A New Precondition for Horizontal Mergers

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Abstract: This paper studies a new precondition for horizontal mergers: the division of joint profits in each observed merger must be in the merger’s core. Otherwise, the fundamental assumption of profit-seeking behavior will be violated. The paper shows that the core is non-empty in all linear markets with asymmetric costs and asymmetric capacities, and it precisely characterizes the upper bound for merging costs or transaction costs beyond which the core condition will be destroyed. It derives the main results using an intuitive new technique, which makes core theory less technical and more accessible to applied colleagues.

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1. Introduction

The turn of the twenty-first century is, like the turn of the twentieth century, being characterized by a wave of horizontal mega mergers\(^1\): the fifteen largest mergers in history (each valued at more than $40 billion) all occurred in the two year period between April, 1998 and March, 2000. This paper studies a fundamental hurdle that all such mergers must clear: the division of each merger’s joint profits must be in the merger’s core.

Although the task of proving the core’s existence is quite technical, the logic behind a non-empty core as a merger precondition is fairly simple. Consider a merger to monopoly (for convenience, a “monopoly merger”) and suppose its core is empty. For any proposed division of monopoly profits, an empty core implies that there will always exist a blocking coalition of firms who could guarantee more profits than the proposed deal. With profit-seeking behavior, it would be absurd for a blocking coalition to accept the deal because it would receive less than its worst profits if it did. Therefore, any proposed monopoly merger contract will be blocked if the monopoly merger has an empty core.

Under profit-seeking assumptions, therefore, a precondition for the monopoly merger is that its core is non-empty. Similarly, a precondition for any other merger is that it has a non-empty core (note that the core of a non-monopoly merger is different from the core of the monopoly merger, which is the same as the core of the market).\(^2\) As an example,

\(^{1}\) See p.21-22 and Figure 2.1 in Ravenscraft and Scherer (1987) for the 1895-1901 merger wave. By the author’s own tracking, the 15 largest mergers (by June 2001) are: AOL-Time Warner ($163B, 1/00), Pfizer/Warner-Lambert ($90B, 2/00), Deutsche-Italia ($82B, 4/99), Mobil-Exxon ($77B, 12/98), Citicorp-Travelers ($73B, 4/98), Glaxo-SmithKline ($72B, 1/00), BTM-MTB ($69B, 4/00), SBC-Ameritech ($62B, 5/98), BankAmerica-NationsBank ($60B, 4/98), ATT-MediaOne ($58B, 5/99), Vodafone-Airtouch ($56B, 1/99), Voicestream-D.Telekom ($50.7B, 7/00), BP-Amoco ($50B, 8/98), ATT-TCI ($48B, 6/98), Daimler-Chrysler ($40B, 4/98). The $129B MCI-Sprint deal was canceled due to antitrust oppositions.

\(^{2}\) See the next section for precise definitions.
consider the three-way A.P.A merger\(^3\) (August, 1998), whose ownership is divided (44%, 29%, 27%) between Canadian Alcan, French Pechiney, and Swiss Algroup. This split must be in the merger’s core. Otherwise, one or two of the members would have blocked the deal.

In addition to providing the new precondition, this paper makes two other contributions. First, it shows that the non-empty core condition holds for any merger in all linear markets with asymmetric costs and asymmetric capacities, and it precisely characterizes the upper bound for merging (or transaction) costs beyond which the core will be empty. Second, it derives the main results using an intuitive new method for core existence that makes core theory less technical and more accessible to applied scholars.

The rest of the paper is organized as follows: Section 2 defines the problem, Section 3 discusses merging costs and empty-core markets, Section 4 establishes the main existence results, and Section 5 provides extensions to non-monopoly mergers. Section 6 provides conclusions and discussions, and the appendix provides proofs.

2. Description of the problem

Consider an oligopoly with a homogeneous good, an inverse demand \(P(\Sigma x_i)\), and \(n\) cost functions, \(C_i(x_i), i = 1, \ldots, n\). This is equivalent to the following normal form game:

\[
\Gamma = \{N, Z_i, \pi_i\},
\]

where \(N = \{1, 2, \ldots, n\}\) is the set of firms; for each firm \(i \in N\), \(Z_i = [0, z_i]\) is \(i\)'s production set, \(z_i \in (0, \infty)\) is its capacity, and \(\pi_i(x) = P(\Sigma x_j) x_i - C_i(x_i)\) is \(i\)'s profit. This paper will assume throughout the following assumption, \(A0:\)

\(A0:\) (i) The inverse demand function is decreasing, and each \(\pi_i(x)\) is continuous in \(x\) and quasi-concave in \(x_i\); (ii) the monopoly supply is an interior solution; and (iii) the

\(^3\) APA has sales revenues of $21.6 billion per year (98), passing Alcoa as the leader in Aluminum.
maximum capacity and cost function for each merger $S \subseteq N$ are given by

\begin{equation}
(2) \quad z_S = \sum_{j \in S} z_j, \text{ and}
\end{equation}

\begin{equation}
(3) \quad C_S(q) = \min \{ \sum_{j \in S} C_j(x_j) \mid q = \sum_{j \in S} x_j \leq z_S, x_j \geq 0, j \in S \}.
\end{equation}

Part (ii) assumes that a monopoly’s capacity constraint, $z_N = \sum z_j$, is never binding. Adding capacity constraints expands the merger literature in two directions. First, it is the first step towards studying the welfare effects and profitability of mergers with capacity constraints in future studies. Second, capacity constraints make merger models more realistic and core theory non-trivial.

The cost function (3) represents a weak form of synergy (although it has been referred to as the “no synergy” case in Farrell and Shapiro (1990)): $S$ could use the most efficient technology of its members up to $z_S = \sum_{j \in S} z_j$. In other words, a merger involves both the exit of all inefficient members and an increase in the efficient member’s capacity.$^4$

Let $\bar{x}$ and $\bar{\pi}$ be the monopoly supply and profits. Then, a monopoly merger contract can be defined as a triplet $(N; \bar{x}; \theta)$ such that $\bar{x}$ maximizes total profits and $\theta$ is a division of the monopoly profits (i.e., $\sum x_j \leq \sum z_j$, $\sum \pi_j(x) \leq \sum \pi_j(\bar{x}) = \bar{\pi}$ for all $\sum x_j \leq \sum z_j$, and $\theta \geq 0$, $\sum \theta_i = \bar{\pi}$).

Because the existing literature focuses on profitability and welfare effects, and because the division of joint profits is the most important issue in merger negotiations, adding the allocation $\theta$ is an important step forward in understanding mergers and market stability. Consider, for example, the 1997 MCI-WorldCom merger which broke up the MCI-British TeleCom deal. WorldCom’s final ($37$ billion) and initial (20% less) offers are examples of two different merger contracts. Without the allocation $\theta$, the rejected initial and

$^4$ Such weak synergy seems to exist in internet mergers (like the AOL-CompuServe case), which update software and utilize the capacities of less efficient members. Other synergies (like Perry and Porter, 1985) will
the accepted final contracts would be the same, although they have completely different impacts on the stability of the market structure.

Consider the core of market (1), which is the same as the core of the monopoly merger. For each $S \neq N$, its guaranteed profit is given by

\begin{equation}
  v(S) = \max_{x_S} \min_{y_T} \sum_{i \in S} \pi_i(x_S, y_T) = \min_{y_T} \max_{x_S} \sum_{i \in S} \pi_i(x_S, y_T),
\end{equation}

where $T = \{i \mid i \notin S\}$ is the set of outsiders, $(x'_S, y_T)$ is a vector $w \in \mathbb{R}^n$ with $w_i = x_i$ if $i \in S$, $= y_i$ if $i \notin S$; the Min is taken over $Z_T = \prod_{j \in T} Z_j$; and the Max over $\{x_S \in \mathbb{R}_+^S \mid \sum_{j \in S} x_j \leq z_S\}$.

A profit vector $\theta \in \mathbb{R}_+^n$ is in the core of market (1) if it is a division of the monopoly profits and if it gives each coalition $S$ no less than its guaranteed profits $v(S)$ (i.e., if $\sum \theta_i = \pi$ and $\sum_{i \in S} \theta_i \geq v(S)$ for all $S \neq N$). Denote the set of all core vectors for (1) as Core($\Gamma$).

Note that (4) and Core($\Gamma$) are well defined, because the functions are continuous, and the choice sets are compact and convex. In more general situations, one would have

\begin{equation}
  v_\alpha(S) = \max_{x_S} \min_{y_T} \sum_{i \in S} \pi_i(x_S, y_T) < v_\beta(S) = \min_{y_T} \max_{x_S} \sum_{i \in S} \pi_i(x_S, y_T),
\end{equation}

which implies $\beta$-core $\subseteq \alpha$-core (Aumann, 1959). Since $\alpha$-core = $\beta$-core always holds in an oligopoly market (Zhao, 1999), there is no need to make the $\alpha$- and $\beta$-distinction in oligopoly studies and one can simply use the term core.

Let $x$ denote the Cournot equilibrium in market (1) (i.e., the pre-merger equilibrium at which each $x_i$ is i's best response to $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$), $\pi_j(x)$ be j's pre-merger profits. Theorem 1 below summarizes two preconditions for a monopoly merger.

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5 The $\alpha$-core and $\beta$-core are defined by replacing the above $v(S)$ by $v_\alpha(S)$ or $v_\beta(S)$. Note that Rajan (1989) defined the $\alpha$- and $\beta$-cores in market with linear demand and symmetric costs, but did not show that the $\alpha$-
Theorem 1 (Necessary conditions for a monopoly merger): Suppose firms are profit-seeking and suppose a monopoly merger \((N; \bar{x}; \theta)\) has occurred. Then, the following two conditions hold: (i) \(\theta_j \geq \pi_j(x)\) for all \(j\); and (ii) \(\sum_{j \in S} \theta_j \geq v(S)\) for all \(S\).

Part (i) is the profitability (i.e., incentive to merge) condition, and part (ii) is the non-empty core condition. If part (i) fails, at least one firm is worse off with the merger; if part (ii) fails, at least one blocking coalition receives less than its worst profits. The failure of either would be absurd under profit-seeking behavior. Therefore, both must hold.

It is important to clarify three remarks. First, the two preconditions are necessary rather than sufficient conditions for the monopoly merger, and they are independent. The next section provides examples with an empty core and a profitable monopoly as well as with a non-empty core and an unprofitable monopoly (see Examples 1 and 2).

Second, the non-empty core condition, simplified as below,

\[
\theta \in \text{Core}(\Gamma),
\]

has often been misunderstood as below: Under A0, one has

\[
v(S) = \max \{ \sum_{i \in S} \pi_i(x_S, z_{-S}) | x_S \}.
\]

Since producing \(z_{-S}\) (at the full capacity) is not a credible action by the outsiders, the core is therefore not a useful concept, and the non-empty core condition must be false. Such argument, however, derives from a misunderstanding of the core. The core doesn’t require any \(S\) to produce \(z_S\). In stead, it only requires that total profits be maximized and divided in such a way that each \(S \neq N\) receives no less than its worst profits \(v(S)\). Given \(z_S\) in (5.2), \(v(S)\) is typically small, implying that the core is possibly large. However, the large size of the core strengthens, rather than weakens, the claim that (5.1) must hold.

Third, it will be useful to refine the core in future works by replacing the above \(z_S\) and \(\beta\)-cores are identical nor that they are trivially non-empty in models with no capacity constraints.
with credible reactions.\footnote{The author thanks the referee for addressing this line of future research. Note that this paper does not address the core-selection problem -- whether it is resolved cooperatively or non-cooperatively need not concern us, what matters in this study is that every observed merger contract must have passed the core-test.} Two common reactions (see Rajan, 1989) by the firms in \( N\backslash S \) are: i) they stay as a coalition and play a Cournot game against \( S \); ii) they break up into singletons and play a Cournot game against \( S \). Let \( v_1(S) = v_\delta(S) \) and \( v_2(S) = v_\gamma(S) \) be the post-deviation profits of \( S \) using actions (i)-(ii); and \( C_\delta \) and \( C_\gamma \) the associated new cores. Because the post-deviation equilibrium outputs with actions (i) and (ii) are the same as those with a duopoly and with \( (n-k+1) \) firms (\( k = |S| \) is the number of deviators), one has \( v_\delta(S) > v_\gamma(S) > v(S) \), which implies \( C_\delta \subset C_\gamma \subset \text{Core} (= \alpha\text{-core} = \beta\text{-core}). \)

Note that a division of monopoly profits within \( C_\delta \) will rule out all deviations, as \( v_\delta(S) \) is the largest for any \( S \neq N \) among all partitions of (or credible actions by) \( N\backslash S \). Hence, \( C_\delta \neq \emptyset \) will be sufficient for the monopoly merger to be formed. In addition, such refined cores will lead to better thresholds for the transaction costs to destroy or support a non-empty core, which are discussed in the next two sections. Although this paper only focuses on the core, its new approach can be used in studying the refined cores. Characterizing the sufficient conditions for a merger (i.e., existence conditions for the new cores) will provide a rich source for future works in both theoretical and empirical industrial organization.

3. Empty-core markets and merging costs

Empirical studies on ocean shipping (Sjostrom, 1989; Pirrong, 1992) suggest that industries for lumpy goods with avoidable costs will have an empty core when market demand is low.\footnote{One such empty-core market is Telser’s (1994) duopoly market with two taxicabs and three passengers: passenger \( j \) is willing to pay \( p_j \) for the trip (\( p_1 = 55 < p_2 = 60 < p_3 = 70 \)). Taxicab 1 has two passenger seats} These studies and the new precondition both imply that the monopoly
merger will not take place in such empty-core markets.

Another source for an empty-core is the merging costs or transaction costs associated with a merger, which include direct merging costs (such as labor costs and legal fees) as well as indirect merging costs (such as campaign donations to seek anti-trust approval). Although the monopoly merging costs in certain industries are effectively infinite due to anti-trust laws, they are clearly finite in those industries in which a monopoly or a monopoly merger is legal (like Microsoft in the operating system market or the Boeing-Douglas deal in the domestic aircraft market). As illustrated below, the core in a monopoly merger with positive merging costs could be empty (Example 1) or non-empty (Example 2). Theorem 4 in the next section precisely characterizes the minimum bound of merging costs which will destroy the non-empty core condition.

**Example 1:** Consider a three-firm market with $P = 6 - \sum x_i$, $C_i(x_i) = 0.8x_i$, $0 \leq x_i \leq 1.5$ for $i = 1, 2, 3$. The pre-merger and monopoly profits are: $\pi_i = 1.69$, $\pi_m = 6.76$. Assume that the monopoly merging cost is $MCM = 1.65$. Then, $v(123) = (\pi_m - MCM) = 5.11$. By $5.11 > \sum \pi_i = 5.07$, the profitability condition holds. Using arguments provided in the next section, one can show that the core is empty.

**Example 2:** Consider a three-firm market with $P = 6 - \sum x_i$, $C_i(x_i) = 0.5x_i$, $0 \leq x_i \leq 2$ for $i = 1, 2, 3$. The pre-merger and monopoly profits are: $\pi_i = 1.89$ all $i$, $\pi_m = 7.56$. Assume that the monopoly merging cost is $MCM = 2$. Then, $v(123) = (\pi_m - MCM) = 5.56$. By $5.56 < \sum \pi_i = 5.67$, the profitability condition fails. Using arguments provided in Section 4, one can

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and an opportunity cost $b_1 = 85$ (i.e., not making the trip would allow its owner to earn $b_1$, possibly by delivering pizzas instead), Taxicab 2 has three passenger seats and a larger opportunity cost of $b_2 = 150$.

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8 In the $90$ billion Pfizer/Warner-Lambert deal (2/2000), the direct merging costs included a $1.8$ billion break-up fee paid to American Home for dropping its agreement with Warner-Lambert. In the $37B MCI-WorldCom deal (11/1997), its direct merging costs included a penalty of $500$ million (paid to British Telecom) for breaking up the British Telecom-MCI deal.
show that the core is non-empty.

4. Core existence results

The new precondition for horizontal mergers heightens the need to check core existence in merger studies. The following theorem summarizes four known methods for checking core existence in an oligopoly market.

**Theorem 2**: (I) Assume parts (i) and (iii) of A0 hold. Then, the core in (1) is non-empty if all \( \pi_i(x), i \in N, \) are concave. (II) Given market (1), let its coalitional TU game \( \Gamma_c = \{N, v(S)\} \) and its core \( C(\Gamma) \) be defined by (4). Then, the following four claims are equivalent: (a) \( C(\Gamma) \neq \emptyset \); (b) \( \Gamma_c \) is balanced; (c) the largest excess of the nucleolus for \( \Gamma_c \) is zero;\(^9\) and (d) the minimum no blocking payoff (MNBP) of \( \Gamma_c \) is no greater than \( v(N) = \overline{\pi} \).

Although the concavity condition in part (I) does not apply in linear markets where profit functions are not quasi-concave, it is the only known core result in a general oligopoly model, and it remains as an open problem to show core existence in non-linear markets beyond the concavity assumption.

Theorem 3 below shows that the core is non-empty in all linear markets, which is given by a \((2n+1)\)-vector, \((a, c, z) \in \mathbb{R}^{2n+1}_{++} , \) with \( P = a - \Sigma x_i, \ C_i(x_i) = c_i x_i, \ 0 \leq x_i \leq z_i, \) all \( i.\)

Without loss of generality, assume \( c_1 \leq ... \leq c_n. \) Then, parts (ii-iii) of A0 become:

\[
(6) \quad c_S = \text{Min}\{c_j \mid j \in S\}, \ C_S(q) = c_S q, \ 0 \leq q \leq z_S = \Sigma_{j \in S} z_j, \text{ and}
\]

\[
(7) \quad (a - c_1)/2 \leq z = \Sigma z_j.
\]

**Theorem 3**: Suppose parts (ii-iii) of A0 hold in a linear Cournot market \((a, c, z).\) Then, the core of the market is non-empty and it has a non-empty relative interior.

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9 Part (I) is given in Zhao (1999). In part (II), (a) \( \Leftrightarrow \) (b) is due to Bondareva (1962) and Shapley (1969); (a) \( \Leftrightarrow \) (c) is due to Schmeidler (1969); (c) \( \Rightarrow \) (a) and (a) \( \Leftrightarrow \) (d) are due to Zhao (2001).
The non-emptiness of a core’s relative-interior should provide insights for future empirical work on cartel and merger agreements, because a core with a non-empty relative interior remains non-empty even in the event of small shocks to the market. Hence, a long-lived cartel suggests that the core has a non-empty relative-interior, while a short-lived cartel suggests that either the core is empty or that it is non-empty with an empty relative-interior.

Theorem 3 is proved by a new technique, called the MNBP method, as discussed below. The MNBP method reveals whether the core has a non-empty interior, and it is based on an intuitive argument that makes core theory less technical and more accessible to applied scholars. In addition, it allows us to characterize how core existence is affected by merging costs, which can not be obtained by any of the other known core-techniques.

To better understand the intuition behind MNBP, consider a game \( \Gamma_c = \{N, v(S)\} \). Observe that one will be able to make \( \sum x_i = v(N) \), \( \sum_{i \in S} x_i \geq v(S) \), all \( S \neq N \), satisfied (i.e., \( C(\Gamma) \neq \emptyset \)) if \( v(N) \) is sufficiently large. This observation leads to a procedure for MNBP given below. First, fix all \( v(S), S \neq N \); then, start with a large \( v(N) \) (so \( C(\Gamma) \neq \emptyset \)) and keep reducing \( v(N) \). The core will eventually become empty after \( v(N) \) falls below a critical level, which is called the MNBP,\(^{10}\) and one has \( C(\Gamma) \neq \emptyset \iff v(N) \geq \text{MNBP} \).

As shown in Theorem 4, the MNBP in symmetric linear markets can be explicitly expressed using demand and cost parameters. In Example 1, \( v(N) = 5.11 < \text{MNBP} = 5.13 \), so \( C(\Gamma) = \emptyset \); in Example 2, \( v(N) = 5.56 > \text{MNBP} = 4.59 \), so \( C(\Gamma) \neq \emptyset \).

Theorem 4 below characterizes the smallest upper bound of merging cost above which the core will be empty. Let \( MCM > 0 \) denote the monopoly merging costs, and let \( c>0 \) and \( z>0 \) denote the identical marginal cost and capacity in a symmetric market (i.e., \( c_i = c, z_i \)

\(^{10}\) Precisely, \( \text{MNBP} = \min \{ \sum_{i \in N} \theta_i \mid \theta \in \mathbb{R}^n_+; \sum_{i \in S} \theta_i \geq v(S) \text{ for all } S \neq N \} \).
Theorem 4: Let the core of the monopoly merger in \((a, c, z)\) be \(C(\Gamma)\). Assume \(\text{MCM} > 0\) and all solutions are interior. (i) In a symmetric market \((a, c, z)\), \(C(\Gamma) \neq \emptyset \iff \) 
\[(8) \quad (a-c)^2/4 - \{n(a-c-z)^2/[4(n-1)]\} > \text{MCM};\]
(ii) in an asymmetric market \((a, c, z)\), \(C(\Gamma) \neq \emptyset\) if the following inequality holds:
\[(9) \quad (a-c_1)^2/4 - \{n(a-c_1-z_{\max})^2/[4(n-1)]\} > \text{MCM}.\]

In part (ii), \(c_n = c_{\max} = \max \{c_i \mid i \in N\}\), and \(z_{\max} = \max \{z_i \mid i \in N\}\) are the maximal marginal cost and the maximal capacity. By the new precondition and by (8) or (9), a larger merging cost, fewer incumbent firms, and smaller capacities will make the monopoly merger less likely to occur. Such an effect of capacity on the core is consistent with the stylized fact that mergers are likely to occur in markets plagued by excess capacities.

As previously noted, the core is trivially non-empty in quantity-setting models without capacity, because \(v(S) = 0\) for all \(S \neq N\). Since capacity is an important element of horizontal mergers, it will be useful to check core existence in earlier quantity- or price-setting merger models\(^{11}\) with capacity added. By the difficulty in proving Theorems 3 and 4 and by the fact that part (I) of Theorem 2 is the only \(\beta\)-core result, core existence in each of the previous non-linear merger models will require one or more separate future studies.

5. Extensions to non-monopoly mergers

This section extends the new precondition to a non-monopoly mergers \(S \neq N\) \((2 \leq |S| < n)\). Let \(N/S = \{i_1, i_2, \ldots, i_t\}\) be the set of outsiders (i.e., \(t = n-|S|\)). Assume that parts (i) and (iii) of A0 hold, and assume that the pre- and post-merger equilibria both exist and are

\(^{11}\) For example, see Kreps and Scheinkman (1983), Davidson and Deneckere (1984), Perry and Porter (1985), Farrell and Shapiro (1990), and Ziss (1995).
unique.\footnote{Under parts (i) and (iii) of A0, existence is guaranteed by \( P'(X) + x_i P''(X) \leq 0 \) for all \( x_i \leq X \) (Novshek, 1985). The requirements for uniqueness are in general more complex; some recent results are surveyed in Zhang and Zhang (1996).} Given \( S \neq N \), let \( (\tilde{x}_S; \tilde{x}_S) = (x_S; \tilde{x}_i, i \in S) \in \mathbb{R}^{i+1} \) be the unique post-merger equilibrium supply (i.e., each \( \tilde{x}_i \) is i's best response to \( \tilde{x}_-i \)), and let \( \bar{\pi}_S \) be the insiders' post-merger joint profits. Then, a merger contract for \( S \) is a triplet \( (S; \tilde{x}_S; \theta_S) \) such that \( \tilde{x}_S \) is \( S \)'s best response to \( \tilde{x}_-S \) and \( \theta \) is a split of its joint profits (i.e., \( 0 \leq \tilde{x}_S \leq z_S \), \( \bar{\pi}_S = \pi_S(\tilde{x}_S, \tilde{x}_-S) \geq \pi_S(x_S; \tilde{x}_T) \) for all \( 0 \leq x_S \leq z_S \); and \( \theta_i \geq 0 \) and \( \sum_{j \in S} \theta_i = \pi_S \)).\footnote{This can be generalized to a triplet \( (S; x_S(\tilde{x}_S); \theta_S(x_S)) \) such that \( x_S(\tilde{x}_S) \) is its best response to \( x_S \) and \( \theta_S(x_S) \) is a split of its joint profits. This general definition takes care of multiple post-merge equilibria.}

In other words, a merger contract specifies a set of merging firms, their optimal post-merger outputs, and an allocation of their joint profits. Splitting their joint profits is equivalent to playing the following normal form TU game cooperatively:

\begin{equation}
\Gamma_S(\tilde{x}_-S) = \{S, Z_i, \pi_i(x_S, \tilde{x}_-S)\},
\end{equation}

where for each member \( i \), \( \pi_i(x_S, \tilde{x}_-S) = (a - \sum_{j \in S} x_j - \sum_{j \in T} \tilde{x}_j) x_i - c_i x_i \) is parameterized by the outsiders' supply \( \tilde{x}_-S \). The core of the merger \( S \) or the core of the game (10) can be similarly defined as in (1).

It is important to emphasize that the core of a merger \( S \neq N \) is only defined for the fixed \( \tilde{x}_-S \) – members only consider how to divide their joint profits, and this does not allow them to negotiate with outsiders.\footnote{This involves three classes of firms: (i) the outsiders in \( N/S \) whose supply is fixed at \( \tilde{x}_-S \); (ii) firms in \( T (T \subseteq S) \) who might deviate from \( S \); and (iii) firms in \( S/T \) (the rest of insiders) who might punish \( T \).} If the members could negotiate with outsiders, the problem of dividing their joint profits would be moot. Allowing for such a possibility is
related to the sufficient condition and is therefore beyond the scope of this study. Theorem 5 below extends Theorem 1 to a non-monopoly merger $S$.

**Theorem 5** (Necessary conditions for a non-monopoly merger): Let $x$ be the Cournot equilibrium in (1). Suppose firms are profit-seeking and a merger $(S; \bar{x}_S; \theta_S)$, $S \neq N$, has occurred. Then the following two conditions hold: (i) $\theta_j \geq \pi_j(x)$ for all $j \in S$; and (ii) $\theta_S$ is in the core of $\Gamma_S(\bar{x}_S)$ given by (10).

The proof for Theorem 5 is similar to that for Theorem 1 --- violating either condition will contradict the profit-seeking assumption. Since the allocation $\theta_S$ is in the core of (10), the post-merger equilibrium is a hybrid one (Zhao, 1992).\(^{15}\)

The general core results of Theorems 2 and 3 can be similarly extended to a merger $S \neq N$. In particular, the core of $\Gamma_S(x_S)$ will be non-empty and have a non-empty relative interior if parts (i) and (iii) of A0 hold and if (1) is a linear market. Theorem 6 below characterizes the upper bound of merging costs below which a non-monopoly merger has a non-empty core. Under the interior solution assumption,$^{16}$ the post-merger equilibrium supplies in a linear market $(a, c, z) \in \mathbb{R}_{++}^{2n+1}$ are:

$$\bar{x}_i = \frac{[a-(t+1)c_i+c_S+\sum_{j \in N/S, j \neq i}c_j]}{(t+2)}, i \in N/S; \quad \bar{x}_S = \frac{[a-(t+1)c_S+\sum_{j \in N/S}c_j]}{(t+2)},$$

so the outsiders’ total supply is $y_S = \sum_{i \in N/S} \bar{x}_i = \frac{[t(a+c_S)-2\sum_{j \in N/S}c_j]}{(t+2)}$.

Let the merging costs for $S$ (i.e., the costs of changing the market structure from the finest partition $\Delta_0 = \{(1), \ldots, (n)\}$ to $\Delta_S = \{S, (i_1), \ldots, (i_t)\}$) be $\text{MCS} > 0$. By the term

\(^{15}\) Precisely, the post-merge equilibrium is a HSDR (hybrid solution with a distribution rule) for the partition $\Delta = \{S, (i_1), \ldots, (i_t)\}$ with $\text{DR}(S) = \beta$-core, see Section 5 of Zhao (1999).

\(^{16}\) For the pre- and post-merger equilibria to be interior solutions, one needs the following two assumptions: \textbf{A1}: At the pre-merge equilibrium, each firm's supply is positive and within its capacity; \textbf{A2}: At the post-merge equilibrium for any $S$, each firm's supply is positive and within its capacity.
"symmetric merging members," we mean \( c_i = c_S \) and \( z_i = z_S \) for all \( i \in S \).

**Theorem 6:** Given \( S \neq N \) in \( (a, c, z) \), let \( y_S = \sum_{i \in N \setminus S} x_i \), \( C(\Gamma_S) \) be its core, \( MCS > 0 \), and all solutions be interior. (i) With symmetric merging members, \( C(\Gamma_S) \neq \emptyset \iff \)

\[
(a-y_S-c_S)^2/4 - \{s(a-y_S- c_S- z_S)^2/[4(s-1)]\} > MCS;
\]

(ii) with asymmetric merging members, \( C(\Gamma_S) \neq \emptyset \) if the following inequality holds:

\[
(a-y_S-c_S)^2/4 - \{s(a-y_S- c_{s_{\text{max}}}- z_{s_{\text{max}}})^2/[4(s-1)]\} > MCS.
\]

In (11) and (12), \( c_S = \text{Min}\{c_i | i \in S\} \), \( c_{s_{\text{max}}} = \text{Max}\{c_i | i \in S\} \), and \( z_{s_{\text{max}}} = \text{Max}\{z_i | i \in S\} \) are respectively the minimal marginal cost, the maximal marginal cost, and the maximal capacity of \( S \). One advantage of Theorems 4 and 6 is that they can be directly applied to linear simulation models.

So far we have characterized the non-empty core condition for a single merger. It is relatively simple to define the core for each merger in a set of simultaneous mergers and extend Theorems 5-6 to such multiple mergers.

6. Discussions and conclusions

This paper has shown that an observed merger must have passed the core test. It has established core existence in all linear markets and characterized the upper bound of merging costs for a non-empty core. These results can be directly applied to simulation models for linear oligopolies, and they suggest that smaller merging costs, more merging members, and larger capacities will increase the possibility for a merger in linear markets.

Because the new pre-condition also applies to other problems with binding contracts like joint venture, joint R&D, and legal cartels, this study heightens the need to apply core theory in industrial organization. In this regard, the paper has raised more questions than the above answers. It will be useful in future studies to check core existence in all such models.
with binding contracts; as noted already, core existence in each of the previous non-linear merger models (with capacity in quantity- or price-settings) will be worthy of a separate future study. The new precondition also begs deeper questions like “what happens when the core is empty?” or “can we find sufficient conditions for a merger?” which need to be addressed in future studies. The author hopes that such core studies and the open questions will attract the attention of more scholars.

It is useful to discuss two remarks about the non-cooperative approach to merger formation. First, the non-empty core precondition defines a set of feasible mergers within which mergers can be formed non-cooperatively. Indeed, the core in mergers without capacity constraints is trivially non-empty. However, for mergers with capacity constraints, one must check for the existence of the core before one can study their non-cooperative formation. Second, the new precondition casts a cloud of doubts over the non-cooperative approach to merger formation, in general, because merger contracts are binding. It is not the intention of this author to criticize the non-cooperative approach, but colleagues using the non-cooperative approach need to explain why the Nash equilibrium concept, a tool for problems with non-binding contracts, is appropriate for analyzing problems with binding contracts, such as mergers. Doubts about the non-cooperative approach will be relieved only when it has been supported by empirical evidence.

APPENDIX

Proof of Theorem 1: The discussions following the theorem form its proof. Q.E.D

Our strategy is to first establish core existence in symmetric markets, and then extend it to asymmetric markets. The proof is completed by establishing the following five lemmas.

17 See, for example, Kamien and Zang (1990), Bloch (1995), Gowrisankaran (1999), and Ray and Vohra (1999).
Given a game \( \Gamma = \{N, v\} \), its core is given by

\[
C(\Gamma) = \{ x \in \mathbb{R}_+^n \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \neq N \},
\]

and its minimum no-blocking payoff (MNBP) is given by

\[
\bar{v} = \bar{v}(\Gamma) = \{ \min \sum_{i \in N} x_i \mid x \in \mathbb{R}_+^n, \sum_{i \in S} x_i \geq v(S) \text{ for all } S \neq N \},
\]

which is the minimum total payoff needed to satisfy all proper coalitions. Note that the MNBP only depends on \( v(S) \) for \( S \neq N \), and \( v(N) \) has no effect on MNBP.

**Lemma 1:** Given a game \( \Gamma = \{N, v\} \), let \( \Gamma' = \{N, v'\} \) be a new game such that \( v'(S) \leq v(S) \) for all \( S \neq N \). Then \( \bar{v}(\Gamma') \leq \bar{v}(\Gamma) \).

**Proof of Lemma 1:** Since \( v(S) \) is reduced to \( v'(S) \), the feasible region for \( \Gamma \) given by (P2) is enlarged to that for \( \Gamma' \). Thus, the minimum value of the same objective function in (P2), \( \sum_{i \in N} x_i \), is reduced from \( \bar{v}(\Gamma) \) to \( \bar{v}(\Gamma') \). \( \square \)

The following lemma (see Footnote 10) represents the intuition that the core is non-empty whenever the grand coalition has enough to split.

**Lemma 2:** Given a game \( \Gamma = \{N, v\} \), let \( \bar{v} = \bar{v}(\Gamma) \) and \( C(\Gamma) \) be its MNBP and its core. Then \( C(\Gamma) \neq \emptyset \iff v(N) \geq \bar{v}(\Gamma) \).

We first establish core existence using the following four assumptions:

- **A3:** For each \( S \neq N \), \( \bar{x}_S = (a - c_S - \sum_{j \in S} z_j)/2 \leq z_S \).
- **A4:** For each \( S \neq N \), \( v(S) = (\bar{x}_S)^2 > 0 \), where \( v(S) \) is given by (4).
- **A5:** \( c_j = c > 0 \) for all \( j \in N \).
- **A6:** \( z_j = z > 0 \) for all \( j \in N \).

With A3, the unconstrained optimal supply of \( S \) in finding its \( v(S) \) is an interior solution (i.e., less than or equal to its capacity). With A4, the price facing any \( S \) is always greater than its marginal cost \( c_S \), this guarantees that each coalition produces a positive amount of output\(^1\). With A5 and A6, firms are symmetric in marginal cost and in capacity.

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\(^1\) This is equivalent to the following inequality: \( \max\{c_i - z_i \mid i \in N\} \leq a - \sum_{i \in N} z_i \).
For any \(1 \leq k \leq n\), let \(S(k) = \{ T \subseteq N \mid |T| = k \}\) denote the set of coalitions who have precisely \(k\) members. By A3-A6, the guaranteed profit for each \(S \in S(k)\) is equal to

\[(P3) \quad v(k) = \frac{(a-(n-k)z-c)^2}{4} = v(\alpha(S)) = v_\beta(S).\]

Now, for each \(k = 1, \ldots, n\), consider the minimum value defined by

\[(P4) \quad MV(k) = \{ \min \sum_{i \in N} x_i \mid x \in \mathbb{R}^n_+ ; \sum_{i \in S} x_i \geq v(k), \text{ for all } S \in S(k) \} \].

The next lemma provides a formula for each \(MV(k)\) and their relations.

**Lemma 3:** Under A3-A6, the following two parts hold.

(i) \(MV(k) = n \frac{v(k)}{k}\);

(ii) \(MV(k+1) > MV(k)\), for \(k = 1, \ldots, n-1\).

**Proof of Lemma 3:** Part (i). \((P4)\) has \(\binom{n}{k} = n(n-1)\ldots(n-k+1)/k!\) constraints, so \(v(k)\) on the right-hand side appears \(n(n-1)\ldots(n-k+1)/k!\) times. Each \(x_i\) on the left-hand side appears \(\binom{n-1}{k-1} = (n-1)\ldots(n-k+1)/(k-1)!\) times. By summing over all constraints, we have

\[[(n-1)\ldots(n-k+1)/(k-1)!] \sum x_i \geq [n\ldots(n-k+1)/k!] v(k)\]  or  \(\sum x_i \geq n\frac{v(k)}{k}\).

Hence, the minimum value of \((P4)\) must be \(MV(k) = n \frac{v(k)}{k}\), and this proves part (i).

Part (ii). By part (i), part (ii) is equivalent to

\[g = v(k+1) - v(k) \frac{(k+1)}{k} > 0.\]

Let \(\theta = a-c\), the above function \(g\) can be rewritten as

\[(P5) \quad g(\theta) = (\theta - (n-k-1)z)^2 - (\theta - (n-k)z)^2 \frac{(k+1)}{k}.\]

Because the solution of \(g' = 0\) is \(\theta^* = nz\), and \(dg^2/d\theta^2 = -2/k < 0\), \(g\) is a "∩" shaped function whose maximum is \(\theta^* = nz\). It follows from A3 (\(\bar{x}_S \leq z_S\) \(|S| = k\)) and A4 (\(\bar{x}_S > 0\)) that \((nk)z < \theta \leq (n+k)z\). Since \(g\) is concave, and

\[g((n-k)z) = z^2 > 0, \quad g((n+k)z) = z^2 > 0,\]

we have \(g(\theta) > 0\) for all \(\theta \in [(n-k)z, (n+k)z]\). This proves part (ii). **Q.E.D**

**Lemma 4:** Under A3-A6, the MNBP defined by \((P2)\) can be given by

\[(P6) \quad \bar{v}(\Gamma) = MV(n-1) = n \frac{v(n-1)}{(n-1)}.\]

**Proof of Lemma 4:** Let \(FR\) and \(FR(k)\) denote respectively the "feasible regions" appeared in \((P2)\) and \((P4)\):
\[ FR = \{ x \in \mathbb{R}^n_+ \mid \sum_{i \in \mathcal{S}} x_i \geq v(k), \text{ for all } S \in \mathcal{S}(k), k = 1, \ldots, n-1 \}, \]

\[ FR(k) = \{ x \in \mathbb{R}^n_+ \mid \sum_{i \in \mathcal{S}} x_i \geq v(k), \text{ for all } S \in \mathcal{S}(k) \}, \quad k = 1, \ldots, n-1. \]

The preceding two definitions yield:

\[ FR = \bigcap_{i=1}^{n-1} FR(k). \]

Now, for each \( k = 1, \ldots, n-1 \), define

\[ FR(k)^* = \{ x \in \mathbb{R}^n_+ \mid \sum_{i \in \mathcal{N}} x_i \geq n \frac{v(k)}{k} \}. \]

From the proof of part (i) in Lemma 3, \( FR(k) \subseteq FR(k)^* \) for each \( k = 1, \ldots, n-1 \). From (P9) and part (ii) in Lemma 3, we have

\[ FR \subseteq FR^* = \bigcap_{i=1}^{n-1} FR(k)^* = \{ x \in \mathbb{R}^n_+ \mid \sum_{i \in \mathcal{N}} x_i \geq n \frac{v(n-1)}{(n-1)} \}. \]

Observe that the minimum value of \( \{ \min_{i \in \mathcal{N}} x_i \mid x \in FR^* \} \) is equal to

\[ \{ \min_{i \in \mathcal{N}} x_i \mid x \in FR^* \} = \frac{nv(n-1)}{(n-1)} = \text{MV}(n-1). \]

Since the symmetric minimum solution (i.e., \( x_i = v(n-1)/(n-1) \), all \( i \)) is included in \( FR \) (i.e., it is a feasible point in the problem of (P2)), it follows from (P11) and (P12) that

\[ \bar{v}(\Gamma) = \{ \min_{i \in \mathcal{N}} x_i \mid x \in FR \} = \{ \min_{i \in \mathcal{N}} x_i \mid x \in FR^* \} = \text{MV}(n-1) = \text{nv}(n-1)/(n-1). \]

Lemma 5: Given \((a, c, z)\), let \( v(S) = v_\alpha(S) = v_\beta(S) \) be given by (4), its MNBP in (P2) be denoted as \( \bar{v}(\Gamma) = \bar{v}(a, c, z) \). Define a market \((a, c', z')\) by \( c' = (c_1, \ldots, c_1) \in \mathbb{R}^n_+ \), and \( z' = (z_{min}, \ldots, z_{min}) \in \mathbb{R}^n_+ \), where \( c_1 = \min \{ c_i \mid i \in \mathcal{N} \} \), \( z_{min} = \min \{ z_i \mid i \in \mathcal{N} \} \), let the MNBP in this new market be denoted as \( \bar{v}(a, c', z') \). Then

\[ \bar{v}(a, c, z) \leq \bar{v}(a, c', z') \leq n(a - z_{min} - c_1)^2/\left[4(n-1)\right]. \]

Proof of Lemma 5: Consider the second inequality. If A3 and A4 are both satisfied in \((a, c', z')\), then Lemma 4 and (P3) yield \( \bar{v}(a, c', z') = n(a - z_{min} - c_1)^2/\left[4(n-1)\right] \). Note that \( v(S) \) will be reduced to a smaller value if either A3 or A4 or both are violated. If A3 is violated, \( v(S) \) will fall below \( (a - (n-s)z_{min} - c_1)^2/4 \) (where \( s \) denote the number of firms in \( S \)), because now \( v(S) \) is achieved at constrained solution. If A4 is violated, \( v(S) \) becomes zero. Thus, by
Lemma 1, the MNBP will be reduced, so

\[ \bar{v}(a, c', z') \leq n(a - z_{\text{min}} - c_1)^2/[4(n-1)]. \]

Now consider the first inequality of (P14). Assume for a moment that A3 and A4 are both satisfied in \((a, c, z)\). From the definitions of A3 and A4, the value for \(S\) in \((a, c, z)\) satisfies

(P15) \[ v(S) = (\bar{x}_S)^2 = (a - c_S - \Sigma_{j \notin S} z_j)^2/4 \leq (a - c_1 - (n-s)z_{\text{min}})^2/4. \]

By earlier arguments, the above \(v(S)\) in \((a, c, z)\) will fall below \((a - c_1 - (n-s)z_{\text{min}})^2/4\), if either A3 or A4 or both be violated. Since \((a - c_1 - (n-s)z_{\text{min}})^2/4\) is the value of \(S\) in \((a, c', z')\), (P15) and Lemma 1 lead to \(\bar{v}(a, c, z) \leq \bar{v}(a, c', z')\). \(\text{Q.E.D}\)

Proof of Theorem 3: It follows from parts (ii-iii) of A0 that the monopoly profit in \((a, c, z)\) is \(v(N) = (a-c_1)^2/4\). By part (i) of Lemma 3, this is equal to \(MV(n)\) defined in (P4) for the new market \((a, c', z')\). It follows from Lemmas 4-5 and part (ii) of Lemma 3 that

(P16) \[ v(N) = MV(n) > MV(n-1) = n(a - z_{\text{min}} - c_1)^2/[4(n-1)] \]

\[ \geq \bar{v}(a, c', z') \geq \bar{v}(a, c, z) = \bar{v}(\Gamma). \]

Hence, by Lemma 2, the core is non-empty. Since \(v(N)\) is greater than the MNBP or \(\bar{v}(\Gamma)\), the relative interior of the core is non-empty (see Footnote 10). \(\text{Q.E.D}\)

Proof of Theorem 4: We first prove part (i). By (P3) and (P6), the monopoly profit in symmetric markets without binding capacities is \(v(N) = (a-c)^2/4\), and the MNBP is

\[ \bar{v}(a, c, z) = \bar{v}(\Gamma) = n(a-c-z)^2/ (4(n-1)). \]

By Lemma 2, \(C(\Gamma) \neq \emptyset \iff \]

\[ [v(N)-\text{MCM}] = [(a-c)^2/4-\text{MCM}] \geq \bar{v}(a, c, z) = n(a-c-z)^2/ [4(n-1)]. \]

Therefore, \(C(\Gamma) = \emptyset \iff n(a-c-z)^2/ [4(n-1)] > [(a-c)^2/4- \text{MCM}]. \)

Part (ii). We only need to show \(\bar{v}(a, c, z) > [v(N)- \text{MCM}]\). Define \((a, c'', z'')\) by \(c'' = (c_{\text{max}}, ..., c_{\text{max}}) \in R^n_+\), and \(z'' = (z_{\text{max}}, ..., z_{\text{max}}) \in R^n_+\), where \(c_{\text{max}} = c_n = \max \{c_i | i \in N\}\), and \(z_{\text{max}} = \max \{z_i | i \in N\}\). Let the MNBP in this new market be \(\bar{v}(a, c'', z'')\). By sufficient capacities, and by (P3) and (P6), one has \(\bar{v}(a, c'', z'') = n (a-c_{\text{max}}-z_{\text{max}})^2/ (4(n-1)). \) By A3 and A4, the value for \(S\) in \((a, c, z)\) satisfies
\[ v(S) = (a - c_S - \sum_{j \in S} z_j)^2 / 4 \geq (a - c_{\text{max}} -(n-s)z_{\text{max}})^2 / 4. \]

This and Lemma 1 lead to \( \bar{v}(a, c, z) \geq \bar{v}(a, c^", z") \). By (9) and the above expressions,

\[ \bar{v}(a, c, z) \geq \bar{v}(a, c^", z") = n (a - c_{\text{max}} - z_{\text{max}})^2 / (4(n-1)) > \]

\[ [(a-c_1)^2/4 - \text{MCM}] = [v(N)-\text{MCM}]. \]

\textbf{Q.E.D}

\textbf{Proof of Theorem 5:} The discussions following the theorem forms its proof. \textbf{Q.E.D}

\textbf{Proof of Theorem 6:} The proof follows from the proof of Theorem 4 by replacing “N” with “S” and replacing “a” with \((a-y_S)\). \textbf{Q.E.D}

\textbf{REFERENCES}


