1 Calculus

1.1 Functions of a single variable

- Function $y = f(x)$
- The first derivative of $f$ with respect to $x$ is

  $$f'(x) = \frac{df}{dx}.$$  

  It gives, at each value $x$, the slope or instantaneous rate of change in $f(x)$.

- The second derivative of $f$ with respect to $x$ is

  $$f''(x) = \frac{d^2f}{dx^2}.$$  

  It gives the rate at which the slope of $f$ changes. It is thus related to the curvature of the function $f$. 

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• Rules of differentiation
  – For constants, $\alpha$:
    \[
    \frac{d}{dx} \alpha = 0, \]
  – For sums:
    \[
    \frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x), \]
  – Power rule:
    \[
    \frac{d}{dx}(\alpha x^n) = n\alpha x^{n-1}, \]
  – Product rule:
    \[
    \frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x), \]
– Quotient rule:

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2},
\]

– Chain rule:

\[
\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)
\]

**Examples.** Calculate the derivatives in each of the following cases:

a. \( y = 5x^{-2} \)

b. \( f(x) = 2x^2 + 3x + 4 \)

c. \( g(x) = 3x - 1 \)

d. \( (f(x)g(x))' \)

e. \( \left( \frac{f(x)}{g(x)} \right)' \)

f. \( f(g(x))' \)
2 Optimization

- Function \( y = f(x) \) is differentiable.
- The function achieves a **local maximum** (respectively **global maximum**) at \( x^* \), if \( f(x^*) \geq f(x) \) for all \( x \) in some neighborhood of \( x^* \) (respectively for all \( x \)).
- The function achieves a **local minimum** (respectively **global minimum**) at \( \tilde{x} \), if \( f(\tilde{x}) \geq f(x) \) for all \( x \) in some neighborhood of \( \tilde{x} \) (respectively for all \( x \)).
- Necessary conditions for local interior optima in a single-variable case: 

  \( f(x) \) is twice continuously differentiable. It reaches a local interior

  1. maximum at \( x^* \) \( \Rightarrow \ f'(x^*) = 0 \) (\( FOC \))

      \( \Rightarrow \ f''(x^*) \leq 0 \) (\( SOC \))

  1. minimization at \( \tilde{x} \) \( \Rightarrow \ f'(\tilde{x}) = 0 \) (\( FOC \))

      \( \Rightarrow \ f''(\tilde{x}) \geq 0 \) (\( SOC \))