Problem 1

(a)

First consider the 3 symmetric firms. For the first firm profit is given by

\[ \pi_1 = P q_1 - 20q_1 \]

\[ = (100 - q_1 - q_2 - q_3 - q_4)q_1 - 20q_1 \]

\[ = 100q_1 - q_1^2 - q_2q_1 - q_3q_1 - q_4q_1 - 20q_1 \]

If we maximize profit we obtain \( q_1 \) as a function of the other firms’ outputs.

\[ \frac{d\pi_1}{dq_1} = 100 - 2q_1 - q_2 - q_3 - q_4 - 20 = 0 \]

\[ -2q_1 = 80 - q_2 - q_3 - q_4 \]

\[ q_1 = \frac{80 - q_2 - q_3 - q_4}{2} \]

Since the first three firms are symmetric we know that \( q_1 = q_2 = q_3 \). This will then give

\[ q_1 = \frac{80 - q_2 - q_3 - q_4}{2} \Rightarrow q_1 = \frac{80 - q_1 - q_1 - q_4}{2} \Rightarrow q_1 = 20 - \frac{q_4}{4} \]

This also implies that \( q_1 + q_2 + q_3 = 60 - \frac{3}{4} q_4 \)

Now consider the profit maximization problem of firm 4.
\[ \pi_4 = P q_4 - (20 + \gamma) q_4 \]
\[ = (100 - q_1 - q_2 - q_3 - q_4) q_4 - 20 q_4 - \gamma q_4 \]
\[ = 100 q_4 - q_1 q_4 - q_2 q_4 - q_3 q_4 - q_4^2 - 20 q_4 - \gamma q_4 \]

If we maximize profit we obtain \( q_4 \) as a function of the other firms' outputs.

\[ \frac{d \pi_4}{dq_4} = 100 - q_1 - q_2 - q_3 - q_4 - 20 - \gamma = 0 \]
\[ - 2q_4 = 80 - q_1 - q_2 - q_3 - \gamma \]
\[ - q_4 = \frac{80 - q_1 - q_2 - q_3 - \gamma}{2} \]

If we substitute in the expression for \( q_1 + q_2 + q_3 \), we obtain

\[ q_4 = \frac{80 - q_1 - q_2 - q_3 - \gamma}{2} \]
\[ \frac{80 - \gamma - \left(60 - \frac{3}{4} q_4\right)}{2} \]
\[ = \frac{20 - \gamma + \frac{3}{4} q_4}{2} \]
\[ - \frac{5}{4} q_4 = 20 - \gamma \]
\[ - q_4 = 16 - \frac{4}{5} \gamma \]

We can then find the optimal levels of the first three firms by substituting in as follows

\[ q_i = 20 - \frac{1}{4} q_4 \]
\[ = 20 - \frac{1}{4} \left(16 - \frac{4}{5} \gamma\right) \]
\[ = 16 + \frac{1}{5} \gamma \]

The price is given by

\[ P = 100 - q_1 - q_2 - q_3 - q_4 \]
\[ = 100 - 64 - \frac{3}{5} \gamma + \frac{4}{5} \gamma \]
\[ = 36 + \frac{1}{5} \gamma \]
The profit for each symmetric firm is given by

\[ \pi_i = \pi q_i - 20q_i \]
\[ = (36 + \frac{1}{5} \gamma)(16 + \frac{1}{5} \gamma) - (20)(16 + \frac{1}{5} \gamma) \]
\[ = (16 + \frac{1}{5} \gamma)(16 + \frac{1}{5} \gamma) \]
\[ = 256 + \frac{32}{5} \gamma + \frac{1}{25} \gamma^2 \]

Profit for firm 4 is given by

\[ \pi_4 = \pi q_4 - 20q_4 - \gamma q_4 \]
\[ = (36 + \frac{1}{5} \gamma)(16 - \frac{4}{5} \gamma) - (20 + \gamma)(16 - \frac{4}{5} \gamma) \]
\[ = (16 - \frac{4}{5} \gamma)(16 - \frac{4}{5} \gamma) \]
\[ = 256 - \frac{128}{5} \gamma + \frac{16}{25} \gamma^2 \]

The restrictions on the model are that price be greater than marginal cost and that quantities are nonnegative. Since marginal cost for the first three firms is equal to 20, this implies that

\[ P = 36 + \frac{1}{5} \gamma > 20 \]
\[ \gamma > -80 \]

Since quantities must be positive we also have that

\[ 16 + \frac{1}{5} \gamma \geq 0 \]
\[ \gamma \geq -80 \]
\[ 16 - \frac{4}{5} \gamma \geq 0 \]
\[ \gamma \leq 20 \]

But if we also require that marginal cost for the fourth firm is positive then \((+20) \leq 0\). This then implies

\[ 20 + \gamma \geq 0 \]
\[ \gamma \geq -20 \]

Combining the constraints and eliminating the redundant ones will yield

\[ 20 \leq \gamma \leq 20. \]
(b)

We now have a market with three firms, two identical and one different. The model can be solved in a manner similar to part a. Denote the firms m, 3 and 4, where m denotes the combined firm.

First consider the 2 symmetric firms. For the first firm profit is given by

$$\pi_m = Pq_m - 20q_m$$
$$= (100 - q_m - q_3 - q_4)q_m - 20q_m$$
$$= 100q_m - q_m^2 - q_3q_m - q_4q_m - 20q_m$$

If we maximize profit we obtain $q_m$ as a function of the other firms' outputs.

$$\frac{d\pi_m}{dq_m} = 100 - 2q_m - q_3 - q_4 - 20 = 0$$
$$- 2q_m = 80 - q_3 - q_4$$
$$- q_m = \frac{80 - q_3 - q_4}{2}$$

Since the merged firm and firm 3 are symmetric we know that $q_m = q_3$. This will then give

$$q_m = \frac{80 - q_3 - q_4}{2}$$
$$- q_m = \frac{80 - q_m - q_4}{2}$$
$$- \frac{3}{2}q_m = 40 - \frac{q_4}{2}$$
$$- q_m = 26.66 - \frac{q_4}{3}$$

Now consider the profit maximization problem of firm 4.

$$\pi_4 = Pq_4 - (20 + \gamma)q_4$$
$$= (100 - q_m - q_3 - q_4)q_4 - 20q_4 - \gamma q_4$$
$$= 100q_4 - q_mq_4 - q_3q_4 - q_4^2 - 20q_4 - \gamma q_4$$

If we maximize profit we obtain $q_4$ as a function of the other firms' outputs.
\[ \frac{d\pi_i}{dq_i} = 100 - q_m - q_3 - 2q_4 - 20 - \gamma = 0 \]
\[- 2q_4 = 80 - q_m - q_3 - \gamma \]
\[- q_4 = \frac{80 - q_m - q_3 - \gamma}{2} \]

If we substitute in for the other firms we obtain

\[ q_4 = \frac{80 - q_m - q_3 - \gamma}{2} \]
\[ = \frac{80 - \gamma - \left(53.333 - \frac{2}{3} q_4\right)}{2} \]
\[ = 26.666 - \gamma + \frac{2}{3} q_4 \]
\[- \frac{4}{3} q_4 = 26.666 - \gamma \]
\[- q_4 = 20 - \frac{3}{4} \gamma \]

We can then find the optimal levels of the other two firms by substituting in as follows

\[ q_i = 26.666 - \frac{1}{3} q_4 \]
\[ = 26.666 - \frac{1}{3} (20 - \frac{3}{4} \gamma) \]
\[ = 20 + \frac{1}{4} \gamma \]

The price is given by

\[ P = 100 - q_m - q_3 - q_4 \]
\[ = 100 - (40 + \frac{1}{2} \gamma) - (20 - \frac{3}{4} \gamma) \]
\[ = 40 + \frac{1}{4} \gamma \]

The profit for each symmetric firm is given by
\[ \pi_i = P q_i - 20q_i, \]
\[ = (40 + \frac{1}{4} \gamma)(20 + \frac{1}{4} \gamma) - (20)(20 + \frac{1}{4} \gamma) \]
\[ = (20 + \frac{1}{4} \gamma)(20 + \frac{1}{4} \gamma) \]
\[ = 400 + 10 \gamma + \frac{1}{16} \gamma^2 \]

Profit for firm 4 is given by
\[ \pi_4 = P q_4 - 20q_4 - \gamma q_4, \]
\[ = (40 + \frac{1}{4} \gamma)(20 - \frac{3}{4} \gamma) - (20 \gamma)(20 - \frac{3}{4} \gamma) \]
\[ = (20 - \frac{3}{4} \gamma)(20 - \frac{3}{4} \gamma) \]
\[ = 400 - 30 \gamma + \frac{9}{16} \gamma^2 \]

The combined profits from part a are
\[ 2\pi_i = (2)(256 + \frac{32}{5} \gamma + \frac{1}{25} \gamma^2) \]
\[ = 512 + \frac{64}{5} \gamma + \frac{2}{25} \gamma^2 \]
\[ = 512 + 12.8 \gamma + \frac{2}{25} \gamma^2 \]

If \( \gamma \) is positive then the sum of the independent profits are higher since they are higher in every term. If \( \gamma \) is negative then the difference between \( 512 + 12.8 \gamma + \frac{2}{25} \gamma^2 \) and \( 400 - 30 \gamma + \frac{9}{16} \gamma^2 \) could shrink. Denote this difference as \( \Delta = 112 - 2.8 - 0.0175 \gamma^2 \). Consider this difference at the lower bound for \( \gamma \) (-20). The profits in the two cases are
\[ \pi(\text{sum}) = 512 + 12.8 \cdot (-20) + \frac{2}{25} \cdot 400 = 288 \]
\[ \pi(\text{merge}) = 400 + 10 \cdot (-20) + \frac{1}{16} \cdot 400 = 225 \]

The difference is 63. At \( \gamma = 0 \), the difference is 112. At \( \gamma = 20 \), the difference is 175.

(c)

We assume that firms 2 and 3 are still independent. Since firm 4 has higher costs than firm 1.
all production at the merged firm will take place using the facilities of firm 1. Thus we have a three-firm Cournot game where the firms are symmetric. We can proceed as in a, essentially ignoring firm 4.

For the merged firm profit is given by

\[ \pi_m = Pq_m - 20q_m = (100 - q_m - q_2 - q_3)q_m - 20q_m = 100q_m - q_m^2 - q_2q_m - q_3q_m - 20q_m \]

If we maximize profit we obtain \( q_m \) as a function of the other firms' outputs.

\[
\frac{d\pi_m}{dq_m} = 100 - 2q_m - q_2 - q_3 - 20 = 0
\]
\[2q_m = 80 - q_2 - q_3 \]
\[q_m = \frac{80 - q_2 - q_3}{2} \]

Since the three firms are symmetric we know that \( q_m = q_2 = q_3 \). This will then give

\[ q_m = \frac{80 - q_2 - q_3}{2} \Rightarrow q_m = \frac{80 - q_m - q_m}{2} \Rightarrow q_m = 20 \]

Price is given by

\[ P = 100 - q_m - q_2 - q_3 = 100 - 60 = 40 \]

The profits for each firm are given by

\[ \pi_i = Pq_i - 20q_i = (40)(20) - (20)(20) = 800 - 400 = 400 \]

So firms 2 and 3 each have profits of 400. In a, the profits of firm 2 were \( \pi_2 = \left(16 + \frac{4}{5}\right)^2 \).

The largest possible value of \( \gamma \) is 20 so the maximum profit for firm 2 in the initial problem is 400. Thus, firm 2 cannot lose from this merger. We now need to compare this to the profits that occurred in a to see how firms 1 and 4 do. The combined profits of firm 1 and firm 4 in part a are
\[ \pi_1 = (16 + \frac{1}{5} \gamma)^2 \\
= 256 + \frac{32}{5} \gamma + \frac{1}{25} \gamma^2 \\
\pi_4 = (16 - \frac{4}{5} \gamma)^2 \\
= 256 - \frac{128}{5} \gamma + \frac{16}{25} \gamma^2 \\
\pi_1 + \pi_4 = 512 - \frac{96}{5} \gamma + \frac{17}{25} \gamma^2 \\
\]

The question is whether this is smaller than 400 for positive \( \gamma \). If it is, there is an incentive to merge. For example, if \( \gamma = 10 \) then the profits in (i) are given by

\[ \pi_1 = (16 + \frac{1}{5} 10)^2 = 324 \\
\pi_4 = (16 - \frac{4}{5} 10)^2 = 64 \\
\pi_1 + \pi_4 = 512 - \frac{96}{5} 10 + \frac{17}{25} 100 = 388 \\
\]

This is clearly less than 400 so the firms would want to merge. We can solve this in more general form by writing the difference in profits as \( \Delta = 112 - \frac{96}{5} \gamma + \frac{17}{25} \gamma^2 \). If we can find all values of \( \gamma \) such that \( \Delta < 0 \), we know all values of \( \gamma \) where there is an incentive to merge. Since this is a quadratic equation we can graph it as follows.
Problem 2
(a) From Chapter Problem 1b, we have the following optimal quantities and market price, which are not affected by fixed costs.

\[ q_1 = q_3 = 20 + \frac{1}{4} \gamma \]
\[ q_4 = 20 - \frac{3}{4} \gamma \]
\[ P = 40 + \frac{1}{4} \gamma \]

Profits are now given by

\[ \pi_1 = 400 + 10 \gamma + \frac{1}{16} \gamma^2 - bF \]
\[ \pi_2 = 400 + 10 \gamma + \frac{1}{16} \gamma^2 - F \]
\[ \pi_4 = 400 - 30 \gamma + \frac{9}{16} \gamma^2 - F \]

Combined profits for firms 1 and 2 with no merger are

\[ 2\pi = \pi(\text{sum}) = (2)(256 + \frac{32}{5} \gamma + \frac{1}{25} \gamma^2 - F) \]
\[ = 512 + \frac{64}{5} \gamma + \frac{2}{25} \gamma^2 - 2F \]
\[ = 512 + 12.8 \gamma + \frac{2}{25} \gamma^2 - 2F \]

The merger is profitable if

\[ \pi(\text{merge}) = 400 + 10 \gamma + \frac{1}{16} \gamma^2 - bF \geq 512 + 12.8 \gamma + \frac{2}{25} \gamma^2 - 2F = \pi(\text{sum}) \]

We can rearrange this expression to obtain

\[ 2F - bF \geq 112 + 2.8 \gamma + 0.0175 \gamma^2 \]
\[ F(2 - b) \geq 112 + 2.8 \gamma + 0.0175 \gamma^2 \]
\[ 0.0175 \gamma^2 + 2.8 \gamma + 112 + F(b - 2) \leq 0 \]
\[ 0.0175 \gamma^2 + 2.8 \gamma + 112 + F(b - 2) \leq 0 \]
\[ 0.0175 (\gamma + 80)^2 + F(b - 2) \leq 0 \]
\[ 0.0175 (\gamma + 80)^2 \leq F(2 - b) \]
\[ (\gamma + 80)^2 \leq \frac{F(2 - b)}{0.0175} \]
\[ \gamma + 80 \leq \sqrt{\frac{F(2 - b)^2}{0.0175}} \]
\[ \gamma \leq \frac{\sqrt{F(2 - b)^2}}{0.0175} - 80 \]

Now to interpret the condition. If \( b \) is relatively close to 1, then the right-hand side of the expression will be large and the left-hand side is more likely to be less than it. If \( F \) is large in general and \( b \) is not close to 2, then the left-hand side is more likely to be less than it. Suppose that \( b \) is equal to 1 so that the merged firm has the same fixed costs as one of the merging firms. Then we can write the above expression as

\[ \gamma \leq \frac{\sqrt{F}}{0.13228756} - 80 \]
Large values of $F$ (with $b = 1$) will make it more likely a merger is profitable. Low values of $\gamma$ will also encourage the merger.

(b)
From Chapter Problem 1c, we have the following optimal quantities and market price, which are not affected by fixed costs.

\[ q_m = q_2 = q_3 = 20 \]
\[ P = 40 \]

Profits are now given by

\[ \pi_m = 400 - bF \]
\[ \pi_3 = \pi_4 = 400 - F \]

Combined profits for firms 1 and 4 with no merger are

\[ \pi_1 + \pi_4 = 512 - \frac{96}{5} \gamma + \frac{17}{25} \gamma^2 - 2F \]

The merger is profitable if

\[ \pi(merge) = 400 - bF \geq 512 - \frac{96}{5} \gamma + \frac{17}{25} \gamma^2 - 2F = \pi(sum) \]

We can rearrange this expression to obtain

\[ \frac{17}{25} \gamma^2 - \frac{96}{5} \gamma + 112 \leq F(2-b) \]

Now to interpret the condition. If $b$ is relatively close to 1, then the right-hand side of the expression will be large and the left-hand side is more likely to be less than it. If $F$ is large in general and $b$ is not close to 2, then the left-hand side is more likely to be less than it. We can repeat the diagram from Chapter Problem 1 part c.

**Incentives to Merge**

\[ \frac{17}{25} \gamma^2 + \frac{96}{5} \gamma + 112 \]

\[ F = 30, b = 1.2 \]
\[ F = 30, b = 1.5 \]
\[ F = 20, b = 1.5 \]

(c)
Mergers that create cost savings by economizing on fixed costs or by eliminating high cost firms are more likely to be profitable.
Problem 3

(a)

We now have a market with three firms, two identical and one (the leader) different. Denote the firms $I$, 3 and 4, where $I$ denotes the combined leader firm.

First consider the response of the two follower firms. For the third firm profit is given by

$$\pi_3 = P q_3 - 20q_3$$

$$= (100 - q_t - q_3)q_3 - 20q_3$$

$$= 100q_3 - q_t q_3 - q_3^2 - 4q_3^2 - 20q_3$$

If we maximize profit taking the other outputs as given, we obtain $q_3$ as a function of the other firms' outputs.

$$\frac{d\pi_3}{dq_3} = 100 - q_t - 2q_3 - q_3^2 - 20 = 0$$

$$\rightarrow 2q_3 = 80 - q_t - q_3$$

$$\rightarrow q_3 = \frac{80 - q_t - q_3}{2}$$

Since firm 4 and firm 3 are symmetric we know that $q_4 = q_3$. This will then give

$$q_3 = \frac{80 - q_t - q_4}{2}$$

$$\rightarrow q_3 = \frac{80 - q_t - q_4}{2}$$

$$\rightarrow \frac{3}{2}q_t = 40 - \frac{q_t}{2}$$

$$\rightarrow q_3 = q_4 = 26.66 - \frac{q_t}{3}$$

The total output is

$$q_3 + q_4 = 53.33 - \frac{2q_t}{3}$$

Now consider the profit maximization problem of the leader firm. This firm will take into account the best response function of the follower firms.

$$\pi_t = P q_t - 20q_t$$

$$= (100 - q_t - q_3 - q_4)q_t - 20q_t$$

$$= 100q_t - q_t^2 - q_3q_t - q_4q_t - 20q_t$$

$$= 100q_t - q_t^2 - (26.666 - \frac{q_t}{3})q_t - (26.666 - \frac{q_t}{3})q_t - 20q_t$$

$$= 100q_t - q_t^2 - (53.333 - \frac{2q_t}{3})q_t - 20q_t$$

$$= 80q_t - q_t^2 - 53.333q_t + \frac{2}{3}q_t^2$$

$$= 26.666 - \frac{1}{3}q_t^2$$

If we maximize profit we obtain $q_t$ as a function of the other firms' outputs.
\[ \pi_i = 26.66\bar{6}q_i - \frac{1}{3}q_i^3 \]
\[ \frac{d\pi_i}{dq_i} = 26.66\bar{6} - \frac{2}{3}q_i \]
\[ \Rightarrow \frac{2}{3}q_i = 26.66\bar{6} \]
\[ \Rightarrow q_i = 40 \]

If we substitute in for the follower firms, we obtain
\[ q_3 = q_4 = 26.66\bar{6} - \frac{1}{3}q_i \]
\[ = 26.66\bar{6} - \frac{1}{3}(40) \]
\[ = 26.66\bar{6} - 13.333 \]
\[ = 13.333 \]

The price is given by
\[ P = 100 - q_1 - q_3 - q_4 = 33 \frac{1}{3} \]

The profits for each symmetric follower firm are given by
\[ \pi_i = Pg_i - 20q_i, \quad i = 3, 4 \]
\[ = (33 \frac{1}{3})(13 \frac{1}{3}) - (20)(13 \frac{1}{3}) \]
\[ = (13 \frac{1}{3})^2 \]
\[ = 177.77 \]

Profit for the leader firm is
\[ \pi_i = Pg_i - 20q_i \]
\[ = (33 \frac{1}{3})(40) - (20)(40) \]
\[ = (13 \frac{1}{3})(40) \]
\[ = 533 \frac{1}{3} \]

This is larger than the sum of the profits from problem 1b, which were
\[ 2\pi_i = (2)(256 + \frac{32}{5}\gamma + \frac{1}{25}\gamma^2) \]
\[ = 512 + \frac{64}{5}\gamma + \frac{2}{25}\gamma^2 \]
\[ = 512 + 12.8\gamma + \frac{2}{25}\gamma^2 \]

since if \( \gamma = 0 \) then 533 1/3 > 512. The product price of 33 1/3 is lower than the price of 36 in 1a or 40 in 1b. The profits of the non-merged of follower firms (177.77) are smaller than in either 1a (256) or 1b (400) when \( \gamma = 0 \).
(b) In this situation we will have two Cournot competitors in the leader group. Thus, we can solve the problem as a Cournot duopoly. The firms will be symmetric. Denote them as R and m for the initial leaders and followers who merge. Profits for the newly merged firm are

\[ \pi_m = P(q_m - 20q_m) \]
\[ = (100 - q_l - q_m)q_m - 20q_m \]
\[ = 100q_m - q_lq_m - q_m^2 - 20q_m \]

If we maximize profit we obtain \( q_m \) as a function of the other firm's output.

\[ \frac{d\pi_m}{dq_m} = 100 - q_l - 2q_m - 20 = 0 \]
\[ \sim 2q_m = 80 - q_l \]
\[ \sim q_m = \frac{80 - q_l}{2} \]
\[ = 40 - \frac{q_l}{2} \]

Since the two firms are symmetric we know that \( q_R = q_m \). This will then give

\[ q_m = \frac{80 - q_l}{2} \]
\[ q_m = \frac{80 - q_m}{2} \]
\[ \sim \frac{3}{2}q_m = 40 \]
\[ \sim q_m = 26.66 \]

The price is given by

\[ P = 100 - 26.666 - 26.666 \]
\[ = 46.666 \]

The profit for each firm is given by

\[ \pi_l = Pq_l - 20q_l \]
\[ = (46 \frac{2}{3})(26 \frac{2}{3}) - (20)(26 \frac{2}{3}) \]
\[ = (26 \frac{2}{3})^2 \]
\[ = 711.11 \]

This is much larger than the sum of follower firm profits each of which was 177.77.

Problem 4

(a) The total profit of the firms 1 and 2 after merger as a Stackelberg leader is 533 1/3. The sum of the profits from being part of a Cournot market is 512. Thus a total fixed cost for both firms of more than 533 1/3 - 512 = 21 1/3 would make the merger not desirable.

(b) In the Stackelberg equilibrium, the follower firms each make 177.77 for a total profit of 355.55. In the two firm Cournot model the profit of each member of the duopoly has a profit of 711.11. If the cost of merging is more than 711.11 - 355.55 = 355.55, then these two firms will not merge.
The following table summarizes the results:

<table>
<thead>
<tr>
<th>Model</th>
<th>Q</th>
<th>P</th>
<th>q1 &amp; q2</th>
<th>q3 &amp; q4</th>
<th>q1</th>
<th>q2</th>
<th>q3</th>
<th>q4</th>
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<tr>
<td>Monopoly</td>
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<td>Cournot (4 firms)</td>
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<td>Stackelberg (2 followers)</td>
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<td>Cournot (2 firms)</td>
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<td>26.66</td>
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<td>26.66</td>
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