{LMP: } \text{Min}\{ F(y) = y \mid y \in R^*_N, Gy \geq h \},
\]

and the objective function will take the high value \(F(D) = D\) if no agreement is reached. It is non-trivial to see if the dual programming problem [see Zeleny (1984) for survey] to the above LMP problem (i.e., a linear vector max problem) can be interpreted as a bargaining problem. (b) Since a bankruptcy problem \((D; E)\) represents a loss-loss situation, a coalition's worst loss

\[
v(S) = [\sum_{j \in S} D_j - (E - \sum_{j \in S} D_j)^+]\]

defines a loss set function \(v: \mathcal{N} \rightarrow \mathbb{R}_+\) (where \(r^+ = r\) if \(r > 0\), and \(= 0\) if \(r \leq 0\)). It would be useful to see if the dual nucleolus of the above function \(v\) is equivalent to the nucleolus of
the related TU game $\Gamma = \{N, u\}$ with $u(S) = (E - \sum_{j \in S} D^j)^+$ [see Aumann and Maschler, 1985].

(c) Let $B(v)$ denote the base polyhedron of the above loss function. It is interesting to see if
the KS (or NA) solution of the linear dual bargaining defined by $B(v)$ (see Section 4)
coincides with the original KS (or NA) solution of the bankruptcy problem $(D; E)$. 
Appendix

Proof of Proposition 1: Part (i). The existence follows from convexity and compactness of the feasible set $Y$. Parts (ii-iii). It is straightforward to verify that the KS solution satisfies efficiency, symmetry, scale invariance, and monotonicity, and that the NA solution satisfies efficiency, symmetry, scale invariance, and contraction independence. The proof for uniqueness is similar to those in Nash (1950) and Kalai-Smorodinsky (1975).  

Q.E.D

Proof of Proposition 3: Given $Y = \{y \in \mathbb{R}^n_+ \mid g \leq y \leq D \}$, one can check that the optimal direction is given by $t_{KS} = (g-D)$, leading to $KS = g$. The rest can be similarly proved by using the special features in rectangular bargaining.  

Q.E.D

Proof of Proposition 4: Part (i). Let us first find $t^i_{KS}(Y) = \min \{(y^i-D^i) \mid y \in Y\}$, where $Y = \{y \in \mathbb{R}^n_+ \mid y\leq D \text{ and } \sum \hat{y}_j \geq L\}$. For any $y \in Y$, by $y \leq D$ and $\sum \hat{y}_j \geq L$, one has

\[(A1) \quad y^i \geq L - \sum_{j \neq i} \hat{y}_j \geq L - \sum \hat{D}_j \]

which leads to

\[(A2) \quad \min \{y^i \mid y \in Y\} = \max \{0, L - \sum \hat{D}_j\}.

By $\omega = (\sum \hat{D}_j - L) \leq D^i$, $L - \sum \hat{D}_j \geq 0$. By (A2), $\min \{y^i \mid y \in Y\} = L - \sum \hat{D}_j$, and one has

\[(A3) \quad t^i_{KS}(Y) = \min \{(y^i-D^i) \mid y \in Y\} = \min \{y^i \mid y \in Y\} - D^i = D - \sum \hat{D}_j = -\omega.

Hence, $t_{KS} = (-\omega)e$, which leads to $KS = ES = [D - (\omega/n)e]$.  

Because $[D - (\omega/n)e]$ is feasible, and because $\Pi(D^i-y^i)$ achieves its maximum at the equal surplus choice, we have $KS = ES = [D - (\omega/n)e] = \arg \max \{\Pi(D^i-y^i) \mid y \in Y\}$. This proves part (i). Now consider part (ii).
By $\omega = (\Sigma D_j - L) \geq D_i^1$, $-L - \Sigma_{j \neq i} D_j^1 \leq 0$. By (A2), $\text{Min} \{ y^i \mid y \in Y \} = 0$. Hence,

(A4) \quad t_{KS}^i(Y) = \text{Min} \{ (y^i - D_i^1) \mid y \in Y \} = \text{Min} \{ y^i \mid y \in Y \} - D_i^1 = - D_i^1, \text{ for all } i.

This leads to $t_{KS} = -D$, so $KS = P = (L/\Sigma D_j^1)$ D.

Part (iii). Suppose $D^k < \omega \leq D^{k+1}$ for some $k$. By $D^1 \leq \ldots \leq D^n$, we have $\omega \geq D_i^1$ for $i \leq k$, and $\omega \leq D_i^1$ for $i \geq k+1$. Hence, (A3) and (A4) lead to

(A5) \quad t_{KS}^i(Y) = \begin{cases} -D_i^1 & \text{if } i \leq k \\ -\omega & \text{if } i \geq k+1. \end{cases}

Solving for $\lambda$ in $\Sigma D_j^1 + \lambda \Sigma_{j=1}^k D_j = L$, one has

(A6) \quad \lambda_{KS} = \omega / \left[ \sum_{j=1}^k D_j^1 + \omega (n-k) \right]. \quad \text{Q.E.D}

**Proof of Proposition 5:** Note first that $\Pi(D_j^1 - y^j)$ achieves its maximum, among all $y$ satisfying $\Sigma y^j = L$, at equal surplus choice $(D^1_j - y_j^1) = \omega/n$ for all $i$, where $\omega = (\Sigma D^1 - L)$. It follows from $D^1 \leq \ldots \leq D^n$ and $\omega/n \leq D^1$ that

(A7) \quad NA^i = D^i - \omega/n \geq 0 \text{ for all } i.

Hence, $y = [D-(\omega/n)e] \in Y = \{ y \in K^0_+ \mid y \leq D \text{ and } \Sigma y^j \geq L \}$, so it is the solution of Max \{ $\Pi(D^1_j - y^j) \mid y \in Y \}$. This proves the first half of Step 1.

Now suppose $\omega/n > D^1$. Since the maximum value of $\Pi(D^1_j - y^j)$ is achieved at the equal surplus choice, the value of $\Pi(D^1_j - y^j)$ can be increased, at any $y$ with $(D^1_j - y^1_j) < \omega/n$, by reducing $y^1$ or by increasing $(D^1_j - y^j)$ up to $\omega/n$. The limit of this process is at $NA^1 = 0$, at which agent 1 saves an amount equal to his d-point value $D^1 = D^1 - NA^1$. Therefore, if $\omega/n > D^1$, the original maximization problem
(A8) \[
\text{Max } \{ \Pi(D^j - y^j) \mid y \in Y \}
\]
is equivalent to the following new maximization problem with one less variable:

\[
\begin{align*}
\text{(A9)} & \quad \text{Max} \quad \Pi_{j=2}^{n}(D^j - y^j)D^1 \\
\text{Subject to } & \quad y^j \leq D^j \text{ for all } j \geq 2; \text{ and} \\
& \quad \sum_{j=2}^{n} y^j = L - D^1.
\end{align*}
\]

Repeating the above arguments for (A9), one gets the rest of the procedure that solves the original Nash maximization problem. This competes the proof. \( \text{Q.E.D} \)

**Proof of Proposition 2:** Case 1. It follows from \( E = \omega = 100 \leq D^1 = 100 \), part (i) of Proposition 4, and Proposition 5 that the allocation of losses is

\[
\text{KS} = \text{NA} = \text{ES} = D - (\omega/n) e = (66\frac{2}{3}, 166\frac{2}{3}, 266\frac{2}{3}),
\]

so the shares of assets are \( x_{\text{KS}} = x_{\text{NA}} = x_{\text{ES}} = D - \text{KS} = (33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3}) \).

Case 2. By \( D^1 = 100 < E = 200 \leq D^2 = 200 \) and part (iii) of Proposition 4, one has

\[
t_{\text{KS}}^1 = -D^1 = -100, \quad t_{\text{KS}}^2 = t_{\text{KS}}^3 = -E = -200.
\]

Solving \( \lambda \) in \( \sum_{j=1}^{3}[D^j + \lambda t_{\text{KS}}^j] = L = 400 \), one has \( \lambda = 2/5 \), so

\[
\text{KS} = D + (2/5) t_{\text{KS}}^j = (60, 120, 220), \text{ and } x_{\text{KS}} = D - \text{KS} = (40, 80, 80).
\]

By applying Proposition 5, one can show

\[
\text{NA} = \text{ES} = (33\frac{1}{3}, 133\frac{1}{3}, 233\frac{1}{3}), \text{ and } x_{\text{NA}} = x_{\text{ES}} = (66\frac{2}{3}, 66\frac{2}{3}, 66\frac{2}{3}).
\]

Case 3. It follows from \( D^3 = 300 \geq E = 300 \) and part (ii) of Proposition 4 that

\[
\text{KS} = P = x_{\text{KS}} = x_p = (50, 100, 150).
\]

\( \text{Q.E.D} \)

**Proof of Footnote 11:** Consider the utility bargaining \((0; X)\) with
(A10) \quad X = \{ x \in \mathbb{R}^n_+ \mid \sum \lambda_j x_j \leq E, x_j \leq d_j \text{ for all } j \}.

Let \( x_{KS} = x_{KS}(0; X) \) be the KS allocation of the asset \( E \), and \( KS = KS(\Sigma d^j - E; d) \) be the KS allocation of the total loss. Then one has

(A11) \quad x_{KS}(0; X) = d - KS(\Sigma d^j - E; d).

The equivalence relation (A11) might have appeared in the literature, but I failed to track down any specific source. Therefore, a brief proof is provided here. Suppose \( d^1 \leq d^2 \leq \ldots \leq d^n \) and \( E \leq d^1 \). By Proposition 4, \( KS(\Sigma d^j - E; d) = d - (E/n) e \). Since the \( d \)-point is zero, one has

(A12) \quad t^i_{KS} = \Max{x^i \mid x \in X} = E

for all \( i \). By (A12) and the definition of KS solution, one has

\[
x_{KS}(0; X) = (E/n)e = d - KS(\Sigma d^j - E; d).
\]

Now suppose \( d^n \leq E \). By Proposition 4, \( KS(\Sigma d^j - E; d) = P = (\Sigma d^j - E)/\Sigma d^j \) \( d \). Since \( d^i \leq E \) for all \( i \), and the \( d \)-point is zero, one sees that \( t^i_{KS} = \Max{x^i \mid x \in X} = d^i \) for all \( i \). Hence

\[
x_{KS}(0; X) = (E/\Sigma d^j) d = d - (\Sigma d^j - E)/\Sigma d^j) d = d - KS(\Sigma d^j - E; d).
\]

The proofs for \( d^k < E \leq d^{k+1} \) and for the NA solution are similar. \( \text{Q.E.D} \)

REFERENCES


Table 1. Solutions in the Talmud Bankruptcy Problem

\[
D = (D^1, D^2, D^3) = (100, 200, 300)
\]

<table>
<thead>
<tr>
<th>L</th>
<th>Cost Sharing Version</th>
<th>Bankruptcy Version</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Allocation of Loss</td>
<td>Allocation of Asset</td>
</tr>
<tr>
<td>500</td>
<td>KS = NA = (66(\frac{2}{7}), 166(\frac{2}{7}), 266(\frac{2}{7}))</td>
<td>KS = NA = (33(\frac{1}{7}), 33(\frac{1}{7}), 33(\frac{1}{7}))</td>
</tr>
<tr>
<td></td>
<td>Mishna = (66(\frac{2}{7}), 166(\frac{2}{7}), 266(\frac{2}{7}))</td>
<td>Mishna = (33(\frac{1}{7}), 33(\frac{1}{7}), 33(\frac{1}{7}))</td>
</tr>
<tr>
<td></td>
<td>P = (83(\frac{1}{7}), 166(\frac{2}{7}), 250)</td>
<td>P = (16(\frac{1}{7}), 33(\frac{1}{7}), 50)</td>
</tr>
<tr>
<td>400</td>
<td>KS = (60, 120, 220)</td>
<td>KS = (40, 80, 80)</td>
</tr>
<tr>
<td></td>
<td>Mishna = (50, 125, 225)</td>
<td>Mishna = (50, 75, 75)</td>
</tr>
<tr>
<td></td>
<td>NA = (33(\frac{1}{7}), 133(\frac{1}{7}), 233(\frac{1}{7}))</td>
<td>NA = (66(\frac{2}{7}), 66(\frac{2}{7}), 66(\frac{2}{7}))</td>
</tr>
<tr>
<td></td>
<td>P = (66(\frac{2}{7}), 133(\frac{1}{7}), 200)</td>
<td>P = (33(\frac{1}{7}), 66(\frac{2}{7}), 100)</td>
</tr>
<tr>
<td>300</td>
<td>KS = P = (50, 100, 150)</td>
<td>KS = P = (50, 100, 150)</td>
</tr>
<tr>
<td></td>
<td>Mishna = (50, 100, 150)</td>
<td>Mishna = (50, 100, 150)</td>
</tr>
<tr>
<td></td>
<td>NA = (0, 100, 200)</td>
<td>NA = (100, 100, 100)</td>
</tr>
</tbody>
</table>
Figure 1. Solutions in rectangular dual bargaining \((D, g)\), where ES and P solutions are, in general, not efficient.

Figure 2. (a) A CS problem whose ES, KS and NA solutions are different; (b) A CS problem whose ES, KS and NA solutions are identical.
Note that such cost sharing problems are different from the public good provision problem defined by $(b; c) \in \mathbb{R}_{++}^n$. It costs $c < \sum b^j$ to provide a public good that gives a player $j$ a benefit $b^j$. The difference between the carpooling and the public good provision problems is the nature of cooperation: achieving the low cost of carpooling requires the participation of all agents, while the low cost of a public good can be achieved by any coalition or any individual player [see Moulin, 1988].

Inequality definitions are: $y \leq D \iff y^i \leq D^i$, all $i$; $y < D \iff y^i \leq D^i, y \neq D^i$; $y << D \iff y^i < D^i$, all $i$.

This is also called MOP (multiobjective programming), see Zeleny [1984] for survey.

A utility bargaining problem $(d; X)$ satisfies: (i) $d \in (\mathbb{R}_{++}^n, X \subseteq \mathbb{R}_{++}^n)$; (ii) $d \leq x$ for all $x \in X$, and $X$ is compact; (iii) there is $x \in X$ such that $x \gg d$. See Nash [1950], Kalai and Smorodinsky [1975], Moulin [1988], and Thomson [1994] for more discussions.

This is called contraction independence in Thomson [1994].

The word Mishna refers to short statements of law in the Talmud. See O'Neill [1982] and Aumann and Maschler [1985] for references.

Given a supermodular function, consider any partition $\{T_1, ..., T_k\}$ of $T$. Network providers in $T_i$ generate a revenue $v(T_i)$ themselves, the full cooperation by $T$ can generate $v(T) \geq \sum_{i=1}^k v(T_i)$.

The matrices are: $G = G_{(n+1) \times n} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ -1 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 \end{bmatrix}$, $h = h_{(n+1) \times d} = \begin{bmatrix} L \\ -D^1 \\ -D^2 \\ \vdots \\ -D^n \end{bmatrix}$.

The following algorithm completes with at most $n$ iterations. Let $\omega(n) = (\Sigma D j - L) > 0$, and $k = 1$.

**Step 1:** If $\omega(n-k+1)/(n-k+1) \leq D_k$, then $N A^i_k = D^i j - \omega(n-k+1)/(n-k+1)$ for all $i \geq k$, and stop; If $\omega(n-k+1)/(n-k+1) > D_k$, then $N A^k = 0$, and move to next step.

**Step 2:** Let $\omega(n-k) =: \omega(n-k+1) - D^k$. If $k+1 = n$, $N A^{k+1} = D^{k+1} - \omega(n-k)$, and stop. If $k+1 < n$, let $k = k+1$, and repeat Step 1.

The economic duality is the following: An SS problem is a vector-max problem: No agent will accept a return less than his value at $d$-point, which is below the efficient line; while a CS problem is a vector-min problem: No one will pay a cost greater than his value at $d$-point, which is above the efficient line.

For example, one might try the recent results on non-convex bargaining [see Zhou, 1997] or other solutions surveyed in Thomson [1994]. One preliminary dual result is given below. A bankruptcy problem $(D; E)$ is uniquely related to a utility bargaining $(0; X)$ with a zero $d$-point and a feasible set $X = \{x \in \mathbb{R}_{++}^n \mid \Sigma x^j \leq E, x^j \leq d^j \text{ for all } j\}$. Then, one can check that the KS and NA solutions for $(0; X)$ are equivalent to those for the dual bargaining problem $(D; \Sigma D^j - E)$ (see Appendix for a proof). Another preliminary dual result is Proposition 6 in Zhao (1999).