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Symmetry and order in the portfolio allocation problem*

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Summary. This research studies the role of multivariate distribution structures on random asset returns in determining the optimal allocation vector for an expected utility maximizer. All our conclusions pertain for the set of risk averters. By carefully disturbing symmetry in the distribution of the, possibly covarying, returns, we ascertain the ordinal structure of the optimized allocation vector. Rank order of allocations is also established when a permutation symmetric random vector is mapped into the returns vector through location and scale shifts. It is shown that increased dispersion in the vectors of location and scale parameters benefit, ex-ante, investors as does a decrease in the rank correlation coefficient between the location and scale parameter vectors. Revealed preference comparative static results are identified for the location and scale vectors of asset returns. For most issues addressed, we arrive at much stronger inferences when a safe asset is available.

Keywords and Phrases: Arrangement increasing, Location and scale, Majorization, Ordinal structure, Permutation symmetry, Revealed preference, Schur-concave.

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1 Introduction

The theory of optimum portfolio allocation responses to distribution shifts is by now extensive, at least so far as the expected utility framework is concerned.

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However, the implications of distribution structure for order in the asset allocation vector is not nearly as well understood. For allocation among two assets with independently distributed returns, Hadar and Seo [10] have shown that the size and slope of a risk averter's coefficient of relative risk aversion are critical.[1] With multivariate independently distributed returns, Landsberger and Meilijson [14] have found that total order among the marginal distributions, in the likelihood ratio sense, induced total order on the optimal allocation vector for agents with monotone nondecreasing utility. Correspondingly, Kijima and Ohnishi [12] have demonstrated that total order, in the reversed hazard order sense, among independently distributed marginals is sufficient to induce total order on allocative choices for optimizing risk averters. Turning to the literature on portfolio comparative statics under dependent distribution structures, Meyer and Ormiston [17] identified conditions such that a univariate distribution shift in an environment of bivariate dependent returns had determinate optimum allocation impacts. Mitchell and Douglas [19] extended that analysis to a multivariate distribution. But the structure of the allocation vector in a dependent risk environment has received almost no attention. The most penetrating study to date has been by Kijima and Ohnishi [12] who studied decisionmakers with increasing utility and also the subset that are risk averters. They found conditions on a dependent bivariate distribution under which more capital is allocated to a specific asset among the two available.

As in Hadar and Seo [10], Landsberger and Meilijson [14], and Kijima and Ohnishi [12], the main focus of our research is to identify structure on a distribution function such that order is induced on the optimal choice vector. Like Kijima and Ohnishi, we consider dependent risk structures. Our approach, however, differs from the earlier works in that we study almost symmetric n-variate stochastic environments where asymmetries enter only through a parameter vector. We take this approach because we intend to control the asymmetries in the stochastic environment so as to identify when such asymmetries assuredly affect the allocation vector. The purpose is to establish a benchmark model from which to derive intuition concerning what is critical about the stochastic structure in determining allocative order when there is dependence among risks. Also in contrast with previous research, we inquire into order on investor preferences over investment environments. A third distinguishing feature in our work is that we use revealed preference arguments to establish compensated comparative statics for arbitrary multivariate distributions.

The analysis begins with a presentation of the problem that we seek to shed light on. This is followed by studies of how asymmetries in a strongly structured distribution induce allocative order for the class of agents with nondecreasing utility. Then we investigate asymmetries that arise in location and scale mappings of symmetrically distributed sources of randomness. In these latter sections, the heterogeneity takes place in the utility function rather than in the distribution

[1] Specifically, when asset returns are ordered in the usual first-degree sense and the coefficient of relative risk aversion exceeds unity, then the majority of funds are allocated to the asset with the dominating marginal.

function. Whether the opportunity to invest in a risk-free bond affects decisions is an important issue, and we use our model to identify some implications for allocations that arise from the existence of such an opportunity. We then inquire into what revealed preference arguments might have to relate about portfolio comparative statics with respect to the vector of mean returns. Our revealed preference analysis yields stronger conclusions when a safe asset is available. In the final main section we place our findings in context with existing results on portfolio allocation in bivariate and multivariate settings, and we strengthen findings for the two-asset model.

2 Portfolio allocation problem

At time 0 in a two-period model, an economic agent with wealth W_0 has n available investment assets paying gross return $\$x_i$ at time 1 per \$1 invested at time 0. At no loss of generality we assume that $W_0 = 1$. We assume that preferences may be represented by a von-Neumann & Morgenstern preference function so that the problem (P) is

$$\operatorname*{Max}_{\vec{a}\in S} E\left[U\left(\sum_{i=1}^{n} a_i x_i\right)\right] \quad \text{s.t.} \quad \sum_{i=1}^{n} a_i = 1\,. \tag{2.1}$$

where $S \in \mathbb{R}^n$ is a compact convex choice set the size of which will be declared when we consider each fully specified optimization problem, where $\vec{a} = (a_1, a_2, \ldots, a_n)$, and where $E[\cdot]$ is the expectation operator over the distribution of $\vec{x} \in \mathbb{R}^n_+$, the nonnegative orthant.[2,3] While this problem may not have a unique solution, we will identify the structure that any solution must have given alternate assumptions on utility function $U(\cdot)$ and on the multivariate distribution of expected returns. A solution vector to problem (P) is identified by $\vec{a}^*$. We denote by U_1^* (respectively U_2^*) the set of all agents solving (P) who have continuously differentiable and nondecreasing (respectively, twice continuously differentiable, nondecreasing and weakly concave) utility functions.

3 Arrangement increasing order and nondecreasing utility

Before we can be assured of order on decisions, it is necessary to impose structure on the stochastic environment. Underlying the main finding in this section is the concept of arrangement increasing. The concept generates stochastic structure sufficient to rank order asset allocations for U_1^* decisionmakers. The two key ideas required to identify our deduction pertain to invariance under a class of

[2] Later, we will use $\vec{x}$ to denote the random component of a location and scale parameterized returns vector. While $\vec{x}$ may then be negative, we assume that the vector of gross returns is always nonnegative. This is true when investment opportunities take the form of limited liability companies and transactions costs are zero.

[3] As there should be no reason for confusion, for convenience we do not use the conventional transpose notation on vectors written horizontally.

interchange operations and to a notion of order when there is a sufficiently structured departure from that class of invariance operations. There are two invariance concepts that will have fruitful implications for us. The first is invariance in a parameter space:

Definition 3.1. (Pečarić et al. [21, p. 377]) Consider the function $g(\vec{x}, \vec{\lambda})$ from $\mathbb{R}^n \times \mathbb{R}^n$ onto $\mathbb{R}$, and let the operation $\vec{\lambda} \circ \tau$ map $\vec{\lambda}$ onto $(\lambda_{\tau(1)}, \lambda_{\tau(2)}, \dots, \lambda_{\tau(n)})$. Then $g(\vec{x}, \vec{\lambda})$ is said to be *permutation invariant* if $g(\vec{x} \circ \tau, \vec{\lambda} \circ \tau) = g(\vec{x}, \vec{\lambda})$ for all permutations.

Thus, $x_1^{\lambda_1} x_2^{\lambda_2}$ is permutation invariant whereas $x_1^{\lambda_1} x_2^{\lambda_2} + \lambda_1 x_1$ is not. Permutation invariance is used to assure some symmetry when the, yet to be introduced, arrangement increasing property is invoked by re-permuting vector elements. A globally more restrictive, yet less structured, form of invariance is permutation symmetry.

Definition 3.2. The distribution function, $F(\vec{x})$, is said to be *permutation symmetric* if for every permutation $\tau = \{\tau_1, \tau_2, \dots, \tau_n\}$ of $\{1, 2, \dots, n\}$ we have $F(\vec{x}) \equiv F(\vec{x}_\tau)$.

An alternative term for the permutation symmetry concept is *exchangeability*. Function $x_1^{\lambda_1} x_2^{\lambda_2}$ is permutation symmetric over a non-trivial domain if and only if $\lambda_1 \equiv \lambda_2$. Our immediate concern is with permutation invariance, but later sections will study implications of permutation symmetric distributions. In comparing Definitions 3.1 and 3.2, note that a permutation invariant distribution function that can be represented by the parameter vector $\vec{\lambda}$ is permutation symmetric if $\vec{\lambda} \equiv \bar{\lambda}\vec{1}$, with $\bar{\lambda}$ a scalar. In that sense, permutation invariance allows for more general representations of the multivariate distribution. On the other hand, permutation invariance is a meaningful concept only if the distribution function can be parameterized whereas permutation symmetry admits consideration of nonparametric distributions.

We turn now to an order concept that is motivated by the permutation invariance property.

Definition 3.3. Parametric function $g(\vec{x}, \vec{\lambda}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to exhibit *permutation order* if $g(\vec{x}, \vec{\lambda}_\tau) \geq g(\vec{x}, \vec{\lambda})$ for all permutations τ of $\{1, 2, \dots, n\}$ which

a) interchange two indices j and k for which $(x_j - x_k)(\lambda_j - \lambda_k) \leq 0$, and
b) leaves all other indices unchanged.

Upon conjunction, permutation invariance and permutation order identify the property of principal concern to this section.

Definition 3.4. A function, $g(\vec{x}, \vec{\lambda}) : (\mathbb{R}^n)^2 \to \mathbb{R}$, is said to be *arrangement increasing* (*AI*) if it exhibits permutation invariance and permutation order. It is said to be an *arrangement decreasing* (*AD*) *function* if $-g(\vec{x}, \vec{\lambda})$ is AI.

These functions have been studied in detail by Marshall and Olkin [15], Hollander et al. [11], and Boland et al. [2], among others. A simple AI function that will be central to our analysis is the inner product specification,

$h_1(\vec{a},\vec{x}) = \vec{a}\cdot\vec{x} = \sum_{i=1}^{n} a_i x_i$. We will now show how Definition 3.4 can be invoked to induce a partial ordering on a vector pair. Consider vector pair $\{\vec{a},\vec{x}\}$ where $\{\vec{a},\vec{x}\} = \{(1,5,4,2),(6,5,8,1)\}$. Define $\{\vec{a}\uparrow,\vec{x}^1\} = \{(1,2,4,5),(6,1,8,5)\}$ where the permutation mapping $\tau(i) : (1 \to 1, 2 \to 4, 3 \to 3, 4 \to 2)$ has been applied to both vectors. Consistent with Definition 3.1, this mapping will leave an AI function invariant. Define $\{\vec{a}\uparrow,\vec{x}^2\} = \{(1,2,4,5),(1,6,8,5)\}$, $\{\vec{a}\uparrow,\vec{x}^3\} = \{(1,2,4,5),(1,5,8,6)\}$, and $\{\vec{a}\uparrow,\vec{x}^4\} = \{1,2,4,5),(1,5,6,8)\}$. Notice that, for $k \in \{2,3,4\}$, $\{\vec{a}\uparrow,\vec{x}^k\}$ can be obtained from $\{\vec{a}\uparrow,\vec{x}^{k-1}\}$ through a permutation order interchange as described in Definition 3.3.[4] Therefore, for $h_1(\vec{a},\vec{x}) = \vec{a}\cdot\vec{x}$ we have $h_1(\vec{a},\vec{x}) = 65 = h_1(\vec{a}\uparrow,\vec{x}^1) \le 70 = h_1(\vec{a}\uparrow,\vec{x}^2) \le 73 = h_1(\vec{a}\uparrow,\vec{x}^3) \le 75 = h_1(\vec{a}\uparrow,\vec{x}^4)$. The best (worst) possible arrangement of a vector pair is the one that maximizes (minimizes) the value of an AI function. It is called the maximal (minimal) arrangement, and $\{\vec{a}\uparrow,\vec{x}^4\}$ is one representation of the maximal arrangement of $\{\vec{a},\vec{x}\}$ while a representation of the minimal arrangement is $\{(1,2,4,5),(8,6,5,1)\}$.

Other useful AI functions include $h_2(\vec{a},\vec{x}) = I_A$ where I_A is the indicator function which equals one when condition A holds and zero otherwise, and where A is the condition that $a_i \le x_i \ \forall\ i \in [1,\ldots,n]$. Functions of the form $h_3(\vec{a},\vec{x}) = \prod_{i=1}^{n} g(a_i,x_i)$ are AI if and only if $g(a_i,x_i)$ is totally positive of order two (TP_2) in a_i and x_i. For twice continuously differentiable functions, this property requires that $\partial^2 \mathrm{Ln}[g(a_i,x_i)]/\partial a_i \partial x_i \ge 0$. Functions of the form $h_4(\vec{a},\vec{x}) = g(\vec{a}+\vec{x})$ are AI if and only if $g(\vec{a}+\vec{x})$ is a Schur-convex function. Notice that the inequalities in Definition 3.3 are not strict. It has been shown by Hollander et al. [11] that the AI property is preserved under convolution. The underpinning of our main result in this section arises from a convolution of AI functions.

Proposition 3.1. *For an n-variate arrangement increasing density function of returns on risky assets with parameter vector $\vec{\lambda} \in \mathbb{R}^n$, and for all nondecreasing $U(\cdot)$, expected utility is an AI function of $\vec{a} \in \mathbb{R}^n$ and $\vec{\lambda} \in \mathbb{R}^n$.*

Proof. For a continuous AI density function represented by $g(\vec{x},\vec{\lambda})$, we wish to show that

$$M(\vec{\lambda},\vec{a}) = \int_{\mathbb{R}^n_+} U(\vec{a}\cdot\vec{x})g(\vec{x},\vec{\lambda})d\vec{x} \tag{3.1}$$

is AI.[5] Clearly, $M(\vec{\lambda},\vec{a})$ is permutation invariant. And so we need only show that it is nondecreasing under a permutation τ of two indices j and k, fixing other indices, for which $(a_j - a_k)(\lambda_j - \lambda_k) \le 0$. Without loss of generality, suppose that $\lambda_j \le \lambda_k$. Then

[4] Each interchange is said to be an AI rearrangement (see, e.g., Boland et al. [2]).

[5] Some AI density functions, such as the multinomial, are defined on counting measures. The argument to follow still carries through if an infinite sum replaces the integration.

$$\begin{aligned} M(\vec{\lambda}, \vec{a}_\tau) - M(\vec{\lambda}, \vec{a}) &= \int_{\mathbb{R}^n_+, x_j \geq x_k} U(\vec{a}_\tau \cdot \vec{x}) g(\vec{x}, \vec{\lambda}) d\vec{x} \\ &+ \int_{\mathbb{R}^n_+, x_j < x_k} U(\vec{a}_\tau \cdot \vec{x}) g(\vec{x}, \vec{\lambda}) d\vec{x} \\ &- \int_{\mathbb{R}^n_+, x_j \geq x_k} U(\vec{a} \cdot \vec{x}) g(\vec{x}, \vec{\lambda}) d\vec{x} - \int_{\mathbb{R}^n_+, x_j < x_k} U(\vec{a} \cdot \vec{x}) g(\vec{x}, \vec{\lambda}) d\vec{x} \\ &= \int_{\mathbb{R}^n_+, x_j \geq x_k} [(U(\vec{a} \cdot \vec{x}) - U(\vec{a}_\tau \cdot \vec{x})][g(\vec{x}, \vec{\lambda}_\tau) - g(\vec{x}, \vec{\lambda})] d\vec{x} \end{aligned} \tag{3.2}$$

where a transposition $x_j \leftrightarrow x_k$ on the relevant domain has been employed to map into the half-orthant $\mathbb{R}^n_+, x_j \geq x_k$.[6] Because, on the domain of integration, $(x_j - x_k)(\lambda_j - \lambda_k) \leq 0$, therefore $g(\vec{x}, \vec{\lambda}_\tau) \geq g(\vec{x}, \vec{\lambda})$ by the AI property. Viewing the integrand of the last expression in (3.2), it is then immediate from the monotonicity of $U(\cdot)$, and the AI property of $\vec{a} \cdot \vec{x}$, that $U(\vec{a} \cdot \vec{x}) \geq U(\vec{a}_\tau \cdot \vec{x})$ if and only if $a_j \geq a_k$. Thus, $M(\vec{\lambda}, \vec{a}_\tau) \geq M(\vec{\lambda}, \vec{a})$ if and only if $a_j \geq a_k$. □

Corollary 3.1. *Consider an n-variate AI density function of returns on risky assets with parameter vector $\vec{\lambda} \in \mathbb{R}^n$. Arbitrarily set $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then the optimal allocation $\vec{a}$ for any $U \in U_1^*$ is such that $a_1^* \geq a_2^* \geq \ldots \geq a_n^*$.*

Proof. The AI property ensures that, among all arrangements of given vector pairs $\{\vec{a}, \vec{\lambda}\}$, the maximal arrangement maximizes expected utility. Being true for all $\vec{a}$, it must be true for any optimum portfolio allocation vector. And so $(a_k^* - a_j^*)(\lambda_k - \lambda_j) \geq 0$. □

To illustrate the applicability of the result, we now present some examples of AI density functions.

Example 3.1. Proposition 3.1 applies if the joint density is the (permutation invariant) multivariate Dirichlet,

$$g(\vec{x}, \vec{\lambda}) = \frac{\Gamma\left(\theta + \sum_{i=1}^n \lambda_i\right)}{\Gamma(\theta) \prod_{i=1}^n \Gamma(\lambda_i)} \left(1 - \sum_{i=1}^n x_i\right)^{\theta - 1} \prod_{i=1}^n x_i^{\lambda_i - 1}, \tag{3.3}$$

where $\{\lambda_i > 0, x_i \geq 0\}\ \forall\ i \in \{1, 2, \ldots, n\}$, $\sum_{i=1}^n x_i \leq 1$, and $\theta > 0$.[7] Notice that $\Gamma(\theta + \sum_{i=1}^n \lambda_i)(1 - \sum_{i=1}^n x_i)^{\theta-1} / [\Gamma(\theta) \prod_{i=1}^n \Gamma(\lambda_i)]$ is permutation symmetric in $\vec{x}$ and $\vec{\lambda}$ whereas $\prod_{i=1}^n x_i^{\lambda_i - 1}$ is permutation invariant.

More generally, the multivariate Liouville distribution studied by Gupta and Richards [6, 7, 8, 9], a family that encompasses distributions of correlated gamma variables, the Dirichlet distribution and the inverted Dirichlet distributions, and has the density representation

[6] When adapting for counting measures, note that $U(\vec{a} \cdot \vec{x}) \equiv U(\vec{a} \cdot \vec{x}_\tau)$ whenever $x_j = x_k$.

[7] The inequality $\sum_{i=1}^n x_i \leq 1$ may not adhere, but if asset returns are bounded then there exists some number K such that $\sum_{i=1}^n x_i \leq K$. Scaling random variables by K demonstrates that the restriction is not onerous.

$$g(\vec{x},\vec{\lambda}) = h(\vec{\beta},\vec{\lambda})\, f\left(\sum_{i=1}^{n} x_i\right)\prod_{i=1}^{n} x_i^{\lambda_i-1}\,, \tag{3.4}$$

with $h(\vec{\beta},\vec{\lambda})$ permutation symmetric in $\vec{\beta}$, is AI in $\vec{x}$ and $\vec{\lambda}$. The AI property is also satisfied by the multivariate F distribution, which has joint density function

$$g(\vec{x},\vec{\lambda}) = \frac{\Gamma\left(\sum_{i=1}^{n}\lambda_i\right)\prod_{i=1}^{n}\lambda_i^{\lambda_i}\prod_{i=1}^{n}x_i^{\lambda_i-1}}{\left(\lambda_0+\sum_{i=1}^{n}\lambda_i x_i\right)^{\sum_{i=0}^{n}\lambda_i}\prod_{i=1}^{n}\Gamma(\lambda_i)}\,, \tag{3.5}$$

where $\lambda_i > 0\ \forall\ i \in \{0,1,\ldots,n\}$ and $x_i \geq 0\ \forall\ i = 1,2,\ldots,n$.

Other AI density functions include the multinomial, negative multinomial, multivariate hypergeometric, negative multivariate hypergeometric, and multivariate Pareto distributions.[8] This class of distributions can be generalized by use of transformations. For example, if an arbitrary monotonic increasing transformation is performed on each variable in an AI density function, then the transformed density function is also AI.[9] Thus the lognormal distribution generated by a normal distribution satisfying the AI property is also AI.[10] □

Example 3.2. Asset returns $\{x_1,x_2,x_3\}$ are initially ascribed the continuously differentiable source marginal distributions $\{F(x_1,\lambda_1),F(x_2,\lambda_2),F(x_3,\lambda_3)\}$. These marginals are then passed through the copula quantile function $C(u_1,u_2,u_3)$ where $u_i = F_i(x_i,\lambda_i)$, $i \in \{1,2,3\}$.[11] To illustrate, we choose the trivariate Cook-Johnson copula as described in Klugman and Parsa [13];

$$C(u_1,u_2,u_3) = \left[u_1^{-\beta}+u_2^{-\beta}+u_3^{-\beta}-2\right]^{-1/\beta}\,,\quad \beta \geq 0\,. \tag{3.6}$$

This copula, defined on $C(u_1,u_2,u_3) : \{[0,1]\}^3 \to [0,1]$, is continuous and monotone nondecreasing. With reference to the constant elasticity of substitution

[8] The multivariate normal distribution with common variance $\sigma_i^2 = \sigma^2\ \forall\ i = 1,\ldots,n$, and common correlation $\rho_{ij} = \rho\ \forall\ i \in \{1,\ldots,n\}$, $\forall\ j \in \{1,\ldots,n\}$, $j \neq i$, $\rho > -1/(n-1)$ is AI where parameter vector $\vec{\lambda}$ is the vector of means. Tong [27, pp. 86–87] provides further details. For the multivariate normal, our Proposition 3.1 is easily demonstrated by other methods. For other AI densities, see Hollander et al. [11] and Boland [3].

[9] See Marshall and Olkin [15, pp. 160–162].

[10] Many AI densities are comprised partly of products of TP_2 functions, i.e., density functions of form $f^*(\vec{x},\vec{\lambda}) = A(\vec{x},\vec{\lambda})\prod_{i=1}^{n} g(x_i,\lambda_i)$ where $A(\vec{x},\vec{\lambda})$ is permutation symmetric in $\vec{\lambda}$ and also permutation symmetric in $\vec{x}$ and where $g(x_i,\lambda_i)$ satisfies $g(x_j,\lambda_j)g(x_k,\lambda_k) \geq g(x_j,\lambda_k)g(x_k,\lambda_j)$ whenever $(\lambda_j-\lambda_k)(x_j-x_k) \geq 0$. Densities of form $f^*(\vec{x},\vec{\lambda})$ are an important subset of the set of affiliated densities, which have been used widely in economic analyses (see, e.g., Milgrom and Weber [18]). Affiliated densities need not be AI while AI densities need not be affiliated.

[11] Sklar [26] has defined an n-dimensional copula (an n-copula) as a function $C(\cdot)$ from the unit n-cube $[0,1]^n$ to the unit interval $[0,1]$ which satisfies the following conditions;

(1.1) $C(1,\ldots,1,a_m,1,\ldots,1) = a_m$ for each $m \leq n$ and all $a_m \in [0,1]$.
(1.2) $C(a_1,\ldots,a_n) = 0$ if $a_m = 0$ for any $m \leq n$.
(1.3) $C(\cdot)$ is n-increasing in the sense that the C-volume of any n-dimensional interval is nonnegative. Thus, if $C(\cdot)$ is 2-increasing then $C(a_2,b_2)+C(a_1,b_1) \geq C(a_2,b_1)+C(a_1,b_2)$ whenever $(a_2-a_1)(b_2-b_1) \geq 0$.

function, parameter β may be viewed as the elasticity of substitution between the source distributions. The copula has the derivative properties required to ensure that $G(x_1, x_2, x_3) = C[F_1(x_1, \lambda_1), F_2(x_2, \lambda_2), F_3(x_3, \lambda_3)]$ is a trivariate distribution. Finally, if $\partial F(x_i, \lambda_i)/\partial x_i$ is TP_2 then the density of expression (3.6) is AI in $\vec{x}$ and $\vec{\lambda}$. $\square$

The pairing inherent in the AI ordering means that there is a one-for-one matching between parameters and assets. Proposition 3.1 works in part because AI distributions impose a measure of separability on the marginal distributions. It is impossible to obtain strong portfolio allocation results for arbitrary distributions and preference structures because of interactions between marginal densities as expressed by covariances and higher cross moments. In AI densities, these second and higher cross moments are either absent or are carefully structured.

It is often the case that a risk-free asset is available for investment, so that the stochastic payoff is $h(a_0, \vec{a}, \vec{x}) = a_0 x_0 + \sum_{i=1}^{n} a_i x_i$ where x_0 is the time 1 value of a time 0 dollar invested in (safe) bonds, and a_0 is the time 0 dollar investment in bonds. This function continues to be AI in $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{x}$, so a variant of Corollary 3.1 continues to apply. Specifically, when there is a risk-free asset, assume that the rates of return on risky assets 1 through n follow an AI density with parameter vector $\vec{\lambda} \in \mathbb{R}^n$. Then all $U \in U_1^*$ will allocate funds among risky assets such that $(a_k^* - a_j^*)(\lambda_k - \lambda_j) \geq 0$ for all $j, k \in \{1, \ldots, n\}$. Thus Corollary 3.1 still holds, but only with reference to allocations among risky assets. In a later section some nontrivial implications of expanding the opportunity set to include a risk-free asset will be established.

A second inference that is immediate from Corollary 3.1 pertains to the preservation of allocative order under a ray expansion in parameter space.

Corollary 3.2. *Let the density function of time* 1 *gross returns on assets purchased at time* 0 *be AI. Then the rank order of the vector of portfolio shares for all* $U \in U_1^*$ *is invariant to a scalar expansion of* $\vec{\lambda}$.

The corollary is not valid when there is a risk-free asset, but the ordinal structure of the risky asset allocation vector is invariant to a scalar expansion of $\vec{\lambda}$.

4 Permutation symmetry and nondecreasing utility

In contrast to the preceding section, Sections 4 and 5 place complete symmetry on the sources of randomness. Instead, the asymmetries arise in the mappings of random draws onto the rates of return that enter the terminal payoff, π, in the utility function. In this section we assume that random returns, while not necessarily interchangeable, are location transformations of interchangeable primitive sources of randomness.

Proposition 4.1. *Let* $\vec{x} \in \mathbb{R}^n$ *be permutation symmetric random variables, and let* $z_i = r_i + x_i \ \forall \ i \in \{1, \ldots, n\}$ *where the* r_i *are scalars. Define* $\Omega(\vec{a}, \vec{r}) = E[U(\vec{a} \cdot \vec{r} + \vec{a} \cdot \vec{x})]$. *Then* $\Omega(\vec{a}, \vec{r})$ *is AI for any nondecreasing* $U(\cdot)$.

Proof. Let $a_i \geq a_j$ and $r_i \leq r_j$. Permutation symmetry implies that

$$E[U(\vec{a} \cdot \vec{r} + \vec{a} \cdot \vec{x})] = E[U(\vec{a} \cdot \vec{r} + \vec{a}_\tau \cdot \vec{x})] \,, \tag{4.1}$$

where τ is the permutation $i \leftrightarrow j$. Because $\vec{a} \cdot \vec{r}$ is AI and $U(\cdot)$ is nondecreasing, we have

$$E[(U(\vec{a}_\tau \cdot \vec{r} + \vec{a}_\tau \cdot \vec{x})] \geq E[U(\vec{a} \cdot \vec{r} + \vec{a}_\tau \cdot \vec{x})] \,, \tag{4.2}$$

and so $\Omega(\vec{a}, \vec{r})$ is AI.[12] □

Corollary 4.1. *Let $\vec{x}$ and $\vec{r}$ be as in Proposition* 4.1. *Arbitrarily set $r_1 \geq r_2 \geq \ldots \geq r_n$. Then the optimal allocation $\vec{a}$ for any $U \in U_1^*$ is such that $a_1^* \geq a_2^* \geq \ldots a_n^*$.*

Proof. Applying the proposition, for the arbitrarily chosen vector $\vec{a}^0$, continuous application of permutation order interchanges yields the permutation which maximizes $\Omega(\vec{a}^0, \vec{r})$ as that where $a_1^0 \geq a_2^0 \geq \ldots \geq a_n^0$. Finally, since $\vec{a}^0$ was arbitrary, the deduction must be true for optimum choice vector $\vec{a}^*$ also. □

We turn now to the welfare effects of order on the parameter vector of mean returns. The concept of importance here is that of majorization, a property on multivariate functions that has found use in the theory of income distribution (see Atkinson [1]), natural resource and regulatory economics (see Quiggin and Chambers [22]), and elsewhere.[13] Let $x_{(j)}$ denote the jth smallest ordinate in vector $\vec{x}$.

Definition 4.1. A vector $\vec{z}^{\$} \in \mathbb{R}^n$ is said to be *weakly majorized* by vector $\vec{z}^{\$\$} \in \mathbb{R}^n$ ($\vec{z}^{\$} \precsim \vec{z}^{\$\$}$) if: (1) $\sum_{j=1}^{k} z_{(j)}^{\$} \geq \sum_{j=1}^{k} z_{(j)}^{\$\$}$, $k = 1, \ldots, n-1$; and (2) $\sum_{j=1}^{n} z_{(j)}^{\$} = \sum_{j=1}^{n} z_{(j)}^{\$\$}$. A Schur-convex function preserves this order, i.e., if $\varphi(\vec{z})$ is Schur-convex, then $\vec{z}^{\$} \precsim \vec{z}^{\$\$} \Rightarrow \varphi(\vec{z}^{\$}) \leq \varphi(\vec{z}^{\$\$})$. The negative of a Schur-convex function is said to be Schur-concave.

For continuously differentiable functions an important means of representing the Schur-convexity property is the Ostrowski condition. For open interval $I \in \mathbb{R}$, function $\psi(\vec{z}) : I^n \to \mathbb{R}$ is Schur-convex if and only if:

(a) it is permutation symmetric, i.e., for permutation τ we have $\psi(\vec{z}) = \psi(\vec{z}_\tau)$, and
(b) $(z_i - z_j)[\psi_i(\vec{z}) - \psi_j(\vec{z})] \geq 0 \;\forall\; \vec{z}, \;\forall\; i,\; j \in (1, \ldots, n)$ where $\psi_k(\vec{z}) = \partial\psi(\vec{z})/\partial z_k$.[14] The latter representation of Schur-convexity is sufficient to establish our next finding.

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[12] Actually, the finding holds in a more general context. Let $z_i = h(x_i, \lambda_i)$ where $h(x_i, \lambda_i) : \mathbb{R}_+^2 \to \mathbb{R}_+$ is nondecreasing in λ_i. Then any $U \in U_1^*$ will allocate funds such that $(a_i^* - a_j^*)(\lambda_i - \lambda_j) \geq 0$. An example is the case of scale expansion parameters, $z_i = x_i \lambda_i$.

[13] The mean-preserving spread concept, due to Rothschild and Stiglitz [24], is an example of majorization.

[14] See Marshall and Olkin [15, pp. 5–7 and p. 57]. The Ostrowski condition clarifies that, unlike convexity, Schur-convexity is preserved under positive monotonic transformations.

Proposition 4.2. *Let $\vec{x}$ and $\vec{r}$ be as in Proposition 4.1. Assume that investors are $U \in U_2^*$ and that optimal allocations are interior, i.e., $a_j^* > 0\; \forall\, j \in \{1, \dots, n\}$. For mean return vectors $\vec{r}^{\alpha}$ and $\vec{r}^{\beta}$, if $\vec{r}^{\beta} \gtrsim \vec{r}^{\alpha}$ then expected utility is (weakly) larger under $\vec{r}^{\beta}$ than under $\vec{r}^{\alpha}$.*

Proof. For the environment in question, upon optimizing we obtain maximized expected utility as a function of the returns vector, and we write the maximized solution to problem (P) as $\varphi(\vec{r})$. Due to Definition 4.1, it is sufficient to establish that $\varphi(\vec{r})$ is Schur-convex. By construction, the function must be symmetric. And so, to conform to Ostrowski's condition, it suffices to demonstrate that $(r_i - r_j)[\partial\varphi(\vec{r})/\partial r_i - \partial\varphi(\vec{r})/\partial r_j] \geq 0$. Applying the envelope theorem to (P), we know that an increase in r_i so that there is deviation off the simplex $\vec{r} \cdot \vec{1} = K$, for K fixed, has the impact $\partial\varphi(\vec{r})/\partial r_i = a_i^* E[U'(\pi)]$. And so

$$(r_i - r_j)\left(\frac{\partial\varphi(\vec{r})}{\partial r_i} - \frac{\partial\varphi(\vec{r})}{\partial r_j}\right) = (r_i - r_j)(a_i^* - a_j^*)E[U'(\pi)]\,. \tag{4.3}$$

Invoking Proposition 4.1, the Ostrowski condition holds and so $\varphi(\vec{r})$ is Schur-convex. □

Thus, along the simplex $K = \vec{r} \cdot \vec{1}$, the more dispersed are the ordinates r_i the better for the asset allocator. The underpinning of this conclusion is in the envelope theorem. By Proposition 4.1, fund allocations will be concentrated on the assets with the larger values of the location parameter. A more dispersed vector of location parameters allows larger ex-ante gains from specialization. Because the welfare gains are proportional to asset holdings, a beneficial perturbation of a relatively large location parameter that is offset by a detrimental perturbation of a smaller location parameter will give rise to a net positive welfare gain to the investor.

Placing this intuition on a more formal footing, suppose that, in the n asset situation, we have $r_1 \to r_1 + \delta$, $r_2 \to r_2 - \delta$, $r_1 \geq r_2$, $\delta > 0$. If $\vec{a}^*$ does not alter then the random component of terminal payout, $\vec{a}^* \cdot \vec{x}$, does not change. But the deterministic component, $\vec{a}^* \cdot \vec{r}$, increases by $(a_1^* - a_2^*)\delta$. It is known (see Marshall and Olkin [15, p. 6]) that any majorization can be constructed from a sequence of simple transfers such as δ above when the initial point is $(r_1, r_2, \dots, r_n) = (\bar{r}, \bar{r}, \dots, \bar{r})$. And so expected utility must increase under majorization $\vec{r}^{\beta} \gtrsim \vec{r}^{\alpha}$ even if $\vec{a}^*$ is fixed, while any re-optimization only improves expected utility. Proposition 4.2 bears similarity to Oi's [20] demonstration that risk neutral decision makers who can react to unstable output prices prefer price instability. In our case the investor actually gains from inaction in response to the new vector of mean returns, and ex-ante gains are even larger when adjustments are permitted.

5 Permutation symmetry and risk aversion

Proposition 4.1 may be viewed as a rule for ranking assets in a portfolio when the rates of return are permutable up to a location shift, i.e., a first-degree stochastic dominance (FSD) type shift. It is rather apparent that if the expected returns on assets differ it will be difficult to obtain parallel results that correspond to shifts of the second-degree stochastic dominance type because the standard trade-off between mean and risk will vary across risk averters. We therefore consider what insights may be obtainable when all risky assets have the same expected return.

Thus, assume there are n risky assets, and the realized return on the jth asset is given by r_j:

$$r_j = \overline{r} + \lambda_j x_j \ , \quad \lambda_j > 0 \ , \quad E[x_j] = 0 \ , \quad \forall\, j \in \{1, \ldots, n\} \ , \tag{5.1}$$

where $\overline{r}$ is the return invariant mean. Further, assume that $(x_1, \ldots, x_n)$ are permutation symmetric random variables. Given initial wealth \$1, which may be invested in any of these risky assets, the budget constraint $\sum_{i=1}^{n} a_i = 1$ implies that realized wealth (π) is given by:

$$\pi = \overline{r} + \sum_{j=1}^{n} a_j \lambda_j x_j = \overline{r} + \sum_{j=1}^{n} c_j x_j \ ; \quad c_j \equiv a_j \lambda_j \ . \tag{5.2}$$

As will be apparent from the proof that follows, the conclusions are invariant to the level of wealth. We can write expected utility as:

$$U^e(\vec{c}) = \psi(c_1, \ldots, c_n) = E\left[U\left(\overline{r} + \sum_{j=1}^{n} c_j x_j\right)\right] \ ; \quad c_j = a_j \lambda_j \ ; \quad \sum_{j=1}^{n} a_j = 1 \ . \tag{5.3}$$

Our scale mapping result can now be demonstrated.

Proposition 5.1. *Consider n assets with returns $r_j = \overline{r} + \lambda_j x_j$ where $E[x_j] = 0$, $\lambda_j > 0\ \forall\, j \in \{1, \ldots, n\}$. Assume that $(x_1, \ldots, x_n)$ are permutation symmetric random variables, and denote realized wealth by π as given in (5.2). Assume that optimal allocations are interior, i.e., $a_j^* > 0\ \forall\, j \in \{1, \ldots, n\}$. Finally, order the assets so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then for $U \in U_2^*$ investors, $\vec{a}^*$ satisfies*

a) $a_n^* \geq \ldots \geq a_1^*$, and
b) $c_n^* \geq \ldots \geq c_1^*$, i.e., $\lambda_i \geq \lambda_j \Rightarrow a_j^* \geq a_i^*(\lambda_i/\lambda_j) \geq a_i^*$.

Proof. For convenience of presentation, we will discard the arrow representation of vectors in this proof. The portfolio problem is equivalent to

$$\begin{aligned} \mu(\lambda_1, \ldots, \lambda_n) &= \operatorname*{Max}_{c_1, \ldots, c_n} \psi(c_1, \ldots, c_n) \\ &\equiv \operatorname*{Max}_{c_1, \ldots, c_n} E\left[U\left(\overline{r} + \sum_{j=1}^{n} c_j x_j\right)\right] + \kappa\left[\sum_{j=1}^{n} (c_j/\lambda_j) - 1\right] \ , \end{aligned} \tag{5.4}$$

where $U \in U_2^*$ and κ is the LaGrange multiplier for budget constraint $\sum_{j=1}^{n} (c_j/\lambda_j) = 1$. From the optimality conditions and the assumption that the solution vector is interior on the unit simplex, it is readily demonstrated that $\kappa > 0$ and also that

$$\frac{d\psi(\theta c_1,\ldots,\theta c_n)}{d\theta} = E[U'(\pi)\omega] < 0 \tag{5.5}$$

where $\omega = \pi - \bar{r}$. And so $\psi(c_1,\ldots,c_n)$ is ray decreasing in $(c_1,\ldots,c_n)$, i.e., $\psi(\theta c_1,\ldots,\theta c_n) < \psi(c_1,\ldots,c_n)$ for $\theta > 1$.

Now consider any allocation vector, $(\hat{a}_1,\ldots,\hat{a}_n)$, such that $\sum_{s=1}^n \hat{a}_s = 1$ and $\hat{c}_s \equiv \lambda_s\hat{a}_s$. The demonstration that conclusion b) must hold at any optimum involves the construction of an alternative to $(\hat{a}_1,\ldots,\hat{a}_n)$ such that $n-2$ allocations are altered by just a scaling factor. Define vector $(a_1^+,\ldots,a_n^+)$ as follows; $a_s^+ = \hat{a}_s\ \forall\ s \neq \{i,j\}$, $a_i^+ = \hat{a}_j\lambda_j/\lambda_i$, $a_j^+ = \hat{a}_i\lambda_i/\lambda_j$. Thus, with $c_s^+ \equiv \lambda_s a_s^+$, it follows that $c_s^+ = \hat{c}_s\ \forall\ s \neq \{i,j\}$ while $c_i^+ = \hat{c}_j$ and $c_j^+ = \hat{c}_i$. So $(c_1^+,\ldots,c_n^+)$ is constructed through map τ, the two-element permutation $i \leftrightarrow j$ of the vector $(\hat{c}_1,\ldots,\hat{c}_n)$. It follows that

$$\begin{aligned} E\left[U\left(\bar{r}+\sum_{s=1}^n \lambda_s a_s^+ x_s\right)\right] &= E\left[U\left(\bar{r}+\sum_{s=1}^n \hat{c}_{\tau(s)} x_s\right)\right] \\ &= E\left[U\left(\bar{r}+\sum_{s=1}^n \hat{c}_s x_s\right)\right], \end{aligned} \tag{5.6}$$

where the latter equality is due to permutation symmetry. Now, in general, $\sum_{s=1}^n a_s^+ \neq 1$ so vector $(a_1^+,\ldots,a_n^+)$ does not necessarily represent a valid portfolio. We will show that it does not represent a valid portfolio if $(c_i^* - c_j^*)(\lambda_i - \lambda_j) > 0$ for the now given (but arbitrarily chosen) pair of assets $\{i,j\}$.

Define vector elements $\tilde{a}_s$ by $\tilde{a}_s = \delta a_s^+$ where $\delta = \left(\sum_{s=1}^n a_s^+\right)^{-1}$. By construction, $\sum_{s=1}^n \tilde{a}_s = 1$ and so vector $(\tilde{a}_1,\ldots,\tilde{a}_n)$ represents a valid portfolio. Therefore, defining $\tilde{c}_s = \lambda_s\tilde{a}_s$, we have

$$\begin{aligned} \psi(\tilde{c}_1,\ldots,\tilde{c}_n) &\equiv E\left[U\left(\bar{r}+\sum_{s=1}^n \lambda_s\tilde{a}_s x_s\right)\right] = E\left[U\left(\bar{r}+\delta\sum_{s=1}^n \hat{c}_{\tau(s)} x_s\right)\right] \\ &> (=)[<] E\left[U\left(\bar{r}+\sum_{s=1}^n \hat{a}_s\lambda_s x_s\right)\right] \end{aligned} \tag{5.7}$$

according as $\delta < (=)[>]1$. Here, the inequality is due to the fact that $\psi(\tilde{c}_1,\ldots,\tilde{c}_n)$ is ray decreasing, i.e., $d\psi(\tilde{c}_s,\ldots,\tilde{c}_n)/d\delta < 0$. Now note that

$$\begin{aligned} \sum_{s=1}^n a_s^+ &= \sum_{s=1}^n \hat{a}_s + [a_i^+ - \hat{a}_j + a_j^+ - \hat{a}_i] \\ &= 1 + \hat{a}_j[(\lambda_j/\lambda_i) - 1] + \hat{a}_i[(\lambda_i/\lambda_j) - 1] \\ &= 1 + \left[\frac{(\lambda_i - \lambda_j)(\lambda_i\hat{a}_i - \lambda_j\hat{a}_j)}{\lambda_i\lambda_j}\right] > 1 \end{aligned} \tag{5.8}$$

if $(\hat{c}_i - \hat{c}_j)(\lambda_i - \lambda_j) > 0$. Therefore, if $(\hat{c}_i - \hat{c}_j)(\lambda_i - \lambda_j) > 0$ then $\delta < 1$. But, by (5.7), this means that there exists the allocation vector $(\tilde{a}_1,\ldots,\tilde{a}_n)$ on the simplex $[0 < a_s < 1, s \in \{1,2,\ldots,n\} : \sum_{s=1}^n a_s = 1]$ such that $\psi(\tilde{c}_1,\ldots,\tilde{c}_n) >$

$\psi(\hat{c}_1, \ldots, \hat{c}_n)$. But this cannot be true if allocation $(\hat{a}_1, \ldots, \hat{a}_n)$ is optimal. Thus, for the optimized vector $(c_1^*, \ldots, c_n^*)$, we have $(c_i^* - c_j^*)(\lambda_i - \lambda_j) \leq 0$ and so conclusion b) holds. Conclusion a) is then immediate. □

Thus without imposing specific conditions, such as normality, on the distribution of returns, we have shown that there is an inverse correlation between the riskiness (variance) of an asset and its portfolio share. This result is insensitive to changes in wealth, while homogeneity of demand, in the rank order sense of Corollary 3.2, also holds.

The results in Proposition 5.1 can be paraphrased as follows: if asset j is the safer of two assets $\{i, j\}$, then not only will the risk averse investor allocate less of her portfolio to the riskier asset, but – on average – demand for that asset is elastic with respect to the magnitude of risk (i.e., the total risk derived from an asset is inversely related to the riskiness of the asset). It is difficult to imagine that stronger results can be derived without specifying a particular utility function, or a particular density function for the random variables. For example, under the distributional assumptions of Proposition 5.1 all correlations are common (we denote the correlation statistic by ρ).

If we assume that the investor chooses the portfolio to minimize variance, we find, after some straightforward calculation:

$$c_i^* = \left(\frac{[1 + (n-1)\rho]\delta_i - \left(\sum_{s=1}^n \delta_s\right)\rho}{D} \right) , \tag{5.9}$$

where $\delta_s \equiv 1/\lambda_s$, $\rho \in (-(n-1)^{-1}, 1)$, and $D \equiv \left([1 + (n-1)\rho](\sum_{s=1}^n \delta_s^2) - (\sum_{s=1}^n \delta_s)^2 \rho\right)$.[15] Relation (5.9) yields the expression

$$c_j^* - c_i^* = \left(\frac{\lambda_i - \lambda_j}{\lambda_i \lambda_j} \right) \left(\frac{1 + (n-1)\rho}{D} \right) . \tag{5.10}$$

Thus, as in Proposition 5.1, we have $(c_j^* - c_i^*)(\lambda_i - \lambda_j) \geq 0$. But stronger results do not seem possible, even under this assumption of variance minimization. For example, $\text{sign}(\lambda_j c_j^* - \lambda_i c_i^*) = \text{sign}[(\lambda_i - \lambda_j)\rho]$, so that expression (5.10) cannot be signed without knowledge of the correlation coefficient. Further, for $\rho < 0$, expression $(c_j^* - c_i^*)$ can approach zero from above. So a tighter lower bound than that specified in Proposition 5.1 seems unattainable without placing further restrictions not only on the preferences, but also on the distribution of risk.

A risk scaling analog to welfare Proposition 4.2 is also possible.

Proposition 5.2. *Assume that the n available assets generate random returns as specified in Proposition 5.1. Denote realized wealth by π as given in (5.2). Assume that investors are $U \in U_2^*$, and that $a_j^* > 0 \; \forall \, j \in \{1, \ldots, n\}$. For risk scaling vectors $\vec{\lambda}^\alpha$ and $\vec{\lambda}^\beta$, if $\vec{\lambda}^\beta \gtrsim \vec{\lambda}^\alpha$ then ex-ante welfare is larger under $\vec{\lambda}^\beta$ than under $\vec{\lambda}^\alpha$.*

[15] Alternatively, $D = n[\{(1 + (n-1)\rho)\sigma^2\} + (\bar{\delta})^2(1-\rho)] > 0$ where $\bar{\delta} = \left(\sum_{s=1}^n \delta_s\right)/n$ and $\sigma^2 \equiv \left[\sum_{s=1}^n (\delta_s - \bar{\delta})^2\right]/n$.

Proof. Upon optimizing we obtain function $\mu(\vec{\lambda})$ as the solution to problem (5.4). Symmetry holds, and so we need only establish the Ostrowski condition for problem (5.4). The first-order conditions are

$$E[U'(\pi)x_i] + \frac{\kappa}{\lambda_i} = 0\,, \quad i \in (1,\ldots,n)\,, \tag{5.11}$$

and manipulation of (5.11) readily establishes that $\kappa > 0$. Application of the envelope theorem to (5.4) yields $\partial\mu(\vec{\lambda})/\partial\lambda_i = -\kappa c_i^*/\lambda_i^2$. Ostrowski's condition for Schur-convexity is demonstrated if $(\lambda_i - \lambda_j)[\partial\mu(\vec{\lambda})/\partial\lambda_i - \partial\mu(\vec{\lambda})/\partial\lambda_j] \geq 0$, i.e., if $\kappa(\lambda_i - \lambda_j)(c_j^*/\lambda_j^2 - c_i^*/\lambda_i^2) \geq 0$. But the latter statement is immediate from Proposition 5.1 b). □

The proposition asserts that, for $K = \vec{\lambda}\cdot\vec{1}$ fixed, the more dispersed the risk scaling parameters the larger will the asset allocator's certainty equivalent return be. Essentially, risk averters should prefer greater variety in the available set of risky assets. A technical motivation for the result can be obtained from studying what the ordered parametric shift $\vec{\lambda}^\beta \gtrsim \vec{\lambda}^\alpha$ induces concerning the nature of the shift in 'decision vector' $\vec{c}^*$. It can be shown that the shift to $\vec{\lambda}^\beta \gtrsim \vec{\lambda}^\alpha$ induces a submajorized shift in $\vec{c}^*$.[16]

Definition 5.1. A vector $\vec{z}^{\$} \in \mathbb{R}^n$ is said to be *weakly submajorized* by vector $\vec{z}^{\$\$} \in \mathbb{R}^n$ ($\vec{z}^{\$} \underset{w}{\lesssim} \vec{z}^{\$\$}$) if $\sum_{i=1}^k z_{[i]}^{\$} \leq \sum_{i=1}^k z_{[i]}^{\$\$}$, $k = 1,\ldots,n$ where $z_{[i]}$ is the ordinate of ith largest magnitude in $\vec{z}$. A nonincreasing and Schur-concave function reverses this order, i.e., if $\varphi(\vec{z})$ is nonincreasing and Schur-concave, then $\vec{z}^{\$} \underset{w}{\lesssim} \vec{z}^{\$\$} \Rightarrow \varphi(\vec{z}^{\$}) \geq \varphi(\vec{z}^{\$\$})$.

Proposition 5.1, together with a careful study of the first-order conditions, demonstrates that $\psi(\vec{c})$ is Schur-concave and also that $\psi(\vec{c})$ is nonincreasing in its arguments. Curiously, the submajorization property also provides insights into portfolio structure and welfare when there is a safe asset.

6 Risk-free asset

Given the fundamental impact that the advent of a safe asset has on efficient portfolio allocations in the classical Markowitz model, it is of some interest to ascertain whether our findings are robust to the introduction of a safe asset. Suppose that, in addition to the n risky assets, there is a safe asset that pays gross return r_0. Let a_0 be the fraction of initial wealth \$1 invested in it. With ith asset random return $z_i = r_i + \lambda_i x_i$, $i \in \{1,\ldots,n\}$, terminal wealth is

$$\pi = \left(1 - \sum_{i=1}^n a_i\right) r_0 + \sum_{i=1}^n a_i z_i$$

[16] See Marshall and Olkin [15, pp. 10 and 59].

$$
\begin{aligned}
&= r_0 + \sum_{i=1}^{n} a_i(r_i - r_0) + \sum_{i=1}^{n} a_i \lambda_i x_i \\
&= r_0 + \vec{c} \cdot \vec{r}^{\,s} + \vec{c} \cdot \vec{x}\ , \qquad (6.1)
\end{aligned}
$$

where $\vec{c}$ has coordinates $c_i \equiv a_i \lambda_i$ as before, and where $\vec{r}^{\,s}$ has coordinates $r_i^s \equiv (r_i - r_0)/\lambda_i$. Note that, in contrast to Section 5, the aggregate allocation to risky assets, $\sum_{i=1}^n (c_i/\lambda_i) = \sum_{i=1}^n a_i$, is not constrained provided that borrowing is allowed.[17] The specification of returns in the risk adjusted excess return, i.e., Sharpe index, form $r_i^s \equiv (r_i - r_0)/\lambda_i$ is very useful because order can be identified on choices when there is heterogeneity concerning the first two moments of asset returns.[18]

Proposition 6.1. *Suppose that there are n risky assets with returns $z_i = r_i + \lambda_i x_i$, $i \in \{1, \ldots, n\}$, where the distribution of $\vec{x}$ is permutation symmetric. And assume that there is a safe asset with gross return r_0. Then, for any $U \in U_1^*$ decision maker,*

A) expected utility is an AI function of the vectors $\vec{r}^{\,s}$ and $\vec{c}$, and
B) the optimal portfolio is such that $(c_i^* - c_j^*)(r_i^s - r_j^s) \geq 0$.

Proof. Let $\vec{c}_\tau$ denote a transposition of $\vec{c}$. Then,

$$E[U(r_0 + \vec{c} \cdot \vec{r}^{\,s} + \vec{c} \cdot \vec{x})] = E[U(r_0 + \vec{c} \cdot \vec{r}^{\,s} + \vec{c}_\tau \cdot \vec{x})] \qquad (6.2)$$

where permutation symmetry has been invoked. Now suppose that, in particular, τ corresponds to an AI rearrangement of $\vec{c}$ with $\vec{r}^{\,s}$, i.e., $c_i \leq c_j$ while $r_i^s \geq r_j^s$. Because $\vec{c} \cdot \vec{r}^{\,s}$ is an AI function, we then have

$$E[U(r_0 + \vec{c}_\tau \cdot \vec{r}^{\,s} + \vec{c}_\tau \cdot \vec{x})] \geq E[U(r_0 + \vec{c} \cdot \vec{r}^{\,s} + \vec{c} \cdot \vec{x})] \qquad (6.3)$$

for any $U \in U_1^*$ decision maker and so part A) is demonstrated. Part B) is immediate because if $r_i^s < r_j^s$ and $c_i^* > c_j^*$ then allocation of c_i^* funds to the jth asset and c_j^* funds to the ith asset would increase expected utility. □

Note that, as in Propositions 3.1 and 4.1, the structural assumptions necessary to apply calculus are not required. Also, the existence of a safe asset is important in the proof because a consideration of the budget constraint was not necessary. In general, $\sum_{i=1}^n (c_{\tau(i)}/\lambda_i) \neq \sum_{i=1}^n (c_i/\lambda_i)$ for permutation τ and so a different amount is invested in the safe asset. The Sharpe index representation of returns would seem to be a most useful point of departure for studying structural issues, as the following Corollary demonstrates. Defining $\chi(r_1^s, \ldots, r_n^s) = \mathrm{Max}_{\vec{c}}\, E[U(r_0 + \vec{c} \cdot \vec{r}^{\,s} + \vec{c} \cdot \vec{x})]$ for $U \in U_1^*$, we have:

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[17] If borrowing is not allowed, then $\sum_{i=1}^n (c_i/\lambda_i) \leq \$1$, but this constraint need not bind. As long as preferences are such that the individual wishes to hold some of the safe asset in his portfolio, then Proposition 6.1 to follow holds even without borrowing.

[18] See Eun and Resnick [4] and Glen and Jorion [5] on uses of the Sharpe index in empirical analysis.

Corollary 6.1. *Let $a_i^* > 0\ \forall\ i \in \{1, 2, \ldots, n\}$. Then $\chi(r_1^s, \ldots, r_n^s)$ is nondecreasing and Schur-convex for all $U \in U_2^*$ such that $U(\cdot)$ is strictly concave.*

Proof. Clearly, $\chi(r_i^s, \ldots, r_n^s)$ is symmetric because a re-labeling of assets has no impact on expected utility. The strict concavity of $U(\cdot)$ together with the interior nature of the solution vector mean that the optimizing arguments are differentiable in parameters. And so we can apply the envelope theorem to $\chi(r_1^s, \ldots, r_n^s)$. We obtain $\partial\chi(\vec{r}^{\,s})/\partial r_i^s = E[U'(\pi)c_i^*] \geq 0$ where π is evaluated at $\vec{c}^*$. And so,

$$\left(\frac{\partial\chi(\vec{r}^{\,s})}{\partial r_i^s} - \frac{\partial\chi(\vec{r}^{\,s})}{\partial r_j^s}\right)(r_i^s - r_j^s) = E[U'(\pi)](c_i^* - c_j^*)(r_i^s - r_j^s)\,. \tag{6.4}$$

By Proposition 6.1, the Ostrowski condition is satisfied. □

Thus, a majorizing (i.e., more dispersed) vector of Sharpe indices would be preferred by risk averting investors. And more can be gleaned concerning investor preferences over investment environments. Writing $\chi(r_1^s, \ldots, r_n^s) \equiv \Theta(r_1, \ldots, r_n, \lambda_1, \ldots, \lambda_n)$, we have

Corollary 6.2. *Let $a_i^* > 0\ \forall\ i \in \{1, 2, \ldots, n\}$. Then $\Theta(r_1, \ldots, r_n, \lambda_1, \ldots, \lambda_n)$ is AD in $(r_1, \ldots, r_n)$ and $(\lambda_1, \ldots, \lambda_n)$ for all $U \in U_2^*$ such that $U(\cdot)$ is strictly concave.*

Proof. From Corollary 6.1, we know that $\chi(\vec{r}^{\,s})$ is nondecreasing and Schur-convex. And so, by Definition 5.1 above, we only need to demonstrate that an AD transposition on $(r_1, \ldots, r_n)$ and $(\lambda_1, \ldots, \lambda_n)$ induces submajorized dominance for $\vec{r}^{\,s}$.

Without loss of generality assume that $r_1 \geq r_2 \geq \ldots \geq r_n$ and that $\lambda_i \geq \lambda_j$ for some $i < j$. Now consider a binary permutation τ whereby $\lambda_{\tau(\nu)} = \lambda_\nu$, $\nu \neq \{i, j\}$, $\lambda_{\tau(i)} = \lambda_j$, $\lambda_{\tau(j)} = \lambda_i$. Define $\vec{r}_\tau^{\,s}$ as the vector with coordinates $r_{\tau(\nu)}^s = (r_i - r_0)/\lambda_{\tau(\nu)}$. Then a) $r_{\tau(\nu)}^s \equiv r_\nu^s$, $\nu \neq \{i, j\}$, $r_{\tau(i)}^s \equiv (r_i - r_0)/\lambda_j \geq \text{Max}[r_i^s, r_j^s]$, and b) $r_{\tau(j)}^s \equiv (r_j - r_0)/\lambda_i \leq \text{Min}[r_i^s, r_j^s]$. Further, c) $(r_{\tau(i)}^s + r_{\tau(j)}^s) - (r_i^s + r_j^s) = (r_i - r_j)(\lambda_i - \lambda_j)/(\lambda_i\lambda_j) \geq 0$. Viewing the partial sum conditions in Definition 5.1 and noting that permutation τ is a binary transposition, conclusions a), b), and c) are sufficient to establish that $(r_1^s, \ldots, r_i^s, \ldots, r_j^s, \ldots, r_n^s) \underset{w}{\lesssim} (r_1^s, \ldots, r_{\tau(i)}^s, \ldots, r_{\tau(j)}^s, \ldots, r_n^s)$. Therefore, $\chi(r_1^s, \ldots, r_i^s, \ldots, r_j^s, \ldots, r_n^s) \leq \chi(r_1^s, \ldots, r_{\tau(i)}^s, \ldots, r_{\tau(j)}^s, \ldots, r_n^s)$. □

In this corollary also, the existence of a safe asset was employed because our proof of the Schur-convexity result does not constrain the aggregate allocation to risky assets to be invariant to AD rearrangements of $\vec{r}$ and $\vec{\lambda}$. In interpreting the Corollary, we might say that expected utility rises as the deviation from the natural alignment of $\vec{r}$ with $\vec{\lambda}$ becomes more pronounced. An appealing way to measure vector alignment is the Spearman rank correlation coefficient, $\rho \in [-1, 1]$. As pointed out by Boland et al. [2], expression $-\sum_{i=1}^n (r_i - \lambda_i)^2$ is AI. Let $t(z_i, \vec{z}) \in \{1, 2, \ldots, n\}$ be the rank function, i.e., $t(z_{(i)}, \vec{z}) = i$. Then, for given vectors $\vec{r}$ and $\vec{\lambda}$, the Spearman rank correlation statistic, $\rho(\vec{r}, \vec{\lambda}) =$

$1 - 6\sum_{i=1}^{n}[t(r_i, \vec{r}) - t(\lambda_i, \vec{\lambda})]^2/(n^3 - n)$, serves as an index for the capacity of the investment environment to deliver welfare to $U \in U_2^*$ investors. For arrangements of given vectors $\vec{r}$ and $\vec{\lambda}$, the smaller the value of ρ the better for investors.

7 Weak axiom of portfolio preference

As reflected in Corollaries 3.1 and 4.1, as well as elsewhere, our analysis thus far has relied to some extent on revealed preference methods. And yet we have not identified comparative statics conclusions analogous to the weak axioms of cost minimization and profit maximization, as in Mas-Colell et al. [16, p. 10]. That the revealed preference approach has the power to extract some inferences on the nature of portfolio decisions is intuitive. But the method needs to be extended to a stochastic context. We extend the method in two ways. First we study perhaps the simplest case whereby the vector of mean returns changes and no safe asset is available.

Proposition 7.1. *Let $\vec{x} \in \mathbb{R}^n$ be a vector of random variables, and let $z_i = r_i + x_i \ \forall \ i \in \{1, \ldots, n\}$. Consider an arbitrary vector of mean returns, $\vec{r}^{\alpha}$, and – for a given investor – let $\vec{a}^{*\alpha}$ be the optimal portfolio with $\vec{a}^{*\alpha} \cdot \vec{1} = 1$. For this same investor, consider any arbitrary* **compensated** *shift of vector returns involving a returns vector $\vec{r}^{\beta}$ and payment T such that $T = \vec{a}^{*\alpha} \cdot (\vec{r}^{\beta} - \vec{r}^{\alpha})$ and ex-post wealth is $\pi(\vec{r}^{\beta}, T, \vec{a}, \vec{x}) = \vec{a} \cdot (\vec{r}^{\beta} + \vec{x}) - T$ with $\vec{a} \cdot \vec{1} = 1$.[19] Finally, let $\vec{a}^{*\beta}$ be optimal under the compensated investment environment $\{\vec{r}^{\beta}, T\}$. Write $\Delta\vec{r} = \vec{r}^{\beta} - \vec{r}^{\alpha}$ and $\Delta\vec{a} = \vec{a}^{*\beta} - \vec{a}^{*\alpha}$. Then, for any $U \in U_1^*$ investor, it must be that $\Delta\vec{a}^* \cdot \Delta\vec{r} \geq 0$.*

Proof. Revealed preference implies that

$$E[U(\vec{a}^{*\beta} \cdot \vec{r}^{\beta} - T + \vec{a}^{*\beta} \cdot \vec{x})] \geq E[U(\vec{a}^{*\alpha} \cdot \vec{r}^{\beta} - T + \vec{a}^{*\alpha} \cdot \vec{x})] \ . \tag{7.1}$$

Substitution and revealed preference demonstrate that

$$\begin{aligned} E[U(\vec{a}^{*\alpha} \cdot \vec{r}^{\beta} - T + \vec{a}^{*\alpha} \cdot \vec{x})] &= E[U(\vec{a}^{*\alpha} \cdot \vec{r}^{\alpha} + \vec{a}^{*\alpha} \cdot \vec{x})] \\ \geq E[U(\vec{a}^{*\beta} \cdot \vec{r}^{\alpha} + \vec{a}^{*\beta} \cdot \vec{x})] \ . \end{aligned} \tag{7.2}$$

Comparing relations (7.2) with (7.1), we see that $\vec{a}^{*\beta} \cdot \Delta\vec{r} \geq T$ for a $U \in U_1^*$ investor, while we have $\vec{a}^{*\alpha} \cdot \Delta\vec{r} = T$ by assumption. And so $\Delta\vec{a}^* \cdot \Delta\vec{r} \geq 0$. □

Notice that the proof did not require any assumptions concerning symmetry among the random variables. The proposition identifies conditions under which the law of compensated demand pertains for portfolio assets in the sense that the cheaper an asset is, relative to its expected return, the larger will be the demand for the asset. For example, if $\Delta\vec{r} = (1, 0, 0, \ldots, 0)$ then it must be that a_1^* increases under the compensated shift $\vec{r}_{\alpha} \to \vec{r}_{\beta}$.

[19] That is, T is the Slutsky compensation.

With reference to Section 6, one might hope that more could be deduced when the investment environment includes a safe asset. We will show next that a safe asset allows us to arrive at (compensated) comparative static results for changes in the vector of means and changes in the vector of risk parameters. Following our previous analysis, assume that there are n risky assets with returns $z_i = r_i + \lambda_i x_i$, $\lambda_i > 0$, $E[x_i] = 0 \; \forall \; i \in \{1, \ldots, n\}$. Let r_0 denote the gross return on the safe asset. Given any portfolio of risky assets $\vec{a}$ then, as in equation (6.1), terminal wealth is given by $\pi = r_0 + \vec{c} \cdot \vec{r}^{\,s} + \vec{c} \cdot \vec{x}$ where vectors $\vec{c}$ and $\vec{r}^{\,s}$ have ordinates $c_i \equiv a_i \lambda_i$ and $r_i^s \equiv (r_i - r_0)/\lambda_i$, respectively. Consider now an arbitrary original vector of risk-adjusted excess returns, $\vec{r}^{\,s,\alpha}$, obtained from the investment environment's array of primitive returns vectors $\{\vec{r}^{\alpha}, \vec{\lambda}^{\alpha}\}$. In this α environment, let $\vec{c}^{\alpha}$ denote the optimal portfolio of risky assets. Assume a *compensated* change in the investment environment to environment β which is characterized by the array of primitive returns vectors $\{\vec{r}^{\beta}, \vec{\lambda}^{\beta}\}$, i.e., $r_i^{s,\beta} \equiv (r_i^{\beta} - r_0)/\lambda_i^{\beta}$. The Slutsky compensation that the investor *pays* for this change in asset return distribution is $T^{\beta} \equiv \vec{c}^{*\alpha} \cdot (\vec{r}^{\,s,\beta} - \vec{r}^{\,s,\alpha})$.[20] The expression for realized wealth at the original portfolio choice $\vec{c}^{*\alpha}$ is then

$$\begin{aligned} \pi &= r_0 - T^{\beta} + \sum_{i=1}^{n} \vec{c}^{*\alpha}(r_i^{s,\beta} + x_i) \\ &= r_0 - \vec{c}^{*\alpha} \cdot (\vec{r}^{\,s,\beta} - \vec{r}^{\,s,\alpha}) + \sum_{i=1}^{n} c_i^{*\alpha}(r_i^{s,\beta} + x_i) \,. \end{aligned} \tag{7.3}$$

This is, by construction, unchanged relative to the realized wealth under the α investment environment. Finally let $\vec{a}^{*\beta}$, i.e., $\vec{c}^{*\beta}$ when risk modified, denote the optimal decision vector in compensated environment β described by $\{\vec{r}^{\beta}, \vec{\lambda}^{\beta}, T^{\beta}\}$. This context allows us to assert:

Proposition 7.2. *Consider a compensated change in the distribution of asset returns from* $\{\vec{r}^{\alpha}, \vec{\lambda}^{\alpha}\}$ *to* $\{\vec{r}^{\beta}, \vec{\lambda}^{\beta}\}$ *with Slutsky compensation* $T^{\beta} \equiv \vec{c}^{*\alpha} \cdot (\vec{r}^{\,s,\beta} - \vec{r}^{\,s,\alpha})$ *where* $\vec{a}^{*\alpha}$ *and* $\vec{c}^{*\alpha}$ *identify, respectively, the original optimal allocations to risky assets and the original optimal risk modified allocations to risky assets. Let* $\vec{a}^{*\beta}$ *and* $\vec{c}^{*\beta}$ *identify the new optimal portfolio. Then, for any* $U \in U_1^*$ *investor,*

1) $(\vec{c}^{*\beta} - \vec{c}^{*\alpha}) \cdot (\vec{r}^{\,s,\beta} - \vec{r}^{\,s,\alpha}) = \Delta\vec{c}^{*} \cdot \Delta\vec{r}^{\,s} \geq 0$,
2) *The compensated demand for an asset is a nondecreasing function of its expected return, i.e.,* $\Delta a_i^* \cdot \Delta r_i \geq 0$.
3) *If the expected return on the risky asset exceeds that of the safe asset, and the asset demand is strictly positive, then the compensated asset demand is a decreasing function of its risk scaling parameter, i.e.,* $\Delta a_i^* \cdot \Delta\lambda_i \leq 0$.
4) *Under the above assumptions, the elasticity of compensated demand with respect to the scaling parameter is no larger than negative one, i.e.,* $(\Delta a_i^* \cdot \Delta\lambda_i)(\lambda_i/a_i^*) \leq -1$.

[20] When there is a safe asset, it does not matter whether the compensation is paid before asset returns are realized or afterwards. However, when there are only risky assets then the compensation must be paid afterwards. This is to ensure that the investor can afford the same portfolio.

Proof. Part 1) follows immediately from revealed preference. Then

$$\begin{aligned} & E\{U[r_0 - \vec{c}^{*\alpha} \cdot (\vec{r}^{s,\beta} - \vec{r}^{s,\alpha}) + \vec{c}^{*\beta} \cdot (\vec{r}^{s,\beta} + \vec{x})]\} \\ & \quad \geq E\{U[r_0 - \vec{c}^{*\alpha} \cdot (\vec{r}^{s,\beta} - \vec{r}^{s,\alpha}) + \vec{c}^{*\alpha} \cdot (\vec{r}^{s,\beta} + \vec{x})]\} \\ & \quad = E\{U[r_0 + \vec{c}^{*\alpha} \cdot (\vec{r}^{s,\alpha} + \vec{x})]\} \\ & \quad \geq E\{U[r_0 + \vec{c}^{*\beta} \cdot (\vec{r}^{s,\alpha} + \vec{x})]\}\ , \end{aligned} \tag{7.4}$$

where the first inequality reflects the optimality of $\vec{c}^{*\beta}$ in the new compensated environment, where the equality is by construction, and where the second inequality reflects the optimality of $\vec{c}^{*\alpha}$ in the initial investment environment. Comparing the first and last expressions in (7.4), since the stochastic realizations are exactly the same we have $\vec{c}^{*\beta} \cdot \vec{r}^{s,\beta} - \vec{c}^{*\alpha} \cdot (\vec{r}^{s,\beta} - \vec{r}^{s,\alpha}) \geq \vec{c}^{*\beta} \cdot \vec{r}^{s,\alpha}$, i.e., $\Delta\vec{c}^{*} \cdot \Delta\vec{r}^{s} \geq 0$.

For part 2), let $\Delta\vec{\lambda} \equiv \vec{0}$ and let $\Delta r_j = 0\ \forall\ i \neq j$. From part 1), we then have $\Delta a_i^* \cdot \Delta r_i \geq 0$.

For part 3), let $\Delta\vec{r} \equiv \vec{0}$ and let $\Delta\lambda_j = 0\ \forall\ i \neq j$. From part 1), we then have that $\Delta\vec{c}^{*} \cdot \Delta\vec{r}^{s} \geq 0$ implies that $(c_i^{*\beta} - c_i^{*\alpha})(\lambda_i^{\alpha} - \lambda_i^{\beta})(r_i - r_0)/(\lambda_i^{\beta}\lambda_i^{\alpha}) \geq 0$. Thus, $\Delta c_i^* \cdot \Delta\lambda_i \leq 0$. Furthermore, because by definition $c_i \equiv a_i\lambda_i$, expression $\Delta c_i^* \cdot \Delta\lambda_i \leq 0$ implies that $[\Delta a_i^* \cdot \lambda_i^{\beta} + a_i^{*\alpha}\Delta\lambda_i](r_i - r_0)\Delta\lambda_i \leq 0$. Given $a_i^{*\alpha} \cdot (r_i - r_0) > 0$, we then have $\Delta a_i^* \cdot \Delta\lambda_i \leq -a_i^{*\alpha}(\Delta\lambda_i)^2/(\lambda_i^{\beta}) \leq 0$.

For part 4), we will consider the shock that arises in part 3) and we will assume that the point of evaluation for the elasticity is the mid-point.[21] But it should be clear that our finding is true regardless of the point of evaluation. From $a_i^* = (a_i^{*\alpha} + a_i^{*\beta})/2$ and $\lambda_i = (\lambda_i^{\alpha} + \lambda_i^{\beta})/2$, we have $a_i^{*\alpha} = a_i^* - 1/2\Delta a_i^*$ and $\lambda_i^{\beta} = \lambda_i + 1/2\Delta\lambda_i$. In part 3) we showed that $[\Delta a_i^* \cdot \lambda_i^{\beta} + a_i^{*\alpha}\Delta\lambda_i](r_i - r_0)\Delta\lambda_i \leq 0$. Upon substitution, we have that

$$\left[\Delta a_i^* \cdot (\lambda_i + 1/2\Delta\lambda_i) + (a_i^* - 1/2\Delta a_i^*)\Delta\lambda_i\right](r_i - r_0)\Delta\lambda_i \leq 0\ . \tag{7.5}$$

If $a_i^*(r_i - r_0) \geq 0$, then relation (7.5) reduces to $(\Delta a_i^*/\Delta\lambda_i)(\lambda_i/a_i^*) \leq -1$. □

It was shown in Proposition 5.1 that, for risk averse investors and when the joint density is permutation symmetric, the demand for an asset is - on average-elastic with respect to the risk scaling parameter. For a larger set of decision makers and without recourse to any symmetry conditions on the primitive distribution, Proposition 7.2 demonstrates that asset demand is elastic everywhere. The tradeoff is that compensation is needed. Thus, as elsewhere in demand theory, when income effects are controlled for then the endeavor of comparative static analysis is much less involved.

8 Relation to existing results

To complete our analysis, we will do two things. First we will formally place our findings in context with the literature. Then we will fill in some gaps that become

[21] That is, we consider the Allen empirical elasticity.

obvious when comparing our results with extant knowledge. Two stochastic partial orders that have arisen in earlier studies of problem (P) are the likelihood ratio order and the reversed hazard order.

Definition 8.1. Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$ be continuous random variables with densities $g(x)$ and $h(y)$, respectively, such that
D8.1 a) $g(t)/h(t)$ is nondecreasing over the union of the supports of x and y (with $k/0$ ascribed the value whenever $k > 0$), or equivalently
D8.1 b) $g(v)h(u) \geq g(u)h(v)$ for all $v \geq u$.
Then x is said to be larger than y in the *likelihood ratio* (*LR*) order, and is denoted by $x \overset{LR}{\geq} y$.

Definition 8.2. Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$ be continuous random variables with distribution functions $G(x)$ and $H(y)$, respectively, such that $G(t)/H(t)$ is nondecreasing over the union of the supports of x and y (with $k/0$ ascribed the value ∞ whenever $k > 0$). Then x is said to be larger than y in the *reversed hazard* (*RH*) order, and is denoted by $x \overset{RH}{\geq} y$.

The likelihood ratio order is a restricted version of the reversed hazard order. In turn, the reversed hazard order is a restricted version of FSD; that is $[x \overset{LR}{\geq} y] \Rightarrow [x \overset{RH}{\geq} y] \Rightarrow [x \overset{FSD}{\geq} y]$, but the reverse implications do not hold. It has been demonstrated that when asset returns are independently distributed, then these two order concepts are sufficient to identify order on the optimal asset allocation vector for classes of decisionmakers. Specifically,

Result 8.1. *In problem (P), suppose that the x_i are independently distributed.*
R8.1 a) *If $x_1 \overset{LR}{\geq} x_2 \overset{LR}{\geq} \ldots \overset{LR}{\geq} x_m$ for $m \leq n$, then $a_1^* \geq a_2^* \geq \ldots \geq a_m^*$ for every $U \in U_1^*$, while*
R8.1 b) *if $x_1 \overset{RH}{\geq} x_2 \overset{RH}{\geq} \ldots \overset{RH}{\geq} x_m$ for $m \leq n$, and if optimum portfolio choices are strictly positive then $a_1^* \geq a_2^* \geq \ldots \geq a_m^*$ for every $U \in U_2^*$.*

For the sake of completeness, proofs are provided in Appendix A. Part a) is due to Landsberger and Meilijson [14] who applied a proposition in Ross [23], while part b) is due to Kijima and Ohnishi [12]. For both parts, the conclusions readily extend to apply for conditionally independent random variables with joint distribution $G(\vec{x}, \varepsilon) = F(\varepsilon) \prod_{i=1}^{n} G_i(x_i \mid \varepsilon)$ where one might allow ε to represent systematic risk, i.e., market risk, so that only the residual risks are independent.

Kijima and Ohnishi also extended their analysis to the context of bivariate dependent distribution structures by focusing on the appropriate closed convex cones of functions.[22]

Definition 8.3. Define $\zeta_{LR} = \{\varphi(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}; \varphi(x_1, x_2) \geq \varphi(x_2, x_1)$ whenever $x_1 \geq x_2\}$.

Definition 8.4. Define $\zeta_{RH} = \{\varphi(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}; \varphi(x_1, x_2) - \varphi(x_2, x_1)$ is nondecreasing in x_1 whenever $x_1 \geq x_2\}$.

[22] See p. 270, Theorem 5.1 (a) and p. 271, Theorem 5.2 (a).

When applied to stochastic structures, these two classes of functions admit dependence between the random variables. Kijima and Ohnishi showed that order on the optimal allocation vector could be inferred when the density function satisfied the ζ_{LR} property or the distribution function satisfied the ζ_{RH} property.

Result 8.2. *Consider problem (P) when $n = 2$ and when the distribution is $F(x_1, x_2)$ with joint density $f(x_1, x_2)$. In this context,*
R8.2 a) *if $f(x_1, x_2) \in \zeta_{LR}$, then $a_1^* \geq a_2^*$ for every $U \in U_1^*$, while*
R8.2 b) *if $F(x_1, x_2) \in \zeta_{RH}$ and $a_i^* > 0$, $i \in \{1, 2\}$ then $a_1^* \geq a_2^*$ for every $U \in U_2^*$.*

Proofs of statements R8.2a) and R8.2b) arise when we demonstrate Proposition 8.1 to follow. When the two marginal distributions are independent, then work by Shanthikumar and Yao [25] readily reveals that R8.1 a) and R8.2 a) are equivalent.[23] Otherwise, the conclusions are distinct. Likewise, R8.1 b) and R8.2 b) are equivalent if and only if the random variables are independent.[24] Notice that Definitions 8.3 and 8.4, and consequently Result 8.2, exploit an asymmetry in dependence structure of the joint distribution to arrive at an asymmetric result, $a_1^* \geq a_2^*$. This is in contrast with Result 8.1. The asymmetry in Definition 8.4 and R8.2 b) is depicted in Figure 1. Although areas C and D have the same measure under a uniform weighting, condition $F(x_1, x_2) \in \zeta_{RH}$ in R8.2 b) requires that the weighted probability measure of rectangle D exceeds that of rectangle C for the shift δx_1^* in x_1 from x_1^* when the upper bound in x_2 has been fixed. And this weighted measure inequality is true for comparison of any pair of rectangles with base at one axis that are mutual reflections in the line $x_1 = x_2$ and that are not cut by that line.

To construct a distribution $F(x_1, x_2) \in \zeta_{RH}$, commence with a permutation symmetric distribution $G(x_1, x_2)$. Define $F(x_1, x_2) = G(x_1, x_2)$ on $x_2 > x_1$, and define $F(x_1, x_2) = G(x_1, x_2) + k(x_1, x_2)$ on $x_2 \leq x_1$ where, on $x_2 \leq x_1$, $k(x_1, x_2)$ is nondecreasing in x_1 with $k(x_1, x_2)\,|_{x_1 = x_2} \equiv 0$ and with $\inf_{x_1, x_2}[G(x_1, x_2) + k(x_1, x_2)] = 0$. For example, one might have $F(x_1, x_2) = G(x_1, x_2)$ on $x_2 \leq x_1$ whenever $G(x_1, x_2) \leq (x_2 - x_1)^2$ and $F(x_1, x_2) = G(x_1, x_2) - (x_2 - x_1)^2$ on $x_2 \leq x_1$ whenever $G(x_1, x_2) > (x_2 - x_1)^2$.

But, in contrast with the case of independence, Result 8.2 would not appear to extend to the multivariate context because the proof of Result 8.1, as given in Landsberger and Meilijson [14] and Kijima and Ohnishi [12], applies the partial ordering property to extend their results for the two independent assets context to the n assets context. The distribution assumptions made in parts a) and b) of Result 8.2 do not describe a partial ordering relationship. Result 8.2 follows with very little work from Definitions 8.3 and 8.4, and the approach adopted in Kijima and Ohnishi differs from that used in developing Result 8.1. In seeking to extend Result 8.2 to the n asset context, it is not possible to invoke partial ordering properties. The problem is a failure of transitivity.

[23] See Theorem 2.12, p. 646.
[24] See Shanthikumar and Yao [25, p. 651].

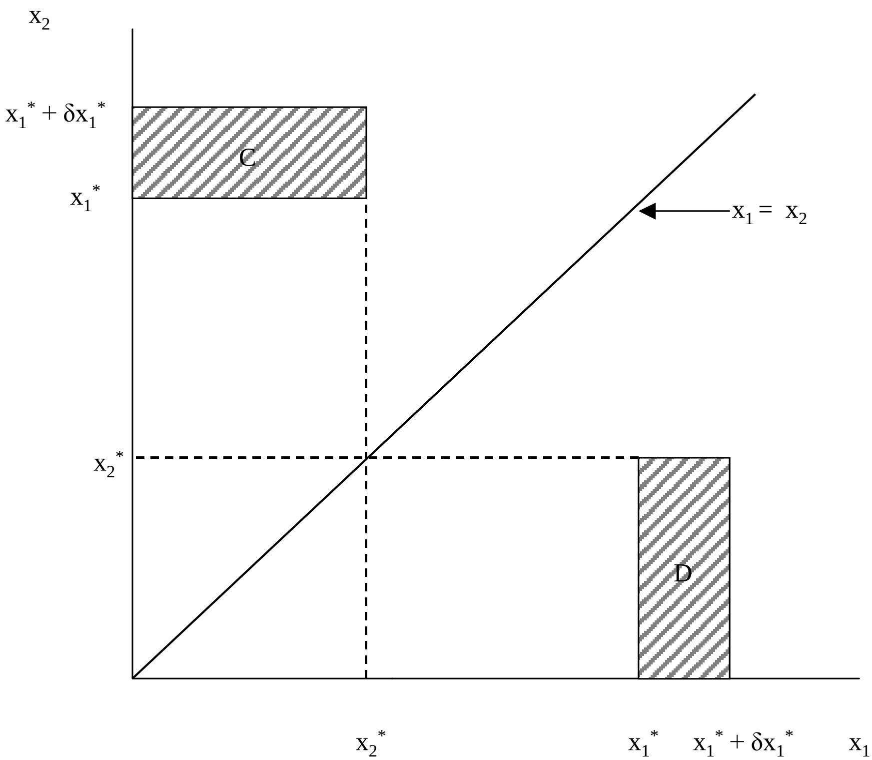

Figure 1. Graphical representation of the $F(x_1, x_2) \in \zeta_{RH}$ property

Proposition 3.1 is a natural extension of Result 8.2 a) in that one might view the class of AI functions as a multivariate extension of class ζ_{LR}. To see this, write $\varphi(x_1, x_2) = g(x_1, x_2, \lambda_1, \lambda_2)$ where $g(x_1, x_2, \lambda_1, \lambda_2)$ is AI in $\vec{x}$ and $\vec{\lambda}$. Then, with $\lambda_1 \geq \lambda_2$, we have $g(x_1, x_2, \lambda_1, \lambda_2) \geq g(x_2, x_1, \lambda_1, \lambda_2)$ whenever $x_1 \geq x_2$. But the extension needs to be structured so that order relations arising from pairwise interchanges are independent of the other random variables. Result 8.2 can also be extended through a further integration by parts. Define by $U \in U_3^*$ the subset of all $U \in U_2^*$ with a continuous and nondecreasing third derivative, i.e., $U'''(\pi) \geq 0 \ \forall \ \pi \in \mathbb{R}$.

Proposition 8.1. *Consider problem (P) when $n = 2$ and when the distribution is $F(x_1, x_2)$ with joint density $f(x_1, x_2)$. Assume that $a_i^* > 0$, $i \in \{1, 2\}$. In this context, if $F(x_1, x_2) \geq F(x_2, x_1)$ for all $x_1 \geq x_2$, then $a_1^* \geq a_2^*$ for every $U \in U_3^*$.*

A proof is provided in Appendix B. To construct a distribution function that satisfies the requisite property, define permutation symmetric distribution $G(x_1, x_2)$. Define $F(x_1, x_2) = G(x_1, x_2)$ on $x_1 < x_2$, and define $F(x_1, x_2) = G(x_1, x_2) + k(x_1, x_2)$ on $x_1 \geq x_2$ where, on $x_1 \geq x_2$, $k(x_1, x_2) \geq 0$, with $k(x_1, x_2) \mid_{x_1 = x_2} \equiv 0$ and with $\sup_{x_1, x_2}[G(x_1, x_2) + k(x_1, x_2)] = 1$.

9 Conclusion

The main thesis of this paper has been that an understanding of symmetries and asymmetries in distribution structure is critical to comprehending optimum portfolio allocations. Specifically, well-structured asymmetries in the distribution will reveal themselves in corresponding asymmetries in the optimum allocation vector. We approached the problem from two directions. First, we introduced strictly controlled asymmetries into the distribution function of the rates of return while preserving symmetry in the payoff function. Then we imposed complete symmetry in the source of randomness of rates of returns, but introduced asset-determined location and scale asymmetries in the gross returns implications of those random draws.

It is clear, however, that we have only identified the more superficial consequences of symmetry for optimum asset allocations. It would seem that mathematical tools which have found application in the study of symmetry, e.g., group and majorization theories, might provide further help in analyzing the determinants of the optimum allocation vector. A more concrete goal is to delineate the precise distribution hierarchical structures for which allocative order can be inferred within partitions of the state space. Specifically, when can it be assumed that optimizers will budget in two stages in the sense that budgets are first allocated to groups of assets (Europe, Asia, healthcare, banking) and then these sub-budgets are allocated among assets in the group? With regard to generalizing the main results established in this paper, the next step might be to investigate methods by which order in the allocation vector can be established when asymmetries in marginals are nonparametric. It would also be of interest to establish the multivariate risk analog of Proposition 3.1 for the AI method. More generally, if a coherent microeconomic theory of portfolio allocation is to be arrived at, it will be necessary to understand the similarities and differences between how order on the stochastic environment induces order on the portfolio allocation vector and how ordered shifts in the stochastic environment induce ordered shifts in the portfolio allocation vector.

Appendix A

Proof of Result 8.1 a): Due to the independence assumption, it suffices to establish the result in a two asset context. Where x_1 has density $g(x_1)$ and x_2 has density $h(x_2)$, the objective is to choose a_1 and a_2, with $a_1 + a_2 = 1$, to maximize $T(\vec{a}) = \iint U(\vec{a} \cdot \vec{x}) g(x_1) h(x_2) dx_1 dx_2$. Confining the domain of integration to a half-space, we have[25]

$$\begin{aligned} &T(\vec{a}_\tau) - T(\vec{a}) \qquad\qquad (A.1)\\ &= \int_{\mathbb{R}^2_+, x_1 \geq x_2} [U(\vec{a}_\tau \cdot \vec{x}) - U(\vec{a} \cdot \vec{x})][g(x_1)h(x_2) - g(x_2)h(x_1)] dx_1 dx_2 \,. \end{aligned}$$

[25] See eqn. (3.2) above.

If $g(\cdot) \overset{LR}{\geq} h(\cdot)$ then the second term in square brackets is positive. Because $U \in U_1^*$, if $T(\vec{a}_\tau) \geq T(\vec{a})$ then it must be that $a_2 \geq a_1$. But the initial allocation is inefficient, and so $a_1^* \geq a_2^*$. □

Proof of Result 8.1 b): Again, independence implies that we need only demonstrate the result for the two asset case. For two assets, it is convenient to re-pose the result in the following way: *Let x_1 and x_2 be independent random variables distributed according to $G(x_1)$ and $H(x_2)$, respectively. Then $E[U(\vec{a} \cdot \vec{x})] \geq E[U(\vec{a}_\tau \cdot \vec{x})] \;\forall\; U \in U_2^*$ if and only if $a_2^* \geq a_1^*$ whenever*

M8.1i) $x_1 \overset{RH}{\leq} x_2$, *or equivalently*

M8.1ii) $g(x_2)H(x_1) \leq G(x_1)h(x_2)$ *whenever* $x_1 \geq x_2$.

The equivalence is a straightforward implication of the definition of the reversed hazard order. Next, integrate (A.1) by parts to obtain

$$
\begin{aligned}
&T(\vec{a}_\tau) - T(\vec{a}) \qquad\qquad \text{(A.2)}\\
&= \int_{\mathbb{R}_+^2, x_1 \geq x_2} [a_2 U'(\vec{a}_\tau \cdot \vec{x}) - a_1 U'(\vec{a} \cdot \vec{x})][G(x_1)h(x_2) - g(x_2)H(x_1)] dx_1 dx_2 \,.
\end{aligned}
$$

The second term in the integrand is nonnegative by M8.1ii). By concavity of the utility function, $U'(\vec{a}_\tau \cdot \vec{x}) \geq U'(\vec{a} \cdot \vec{x})$ if and only if $a_2 \geq a_1$. And so $a_2 U'(\vec{a}_\tau \cdot \vec{x}) \geq a_1 U'(\vec{a} \cdot \vec{x})$ if and only if $a_2 \geq a_1$. In this case, we have that the initial allocation was efficient. And so that any optimum must satisfy $a_2^* \geq a_1^*$ whenever $x_2 \overset{RH}{\geq} x_1$. □

Appendix B

We will build the proof of Proposition 8.1 in three steps. First, Result 8.2 a) will be confirmed. Then Result 8.2 b) will be demonstrated. Finally, Proposition 8.1 will be verified.

Proof of Result 8.2 a): Posing the problem in the manner of Result 8.1 a), the problem is to establish the sign of

$$
\begin{aligned}
&T(\vec{a}_\tau) - T(\vec{a})\\
&= \int_{\mathbb{R}_+^2, x_1 \geq x_2} [U(\vec{a}_\tau \cdot \vec{x}) - U(\vec{a} \cdot \vec{x})][f(x_1, x_2) - f(x_2, x_1)] dx_1 dx_2 \,. \qquad \text{(B.1)}
\end{aligned}
$$

Because $f(x_1, x_2) \in \zeta_{LR}$, we know that the second bracketed term in the integrand is positive. And, because $U \in U_1^*$, by Result 8.1 a) it is necessary and sufficient that $a_2 \geq a_1$ for the first bracketed term in the integrand to be positive. And so $a_1^* \geq a_2^*$. □

Proof of Result 8.2 b): Integrate (B.1) by parts and, after some work, one obtains

$$
\begin{aligned}
T(\vec{a}_\tau) - T(\vec{a}) \;=\; & -\int_{\mathbb{R}_+^2, x_1 \geq x_2} [a_1 U'(\vec{a}_\tau \cdot \vec{x}) - a_2 U'(\vec{a} \cdot \vec{x})]\\
& \times \left[\frac{\partial F(x_1, x_2)}{\partial x_1} - \frac{\partial F(x_2, x_1)}{\partial x_1} \right] dx_1 dx_2 \,. \qquad \text{(B.2)}
\end{aligned}
$$

By definition, if $F(x_1,x_2) \in \zeta_{RH}$ then we know that the latter term in brackets within the integral is nonnegative. If $a_1 \geq a_2$, then the first term in brackets in the integral is nonnegative, we have $T(\vec{a}_\tau) \leq T(\vec{a})$, and so $a_1^* \geq a_2^*$. $\square$

Proof of Proposition 8.1: Integrate (B.2) by parts to obtain

$$\begin{aligned}
&T(\vec{a}_\tau) - T(\vec{a}) \qquad\qquad \text{(B.3)}\\
&= -\int_{x_2=0}^{x_2=x_1} \left[[a_1U'(\vec{a}_\tau \cdot \vec{x}) - a_2U'(\vec{a}\cdot\vec{x})][F(x_1,x_2) - F(x_2,x_1)]\right]_{x_1=x_2}^{x_1\to\infty} dx_2\\
&\qquad + \int_{\mathbb{R}_+^2, x_1 \geq x_2} a_1a_2[U''(\vec{a}_\tau\cdot\vec{x}) - U''(\vec{a}\cdot\vec{x})][F(x_1,x_2) - F(x_2,x_1)]dx_1dx_2\ .
\end{aligned}$$

The first term is clearly signable for all $U \in U_2^*$ because a) by $U \in U_2^*$, the sign of $a_1U'(\vec{a}_\tau\cdot\vec{x}) - a_2U'(\vec{a}\cdot\vec{x})$ is the same as the sign of $a_1 - a_2$ on the half space $x_1 \geq x_2$, b) $F(x_1,x_2)\ |_{x_1=x_2} \equiv F(x_2,x_1)\ |_{x_1=x_2}$, and c) we have assumed that $F(x_1,x_2) \geq F(x_2,x_1)$ for all $x_1 \geq x_2$ so that $\mathrm{Lim}_{x_1\to\infty}[F(x_1,x_2) - F(x_2,x_1)] \geq 0$.

Turning to the second term in (B.3), it has the sign of $U''(\vec{a}_\tau\cdot\vec{x}) - U''(\vec{a}\cdot\vec{x})$, i.e., the sign of $a_2 - a_1$, when $U'''(\pi) \geq 0$. And so, for any $U \in U_3^*$, it must be that $a_1^* \geq a_2^*$. $\square$

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