An ordinal approach to characterizing efficient allocations

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Abstract: The invisible hand theorem relates nothing about the attributes of the optimal allocation vector. In this paper, we identify a convex cone of functions such that order on vectors of exogenous heterogeneity parameters induces component-wise order on allocation vectors for firms in an efficient market. By use of functional analysis, we then replace the vectors of heterogeneities with asymmetries in function attributes such that the induced component-wise order on efficient allocations still pertains. We do so through integration over a kernel in which the requisite asymmetries are embedded. Likelihood ratio order on the measures of integration is both necessary and sufficient to ensure component-wise order on efficient factor allocations across firms. Upon specializing to supermodular functions, familiar stochastic dominance orders on normalized measures of integration provide necessary and sufficient conditions for this component-wise order on efficient allocation. The analysis engaged in throughout the paper is ordinal in the sense that all conclusions drawn are robust to monotone transformations of the arguments in production.

Keywords: arrangement monotone, functional analysis, market structure, ordinal analysis, simplex, symmetry

JEL classification: D2, C6, L1

1. Introduction

Recent years have seen much success in developing an understanding of how optimizing firms respond to a change in the economic environment. Of particular significance has been the body of work by Topkis (1978), Vives (1990), Milgrom and Roberts (1990, 1996), Milgrom and Shannon (1994), and Athey (2000) on the optimal behavior of firms when choice variables and environmental parameters take values on lattice or allied algebraic structures. Appealing aspects of this suite of methods include the facility with which it accommodates vector valued decision variables and environmental parameters, and also the generality of the functional forms to which it applies.

If the comparative statics of decision vectors are of interest to economists, then the level vectors should also be of interest. As an example, let two optimizing firms produce a single homogeneous good. If firm A uses more labor than firm B then will the technology also require that firm A also use more capital than firm B? Little is known about the classes of functions that impose a certain order on the optimum level vectors of factors used by firms in a market. The question should be of importance in understanding both industry structure and the determinants of comparative performance within an industry. It should also be of importance as a foundation in identifying the rents that accrue to factors in production.

The intent of this paper is to establish conditions under which discernible order on the heterogeneities among firms maps into discernible order on the decisions that these firms would make in an efficient equilibrium. The issue addressed, and also the approach taken, is closest to the assortative matching literature due to Becker (1973). In his model, couples match off in the marriage market in a manner that maximizes aggregate household production. If male and female productivity types can be totally ordered in a single dimension and if the household production technology is supermodular, then efficient sorting will involve high types matching with high types and low with low. Kremer (1993) and Lazear (2001) have applied assortative matching to understand equilibrium production structures and surplus distribution, while Shimer and Smith (2000) have studied the robustness of assortative matching under search frictions. In

each of these models, however, quite specific functional structures are assumed, while the dimensionality of the functions is low. This paper will provide a general framework for considering the matching of factors in production in the absence of frictions.

In the first main section, Section 2, we will explain a pre-ordering, as well as the associated class of functions, that is tailored to elicit order on decision vectors in an efficient equilibrium. In the class of functions, which go under the label 'multivariate arrangement increasing' or MAI, heterogeneities among firms are parameterized as vectors. The class contains the Schur-concave functions when arguments sum. It also contains functions that may be represented as the sum of evaluations of a supermodular function, as well as functions that may be represented as the product of eveluations of a log-supermodular function. As with the lattice theory of optimization, our approach is analytically robust in that it requires no assumptions concerning concavity, differentiability, or even continuity. Further, it is particularly well-suited to resource-constrained allocation problems such as the type of problem one would hope that a decentralized market solves. But the approach also has much to relate about agent-level decisions such as portfolio allocation or the allocation of time across opportunities.

In the second main section, we extend the analysis to identify a class of non-parametric functions for which the insights of Section 2 pertain. This is done by building embedded structural asymmetries into the multivariate function such that order on analytic properties of the function map into order on the vectors of decisions that are supported in an efficient equilibrium. The line of approach is a linear functional analysis of convex cones in the manner of Athey's (2000) study of comparative statics under uncertainty, but in quite a different context and with some distinguishing features. Her use of linear functionals has the end of developing a robust framework for identifying comparative statics inferences under stochastic dominance innovations. Our use has the end of providing a robust approach to 'building' structural asymmetries into multivariate deterministic functions. One distinguishing feature pertains to the use of separating hyperplanes. Whereas in her analysis the concern is with identifying the

existence of a separating hyperplane, in our study symmetry considerations place restrictions on the sort of separating hyperplane at issue. As we seek to rank efficient factor choices across firms, bisectors in the allocation simplex assume particular importance. Reflections through these bisectors are used to construct asymmetries in the multivariate function. The asymmetries arise because the replications of a 'fundamental' reference region in the factor domain are imperfect.[1]

The final main section focuses on the supermodular sub-class of MAI functions. We suppose that aggregate production in a market can be represented as the linear sum of productions across heterogeneous firms where the heterogeneities are arguments in the common production technology and where production is supermodular in factors. We then show how, after a normalization, univariate and multivariate dominance partial orderings on sources of heterogeneities across firms map into order in efficient factor allocations across firms. The approach taken establishes the strong analogies between our theory of efficient allocation in equilibrium and the theory of comparative statics. The paper concludes with a discussion on open research issues.

2. Arrangement monotonicity

Our interest is in asymmetries across arguments of a function such that determinate implications for the efficient allocation of fixed stocks of resources can be deduced. There are n active firms in a market where each firm produces a single homogeneous good. The market's resource stocks are given by the array $X = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_r\} \in (\mathbb{R}^n)^r$ where $\vec{x}_i \cdot \vec{1} = \overline{X}_i \ \forall\, i \in \Omega_r = \{1, 2, \ldots, r\}$ and where each $\overline{X}_i$ is a constant.[2] Each vector $\vec{x}_i$, $i \in \Omega_r$, describes how the ith

[1] As such, our approach has much in common with the theory of building mathematical objects from fundamental regions through reflection group operations. Benson and Grove (1971) present the foundations of this topic, while Ronan (1989) provides a more specialized treatment.

[2] As will become apparent, the nature of the theory to be developed implies that the levels of stock endowments are of no analytical importance.

resource is allocated across the n firms. The array is denoted as the allocation array where each row is a firm's decision vector. The jth coordinate in the vector $\vec{x}_i$ is identified as $x_{i,j}$ where the array entry represents the jth firm's utilization of the ith resource. The origin of heterogeneities is in the array $\Theta = \{\vec{\theta}_1, \vec{\theta}_2, \ldots, \vec{\theta}_s\} \in (\mathbb{R}^n)^s$, which is called the character array. The economic distinction between the arrays is that whereas each $\vec{x}_i$, $i \in \Omega_r$, may vary on the simplex $\vec{x}_i \cdot \vec{1} = \overline{X}_i$ $\forall\, x_i \geq 0$, the character array is exogenous to decisions made in the market. The concatenated array $Y = X{:}\Theta$ is labeled the market array.

To illustrate, for $n = 3$ let

$$X_0 = \{\vec{x}_1, \vec{x}_2\} = \begin{pmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 0 \end{pmatrix}, \quad \Theta_0 = \{\vec{\theta}_1, \vec{\theta}_2, \vec{\theta}_3, \vec{\theta}_4\} = \begin{pmatrix} 2 & 1 & 3 & 3 \\ 3 & 0 & 4 & 2 \\ 4 & 2 & 1 & 1 \end{pmatrix}, \tag{2.1}$$

with

$$Y_0 = \{\vec{x}_1, \vec{x}_2 : \vec{\theta}_1, \vec{\theta}_2, \vec{\theta}_3, \vec{\theta}_4\} = \begin{pmatrix} 1 & 3 & : & 2 & 1 & 3 & 3 \\ 4 & 1 & : & 3 & 0 & 4 & 2 \\ 2 & 0 & : & 4 & 2 & 1 & 1 \end{pmatrix}. \tag{2.2}$$

Read horizontally, the array data is firm-specific. The jth firm's decision and character vectors are given by $\vec{x}(j)$ and $\vec{\theta}(j)$, respectively. For example, firm 2 has character vector $\vec{\theta}(2) = (3, 0, 4, 2)$ and decision vector $\vec{x}(2) = (4, 1)$. Read vertically, the array data is either a resource allocation profile across firms (the first two columns) or a character profile across firms.

Our immediate concern is with how, in an efficient market, arrays X and Θ relate. It transpires that the *multivariate arrangement increasing order* on Θ elicits a form of order on X when the available production technology is of a certain structure. For vector $\vec{x}$, define $\vec{x}\uparrow$ as the rearrangement of $\vec{x}$ such that the kth smallest among the ordinate values is the kth ordinate. Thus, if $\vec{x} = (3, 2.5, 4)'$ then $\vec{x}\uparrow = (2.5, 3, 4)'$. The following concept is a variant by Kim and

Proschan (1995) of a definition by Boland and Proschan (1988):[3]

Definition 2.1. For a given pair of vectors with n-dimensional vector arguments $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_w\}$ and $\{\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_w\} \in \mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n = (\mathbb{R}^n)^w$, define $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_w\} \overset{a}{=} \{\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_w\}$ if there exists a permutation π of $(1, 2, \ldots, n)$ such that $\vec{x}_{k_\pi} = \{x_{k_{\pi(1)}}, x_{k_{\pi(2)}}, \ldots, x_{k_{\pi(n)}}\}' = \{z_{k_1}, z_{k_2}, \ldots, z_{k_n}\}'$ $= \vec{z}_k$ for each $k = 1, 2, \ldots, w$. Define $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_w\} \overset{a}{\leq} \{\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_w\}$ if and only if there exists a finite number, t, of elements $\{\vec{y}_2^{\,1}, \vec{y}_3^{\,1}, \ldots, \vec{y}_w^{\,1}\}, \ldots, \{\vec{y}_2^{\,t}, \vec{y}_3^{\,t}, \ldots, \vec{y}_w^{\,t}\} \in (\mathbb{R}^n)^{w-1}$ such that

(a) $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_w\} \overset{a}{=} \{\vec{x}_1\uparrow, \vec{y}_2^{\,1}, \ldots, \vec{y}_w^{\,1}\}$ and $\{\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_w\} \overset{a}{=} \{\vec{x}_1\uparrow, \vec{y}_2^{\,t}, \ldots, \vec{y}_w^{\,t}\}$, and

(b) for each $j = 1, 2, \ldots, t$ there exists a pair of coordinate indices, c and d with $c < d$, such that $\{\vec{x}_1\uparrow, \vec{y}_2^{\,j}, \ldots, \vec{y}_w^{\,j}\}$ may be obtained from $\{\vec{x}_1\uparrow, \vec{y}_2^{\,j-1}, \ldots, \vec{y}_w^{\,j-1}\}$ by interchanging the c and d coordinates of <u>*every*</u> vector $\vec{y}_k^{\,j-1}$ such that $y_{k,c}^{j-1} > y_{k,d}^{j-1}$ where $y_{k,v}^{j-1}$ is the vth coordinate in vector $\vec{y}_k^{\,j-1}$ (such an operation of obtaining $\{\vec{x}_1\uparrow, \vec{y}_2^{\,j}, \ldots, \vec{y}_w^{\,j}\}$ from $\{\vec{x}_1\uparrow, \vec{y}_2^{\,j-1}, \ldots, \vec{y}_w^{\,j-1}\}$ is said to be a *basic rearrangement*). If $X = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_w\} \overset{a}{\leq} \{\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_w\} = Z$, then we say that Z is larger in the multivariate arrangement increasing order than X.

Example 2.1. For Θ as given in (2.1), note that $\vec{\theta}_1 = (2, 3, 4)'$ is in increasing order when read from top to bottom. This is not, however, true of $\vec{\theta}_2 = (1, 0, 2)'$, $\vec{\theta}_3 = (3, 4, 1)'$, or $\vec{\theta}_4 = (3, 2, 1)'$. Notice, though, that the first two arguments of $\vec{\theta}_1$ are aligned with the corresponding arguments of $\vec{\theta}_3$, i.e, $(2-3)(3-4) > 0$. To align all of the first two arguments up across all of the four character dimensions, transpose the 1 with the 0 in $\vec{\theta}_2$ <u>*and*</u> transpose the 3 with the 2 in $\vec{\theta}_4$ to obtain

[3] The concept of multivariate arrangement increasing is a generalization of a bivariate ordering due to Hollander, Proschan, and Sethuraman (1977). Technically, the order is a pre-ordering rather than a partial ordering because anti-symmetry only holds up to a vector permutation. But the distinction is of no consequence in our analysis because symmetry will allow us to treat vector permutations as being identical.

$$\Theta_1 = \begin{pmatrix} 2 & 0 & 3 & 2 \\ 3 & 1 & 4 & 3 \\ 4 & 2 & 1 & 1 \end{pmatrix}. \tag{2.3}$$

The array is still not perfectly aligned because the 2nd and 3rd arguments in the columns for $\vec{\theta}_3$ and $\vec{\theta}_4$ do not align with those in the columns for $\vec{\theta}_1$ and $\vec{\theta}_2$. That is, $(3-4)(4-1)<0$, $(3-4)(3-1)<0$, $(1-2)(4-1)<0$, and $(1-2)(3-1)<0$. Two more sets of transpositions will exhaust all opportunities to improve alignment. Specifically, $\Theta_1 \to \Theta_2 \to \Theta_3$ where

$$\Theta_2 = \begin{pmatrix} 2 & 0 & 3 & 2 \\ 3 & 1 & 1 & 1 \\ 4 & 2 & 4 & 3 \end{pmatrix}, \quad \Theta_3 = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 3 & 1 & 3 & 2 \\ 4 & 2 & 4 & 3 \end{pmatrix}. \tag{2.4}$$

When all opportunities to transpose in a manner that increases the arrangement pre-order have been exhausted, as in Θ_3, then the array is said to be *maximally arranged*.[4]

To best explain the order, and how it relates to other concepts of order, we pose the definition in an alternative manner:

Lemma 2.1. $\{\vec{x}_1^b, \vec{x}_2^b, \ldots, \vec{x}_w^b\} \overset{a}{\leq} \{\vec{x}_1^c, \vec{x}_2^c, \ldots, \vec{x}_w^c\}$ *if and only if* $\{\vec{x}_i^b, \vec{x}_j^b\} \overset{a}{\leq} \{\vec{x}_i^c, \vec{x}_j^c\}$ $\forall\, (i,j) \in \Omega_w \times \Omega_w,\ i \neq j$.

All proofs have been placed in the Appendix. As we will shortly show, it is not a coincidence that this pair-wise presentation of the definition bears a striking resemblance with a description of the supermodularity property. Milgrom and Roberts (1990, Theorem 2) have demonstrated that a multivariate function where all second-order cross-derivatives exist

[4] See footnote 3 on the distinction between a pre-order and a partial order.

everywhere is supermodular if and only if all these derivatives are (weakly) positive on the domain of support.

The form of production technology that is monotone under the MAI order is given as

Definition 2.2. A function $g(Z):(\mathbb{R}^n)^w \to \mathbb{R}$ is said to be (weakly) *multivariate arrangement increasing* (*decreasing*) if $g(X) \leq (\geq)\, g(Z)$ whenever $X \overset{a}{\leq} Z$.

Multivariate arrangement increasing (decreasing) functions are said to be MAI (MAD) functions. A member of the set of MAI (MAD) functions may be denoted as $g(\cdot) \in \text{MAI}$ (MAD). Of course, the negative of an MAI function is MAD.

Since Definition 2.2 asserts that the MAI order induces a weak improvement in the function value, the function may have a common value over a set of partially ordered array arrangements. Thus, any optimum allocation should be viewed as a set that may not be a singleton. To reduce repetition of qualifiers concerning inferences that will be drawn about efficient factor allocations, we make[5]

Assumption 2.1. Among the partially ordered arrangements that support equal values of economic surplus, only the largest in the MAI order will be considered.

Some examples of MAI functions will provide a sense of why the attribute should be of interest to economists.

Example 2.2. Schur-concave functions of the form

[5] Alternatively, we could have studied only functions that are strictly monotone in the MAI order. Or a set concept of optimality could have been introduced in the manner of Veinott's strong set order (Milgrom and Shannon 1994).

$$F(X) \;=\; F(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_w) \;=\; g\left(\Sigma_{i=1}^{w} x_{i,1}, \Sigma_{i=1}^{w} x_{i,2}, \ldots, \Sigma_{i=1}^{w} x_{i,n}\right) \in \text{MAD}. \tag{2.5}$$

Here X is a cross product of simplices on the intervals of positive real numbers, $X = \{\vec{x}_1, \ldots, \vec{x}_w\} \in \times_{i=1}^{w} \{\vec{x}_i \in \overline{\mathbb{R}}_+^n : \Sigma_{j=1}^{n} x_{i,j} = \overline{X}_i \;\; \forall\, i \in \Omega_w\}$. This functional specification has been employed by Atkinson (1970) in his seminal work on income distribution, and also by Chambers and Quiggin (2000) in their studies of the firm under uncertainty.

Example 2.3. When $g(\cdot): \mathbb{R}^n \to \mathbb{R}$ is supermodular, then the sum[6]

$$F(X) \;=\; F(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_w) \;=\; \sum_{j=1}^{n} g(x_{1,j}, x_{2,j}, \ldots, x_{w,j}) \in \text{MAI}. \tag{2.6}$$

Since the property 'increasing' is ordinal, if $g(\cdot): \mathbb{R}^w \to \mathbb{R}_+$ is log-supermodular, it follows from (2.6) that the product

$$F(X) \;=\; \prod_{j=1}^{n} g(x_{1,j}, x_{2,j}, \ldots, x_{w,j}) \tag{2.7}$$

is MAI.

Notice that all these functions are permutation invariant in the sense that $F(\vec{x}_1^{\pi}, \vec{x}_2^{\pi}, \ldots, \vec{x}_w^{\pi}) \equiv F(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_w) \;\forall\, \pi$ where π is any of the $n!$ permutations of the vector arguments. In our context, this means that how the firms are labeled is irrelevant.

Example 2.4. In this example, and in contrast to much of the paper's content, we study the decision of an individual. The issue is decision-making under uncertainty, and we will use the MAI concept to model a multivariate distribution. The risk averse individual allocates a fixed stock of her wealth across investment opportunities. The dimensionality of the heterogeneities is given by the w states of nature, and the arrangement order at issue concerns how asset returns in a

[6] Supermodular functions are formally defined in Section 4.

given state are ordered. Also in contrast with the remainder of this paper, the inference drawn in this example pertains to welfare rather than allocation. The asset portfolio holder allocates \$1 wealth among n assets with the objective of maximizing a function that is linear in the first two moments of portfolio returns. The return on the ith asset, $i \in \Omega_n$, is given by $\vec{\theta}_i^b \in \mathbb{R}^w$ where each of the w states is equiprobable. The multivariate distribution may be represented as the array Θ^b $(\vec{\theta}_1^b, \vec{\theta}_2^b, \ldots, \vec{\theta}_n^b) \in (\mathbb{R}^w)^n$. The investor's optimum allocation is assumed to be $\vec{\alpha}_*^b$, an interior point on the simplex $\{\vec{\alpha} \in \mathbb{R}_+^n : \vec{\alpha} \cdot \vec{1} = 1\}$. Allocation to the ith asset in the optimal vector is identified as $\alpha_{*,i}^b$ under character array Θ^b. The vector of mean returns is given by $\vec{\mu}^b = \mu \vec{1}$, i.e., mean returns are common, while the coefficient of risk aversion is written as the positive number λ. Expected utility may then be expressed as

$$\mathcal{L}(\vec{\alpha}_*^b : \Theta^b) = \mu - \tfrac{1}{2}\lambda \sum_{i=1}^n \sum_{j=1}^n \alpha_{*,i}^b \alpha_{*,j}^b \operatorname{Cov}(\vec{\theta}_i^b, \vec{\theta}_j^b). \tag{2.8}$$

We seek to compare ex-ante investor welfare under Θ^b with that under the alternative distribution as represented by $\Theta^c = (\vec{\theta}_1^c, \vec{\theta}_2^c, \ldots, \vec{\theta}_n^c)$ where the ex-ante optimal allocation vector is then given by $\vec{\alpha}_*^c$. This vector is also assumed to be interior on the simplex. Applying Lemma 2.1 to the covariance terms, we have $\mathcal{L}(\vec{\alpha}_*^c : \Theta^b) \geq \mathcal{L}(\vec{\alpha}_*^c : \Theta^c)$ for $\Theta^b \overset{a}{\leq} \Theta^c$. Indeed, relation $\Theta^b \overset{a}{\leq} \Theta^c$ is both necessary and sufficient to infer $\mathcal{L}(\vec{\alpha}_*^c : \Theta^b) \geq \mathcal{L}(\vec{\alpha}_*^c : \Theta^c)$ when $\vec{\alpha}_*^c$ can assume any (not necessarily interior) value on the allocation simplex. Since $\vec{\alpha}_*^c$ need not be ex-ante optimal under multivariate distribution Θ^b, the chain of inequalities can be extended to $\mathcal{L}(\vec{\alpha}_*^b : \Theta^b) \geq \mathcal{L}(\vec{\alpha}_*^c : \Theta^b)$ $\geq \mathcal{L}(\vec{\alpha}_*^c : \Theta^c)$ so that the investor has an ex-ante preference for the distribution that is not as well arranged. Put simply, multivariate distribution Θ^b is possessed of less systemic risk than Θ^c.

Example 2.5. To illustrate the relevance of the order for understanding allocation vectors, suppose that three firms produce a single good using two inputs and that all three markets are efficient. The *market production function* is $F(\vec{x}_1, \vec{x}_2 : \Theta) : \mathbb{R}^{18} \to \overline{\mathbb{R}}_+$ where Θ is an array of four

character vectors as in Example 2.1, and where $F(\cdot) \in \text{MAI}$. Observe that the second and third rows are ordinally aligned in Θ_2, i.e., $3 < 4$, $1 < 2$, $1 < 4$, and $1 < 3$. However, the alignment does not extend to the allocation array when the market array is $X_0 : \Theta_2$, as given in Equation (2.2), because $4 > 2$ and $1 > 0$. Upon invoking Assumption 2.1, market array $X_0 : \Theta_2$ cannot represent an efficient equilibrium because one can (weakly) increase output by transpositions $4 \leftrightarrow 2$ and $1 \leftrightarrow 0$ to obtain the market array

$$Y_1 = \begin{pmatrix} 1 & 3 & : & 2 & 0 & 3 & 2 \\ 2 & 0 & : & 3 & 1 & 1 & 1 \\ 4 & 1 & : & 4 & 2 & 4 & 3 \end{pmatrix}. \tag{2.9}$$

These transpositions do not violate any resource constraints because the value $\vec{x}_i \cdot \vec{1}$ is invariant to the order of summation. Array $X_0 : \Theta_2$ cannot represent an efficient equilibrium because an alternative position on the cross-product of simplices $\vec{x}_1 \times \vec{x}_2$ dominates it. Whatever the efficient equilibrium is, it is not $X_0 : \Theta_2$.

To generalize on Example 2.5, commence with an inspection of the character array only, Θ, and identify some set of rows (there may be more than one such set of rows) that, by themselves, are maximally arranged according to Definition 2.1. Eliminate all other rows so that what remains is really a sub-array of the character array. Label the residual reduced array as an *aligned character sub-array* (ACsA). The corresponding market array is called an *extended* ACsA. Absent other conditions, this extended ACsA need not be aligned in all arguments.

As an illustration, and with reference to Example 2.5, for character array Θ_2 there are two non-trivial ACsA. These are row pair 1 and 3 as well as row pair 2 and 3, and are presented in sequence below

$$\begin{pmatrix} 2 & 0 & 3 & 2 \\ 4 & 2 & 4 & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 1 & 1 \\ 4 & 2 & 4 & 3 \end{pmatrix}. \tag{2.10}$$

For row pair 2 and 3, when the market array is $X_0 : \Theta_2$ then the extended ACsA is

$$\begin{pmatrix} 4 & 1 & : & 3 & 1 & 1 & 1 \\ 2 & 0 & : & 4 & 2 & 4 & 3 \end{pmatrix}. \tag{2.11}$$

As this ACsA is not maximally arranged, it will not arise in an efficient economy. Whatever the equilibrium in a first-best allocation, it is not consistent with (2.11). To formalize, we make

Assumption 2.2. Allocation is efficient.

In the general setting, the observation concerning induced order in the extended ACsA may be stated as

Theorem 2.1. *Under assumptions 2.1 and 2.2, if the market production function is MAI then an extended ACsA in equilibrium must be maximally arranged.*

Put another way, if allocation is efficient then the allocation component of the extended ACsA inherits the order that all ACsAs are constructed to possess.

Example 2.6. Upon occasion, a vector of characters will become a decision vector. Suppose that firms in a market have been allocated SO_2 emissions permits, but cannot trade them, and that the second column in array (2.11) represents the endowed emissions permits for two firms. The character array, now comprised of the five right-most columns in (2.11), is not an ACsA. If, however, a market in these permits is legalized, then the second column enters the allocation component of the extended sub-array rather than the character component and the narrower character component is an ACsA. Theorem 2.1 now applies, so that if an allocation is (permit endowment constrained) efficient then the lower firm (firm 3) will always use more of the first

and the second (i.e., permit) factors in production than will the upper firm (firm 2). That is, $(x_{1,3}, x_{2,3}) = \vec{x}(3) \geq \vec{x}(2) = (x_{1,2}, x_{2,2})$ where vector order relation $\geq$ is understood to mean the usual coordinate-wise order. In this example $\vec{x}(j) \in \overline{\mathbb{R}}_+^2$, and in general $\vec{x}(j) \in \overline{\mathbb{R}}_+^r$ since each firm uses r factors in production. This component-wise ordering on factor use across firms will be referred to as the strong order on factor use.

3. Representation through measures

Technological heterogeneities are unlikely to present themselves in explicit parametric form. To establish the generality of the insights provided by definitions 2.1 and 2.2, we will extend the theory of MAI functions by constructing non-parametric functions such that the inferences in Theorem 2.1 remain valid.[7] It has been observed by Marshall and Olkin (1979, p. 162) that arrangement increasing functions are convex cones.

Definition 3.1. [Aliprantis and Border (1999, p. 180)] Set T is a *convex cone* if it is convex, and if $x \in T$ implies $\alpha x \in T$ for every $\alpha \in \overline{\mathbb{R}}_+$.

This observation is important because it allows us to view the character parameters in MAI functions as degenerate measures that may be generalized. Specifically, instead of Θ we may conceive of a continuous, positive, finite measure $G(\Theta): (\mathbb{R}^n)^s \to \mathbb{R}$ on the s character vectors, and use it to ascribe a linear functional $H[X: G(\Theta)]: (\mathbb{R}^n)^r \times G(\Theta) \to \mathbb{R}$ for the production technology in r factors where[8]

[7] The authors know of one paper, by Proschan and Leysieffer (1992), that studies MAI functions (when $r = s = 1$) with tools from functional analysis. Their concern is with problems in statistical analysis, and their findings do not overlap with ours. The concept of rank densities that they work with may, however, be of utility in future studies of economic allocation.

[8] Throughout, we treat as equivalent all measures that are equal almost everywhere.

$$H[X:G(\Theta)] = \int F(X:\Theta)\, dG(\Theta). \tag{3.1}$$

Concerning Θ, we assume that it has support on $[\theta_l, \theta^u]^n \subset \mathbb{R}^n$, $\theta_l < \theta^u$. Upon suppressing $G(\Theta)$ in presentation, the function is written as $H[X]:(\mathbb{R}^n)^r \to \mathbb{R}$. In the functional analysis to follow, we refer to $F(X:\Theta)$ as the kernel and $G(\Theta)$ as the multivariate measure. It is sought to ascertain properties on the measure such that the asymmetries underpinning Theorem 2.1, as it applies to an MAI kernel, persist in $H[X:G(\Theta)]$, i.e., in $H[X]$.

To facilitate development of the main insights, it will be assumed in the initial analysis that $s = 1$ so that $\Theta = \vec{\theta}_1$. We will also assume that the measure is comprised of the product of independent measures.

Assumption 3.1. For $s = 1$, $G(\Theta) = \prod_{j=1}^{n} G_j(\theta_{1,j})$ where $\theta_{1,j}$ is the variable of integration for $G_j(\theta_{1,j})$ and each $G_j(\theta_{1,j})$ is a continuously differentiable, positive, finite measure on $[\theta_l, \theta^u]$ with $dG_j(\theta)/d\theta = g_j(\theta) \geq 0$ and $G_j(\theta_l) = 0$.

Thus, heterogeneities (i.e., asymmetries) involving interdependence between factor allocations are expressed only in the kernel.[9] An order relation on (probability) measures that has proven to be of considerable utility in contract theory and elsewhere in economics is the monotone likelihood ratio order.[10] It will also prove useful in our study of efficient allocation.

Definition 3.2. Under Assumption 3.1, let

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[9] Our proofs involve reflections across a bisector that partitions a fundamental region on the measure's support. This fundamental region is given as $\theta_1 < \theta_2 < \ldots < \theta_n$ or one of its $n!$ permutations. The independence assumption, while very useful in modeling reflections across bisectors, is not essential. Using an approach in Shanthikumar and Yao (1991), we could also study how dependent measures affect allocation. Dependent measures are considerably more difficult to work with, especially for measures with more than two character arguments.

[10] See, e.g., Milgrom (1981) or Landsberger and Meilijson (1990).

D3.2 a) $g_1(\theta)/g_2(\theta)$ be nondecreasing over the union of the supports of the measures (with $k/0$ ascribed the value ∞ whenever $k \in \mathbb{R}_+$, or equivalently

D3.2 b) $g_1(v)g_2(u) \geq g_1(u)g_2(v)$ for all $v \geq u$.

Then $G_1(\theta)$ is said to be larger than $G_2(\theta)$ in the *likelihood ratio* (*LR*) order, and is denoted by $G_1(\theta) \overset{lr}{\geq} G_2(\theta)$.

Notice that while $G_j(\theta) \geq 0$ is required, in contrast with probability measures we do not require that $G_j(\theta^u)$ be common across Ω_n. This makes no difference, however, in Definition 3.2 because properties D3.2 a) and D3.2 b) are scale invariant so that the distinction between a probability measure and a finite measure is not germane. Our central result in this section is

Theorem 3.1. *Under assumptions 2.1, 2.2, and 3.1, let* $s = 1$ *and let* $H[X:G(\Theta)]$ *be as given in (3.1). Then* $\vec{x}(j) \geq \vec{x}(k)$ $\forall\, F(X:\Theta) \in$ MAI *if and only if* $G_j(\theta) \overset{lr}{\geq} G_k(\theta)$.

The multivariate measure does admit evaluations of the 'character parameters' such that the kernel would suggest that $\vec{x}(j) \geq \vec{x}(k)$ is not optimal. But the $\overset{lr}{\geq}$ relation across variates in the multivariate measure ensures that the weightings on these out-turns are more than offset by weightings on the hidden character parameters such that $\vec{x}(j) \geq \vec{x}(k)$ is optimal. To be completely clear about how Theorem 3.1 may be viewed as an extension of Theorem 2.1, suppose that, up to a permutation of Ω_n, the independent univariate measures may be ordered as $G_1(\theta) \overset{lr}{\geq} G_2(\theta) \overset{lr}{\geq} \ldots \overset{lr}{\geq} G_n(\theta)$. Then $H[X:G(\Theta)]$ may be considered to be MAI in a functional sense on the array $X:\theta_1$ whereby function $G_j(\theta)$ replaces the jth evaluation of the single vector in the character array.

Example 3.1. The subject of analysis need not be an efficient market in which exchange occurs. The key supposition, Assumption 2.2, is also satisfied when an efficiently operating market is

replaced by a rational individual seeking to optimize the allocation of a finite resource stock, such as the hours in a day or available wealth. Landsberger and Meilijson studied the portfolio allocation problem

$$\underset{\vec{x}}{\text{Max}} \int U(\vec{x}\cdot\vec{\theta}_1)\,\Pi_{i=1}^{n} dG_i(\theta_{1,i}), \qquad \text{s.t. } x_i > 0 \;\forall\, i \in \Omega_n,\; \vec{x}\cdot\vec{1} = 1, \tag{3.2}$$

where function $U(\cdot)$ is increasing and where the $G_j(\theta)$ are now probability measures. Kernel $F(X:\Theta) = U(\vec{x}\cdot\vec{\theta}_1)$ is then MAI. Consequently, their finding that $G_1(\theta) \overset{lr}{\geq} G_2(\theta) \overset{lr}{\geq} \ldots \overset{lr}{\geq} G_n(\theta)$ implies the allocation $x_1 \geq x_2 \geq \ldots \geq x_n$ follows from Theorem 3.1 above.

4. Supermodular market

Among the sub-classes of MAI functions outlined in examples 2.2 and 2.3, the sub-class that lends itself most readily to an analysis of efficient equilibrium was given in equation (2.6), i.e., the sum of character differentiated supermodular functions. This should not be surprising given the strong implications of the supermodularity assumption in the theory of comparative statics. Before engaging in a formal analysis of efficient allocation when market-level output can be represented as in specification (2.6), we will remind the reader of some of the principal technical features of lattice theory as it applies to optimization.[11]

Definition 4.1. A *lattice*, L, is a set that is closed under two binary operations. These are the least upper bound operation $\vee$ and the greatest lower bound operation $\wedge$, such that for all elements $a, b, c \in L$ the following postulates are satisfied: i) to every ordered pair (a,b) of elements is assigned a unique element $a \vee b$ and also a unique element $a \wedge b$, ii) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$, iii) $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, iv) $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$.

[11] The definitions below are summarized from Milgrom and Shannon (1994) and also Topkis (1998).

Definition 4.2. Given a lattice L and a partially ordered set T, the function $h(\cdot): L \to \mathbb{R}$ is *supermodular* if for all $a, b \in L$, $h(a) + h(b) \leq h(a \vee b) + h(a \wedge b)$. Function $h(\cdot): L \times T \to \mathbb{R}$ has *increasing differences* in $(x, t) \in L \times T$ if for $x_+ \geq x$, $g(x_+, t) - g(x, t)$ is monotone non-decreasing in t where $\geq$ is the lattice order.

Milgrom and Shannon (1994) have shown that if $h(\cdot): L \times T \to \mathbb{R}$ is supermodular on L with increasing differences on $L \times T$ then the set of maximizing arguments on L is monotone non-decreasing in evaluations of T. When $L \times T$ is the metric space $\mathbb{R}^{r+s}$, then their result applies to any continuously differentiable function such that all $(r+s)(r+s-1)/2$ second-order cross-derivatives are positive. In our study of efficient allocation equilibria, the algebraic structure of our choice set is the direct product of simplices. It is readily shown that this direct product is not a lattice. Nonetheless, it will be shown that their finding bears striking similarities with the conclusions that we will draw for the attributes of allocation equilibria when market production is of the form (2.6). For the allocation problem and when heterogeneities among firms are represented through vector arguments, as in Section 2 above, then the MAI pre-order (up to re-permutations) on the character array provides the partially ordered set, T, of interest. We seek now, however, to identify pertinent attributes of partially ordered sets, and the associated ordering, when the character array has been replaced by a multivariate measure. That is, we seek a result in the manner of Theorem 3.1 when the MAI market surplus function has been specialized to be supermodular.

To formalize the technical environment underpinning the market, we define

Definition 4.3. Let the jth firm's production function be of the form $h(x_{1,j}, \ldots, x_{r,j}: \theta_{1,j}, \ldots, \theta_{s,j})$ where $h(\cdot)$ is supermodular in $\vec{x}(j)$ and has increasing differences in $\left(\vec{x}(j), \vec{\theta}(j)\right)$. Then the market is said to be a *supermodular market*.

Notice the additive separability of form (2.6) in factors so that firms do not impose production externalities on each other, except through the aggregate allocation constraint. This is in contrast with an arbitrary Schur-concave function, as given in Example 2.2 above, and it is this additive separability that makes the supermodular market a particularly convenient form of MAI market surplus function to work with.

Before developing our results, we need to clarify our notation to identify increasing differences between allocations and sources of heterogeneity among firms. Our interest, as in Athey's study of stochastic dominance, is in convex cones of functions. For the moment, we will confine our attention to the situation where there is just one character argument. Write $p_1 = 1$ if a function is continuously differentiable and (weakly) increasing in that character argument, $p_1 = -1$ if it is continuously differentiable and decreasing, and $p_1 = 0$ if it is not assumed to possess either property. Generally, write $p_k = 1$ if the function's k^{th} derivative in the character argument exists and is uniformly positive, $p_k = -1$ if it exists and is uniformly negative, and $p_k = 0$ if it is not assumed to possess either property. Now write the vector $\vec{p} = (p_1, p_2, p_3, \ldots)$ where the vector terminates at the kth ordinate whenever $p_l = 0 \ \forall\, l > k$. Written in this way, vector $\vec{p}$ represents a cone of functions where the cone is generated from the function's analytic properties along one dimension. Thus, $\vec{p} = (1, -1)$ is understood to represent the cone of twice continuously differentiable, increasing and concave functions in one dimension.

Definition 4.4. Consider a supermodular market where the firm production technology, $h(\cdot)$, has just one character argument. Function $h(x_{1,j}, \ldots, x_{r,j} : \theta_{1,j})$ is said to possess property $P-\vec{p}$ if $dh(x_{1,j}, \ldots, x_{r,j} : \theta_{1,j}) / dx_{i,j} \in \vec{p} \ \forall\, i \in \Omega_r, \ \forall\, \vec{x}(j) \in \overline{\mathbb{R}}^r_+$.

Associated with each property $\vec{p}$ there is a stochastic ordering that increases the expected value of that class of functions. For example, if $\vec{p} = (0, -1)$, then the order in question is the mean-preserving contraction. If, instead, $\vec{p} = (1, -1)$ then the order in question is second-degree

stochastic dominance. While the independent univariate measures written in Assumption 3.1 do not conform to the unit integration requirement of probability measures, a normalization that arises naturally for supermodular kernels will ensure that they do. Writing $\hat{G}(\theta) \equiv G(\theta)/G(\theta^u)$, we denote the dominance order relation $\hat{G}_j(\theta) \overset{P-\vec{p}}{\geq} \hat{G}_k(\theta)$ whenever $\int_{\theta_l}^{\theta^u} h(\theta)\, dG_j(\theta)/G_j(\theta^u) \geq \int_{\theta_l}^{\theta^u} h(\theta)\, dG_k(\theta)/G_k(\theta^u)\ \forall\, h(\theta) \in \vec{p}$. This notation is of relevance in our approach to allocation because dominance relations among normalized independent measures provide the partial ordering that we require to implement a measure generalization of Theorem 2.1 that has been specialized for a firm-level production function which is supermodular in factors.

Theorem 4.1. *Under assumptions 2.1, 2.2, and 3.1, let* $s = 1$ *and let* $H[X:G(\Theta)]$ *be as in (3.1) where the kernel is that for a supermodular market. Further, suppose that the supermodular function* $h(\cdot)$ *has property* $P-\vec{p}$. *Then* $\vec{x}(j) \geq \vec{x}(k)$ *for all such kernels if and only if* $\hat{G}_j(\theta) \overset{P-\vec{p}}{\geq} \hat{G}_k(\theta)$.

Example 4.1. Let $h(x_{1,j}, x_{2,j}:\theta_{1,j})$ be supermodular in $(-x_{1,j}, x_{2,j}, \theta_{1,j})$. Then property $\hat{G}_j(\theta) \overset{P-(1)}{\geq} \hat{G}_k(\theta)$ implies $x_{2,j} \geq x_{2,k}$ and $x_{1,j} \leq x_{1,k}$. If $\hat{G}_1(\theta) \overset{P-(1)}{\geq} \hat{G}_2(\theta) \overset{P-(1)}{\geq} \ldots \overset{P-(1)}{\geq} \hat{G}_n(\theta)$ where $x_{1,j}$ and $x_{2,j}$ are the jth firm's respective labor and capital allocations, then the firm's univariate character measure, $G_j(\theta)$, may be thought of as an indicator for the firm's use of technologies that substitute capital in for labor.

We close this section by generalizing the character array to contain an arbitrary finite number of character vectors so that the kernel has $r+s$ arguments rather than the $r+1$ arguments studied in Theorem 4.1. Suppose that firm-level production for the jth firm is given by $h(x_{1,j}, \ldots, x_{r,j}:\theta_{1,j}, \ldots, \theta_{s,j})$ which is supermodular in all arguments. Extending the measure on firm characters accordingly, let the possibility of dependence in characters within a firm exist, but preclude dependence across firms.

Assumption 4.1. $G(\Theta) = \prod_{j=1}^{n} G_j(\theta_{1,j}, \ldots, \theta_{s,j})$ where each $G_j(\theta_{1,j}, \ldots, \theta_{s,j})$ is a continuously differentiable, positive, finite measure on $[\theta_l, \theta^u]^s$ with $G_j(\theta_l, \ldots, \theta_l) = 0 \ \forall j \in \Omega_n$.

With $\hat{G}_j(\theta_{1,j}, \ldots, \theta_{s,j}) = G_j(\theta_{1,j}, \ldots, \theta_{s,j}) / G_j(\theta^u, \ldots, \theta^u)$, the pertinent dominance relation is the supermodular order as defined in Shaked and Shanthikumar (1997).[12]

Definition 4.5. Positive, finite, normalized measure $\hat{G}_j(\theta_{1,j}, \ldots, \theta_{s,j})$ is said to be larger than positive, finite, normalized measure $\hat{G}_k(\theta_{1,k}, \ldots, \theta_{s,k})$ in the *supermodular order* [and written as $\hat{G}_j(\theta_1, \ldots, \theta_s) \overset{sm}{\geq} \hat{G}_k(\theta_1, \ldots, \theta_s)$] if $\int m(\theta_1, \ldots, \theta_s)\, d\hat{G}_j(\theta_1, \ldots, \theta_s) \geq \int m(\theta_1, \ldots, \theta_s)\, d\hat{G}_k(\theta_1, \ldots, \theta_s)$ for all supermodular functions $m(\theta_1, \ldots, \theta_s)$ for which the integrations exist.

This class of measures is precisely that which induces allocative order across firms when allocation is efficient.

Theorem 4.2. *Under assumptions 2.1, 2.2, and 4.1, let* $H[X : G(\Theta)]$ *be as in (3.1) where the kernel is that for a supermodular market. Suppose too that* $d^3 h(\cdot) / dx_{i,j} d\theta_{k,j} d\theta_{l,j} \geq 0 \ \forall i \in \Omega_r$, $\forall (k, l) \in \Omega_s \times \Omega_s$, $k \neq l$. *Then* $\vec{x}(u) \geq \vec{x}(v)$ *whenever* $\hat{G}_u(\theta_1, \ldots, \theta_s) \overset{sm}{\geq} \hat{G}_v(\theta_1, \ldots, \theta_s)$, $(u, v) \in \Omega_n \times \Omega_n$, $u \neq v$.

Example 4.2. The indicator product function $I_{\{\theta_1 \geq \bar{\theta}_1\}} I_{\{\theta_2 \geq \bar{\theta}_2\}}$ is supermodular. Together with $-I_{\{\theta_1 \geq \bar{\theta}_1\}}$ and $-I_{\{\theta_2 \geq \bar{\theta}_2\}}$, the function set $I_{\{\theta_1 \geq \bar{\theta}_1\}} I_{\{\theta_2 \geq \bar{\theta}_2\}}$, $(\bar{\theta}_1, \bar{\theta}_2) \in \mathbb{R}^2$, can be used as an infinite dimensional basis for constructing arbitrary supermodular kernels in $\mathbb{R}^2$.[13] Let

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[12] Because monotonicity is not imposed on the reference class of functions, $m(\cdot)$, the supermodular order must fix the marginal distributions so that we need not be concerned with the monotonicity status of $h(\cdot)$ in $\theta_{i,j}$, $i \in \Omega_s$.

[13] See Theorem 2.5 in Müller and Scarsini (2000) or Table 1 in Athey (2000).

$$\Theta^b = \{\vec{\theta}_1, \vec{\theta}_2\} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \tag{4.1}$$

and perform the MAI ordered transposition to

$$\Theta^c = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}. \tag{4.2}$$

This increase along the partial ordering weakly increases the value of $\Sigma_{j=1}^{2} I_{\{\theta_{1,j} \geq \bar{\theta}_1\}} I_{\{\theta_{2,j} \geq \bar{\theta}_2\}}$ $\forall$ $(\bar{\theta}_1, \bar{\theta}_2) \in \mathbb{R}^2$ while the independent univariate measures have not changed. Let $h(x_{1,j}, \ldots, x_{r,j} : \theta_{1,j}, \ldots, \theta_{s,j})$ satisfy the conditions in Theorem 4.2. Then, after $\Theta^b \to \Theta^c$, firm 2 in an efficient supermodular market will use more of all resources than will firm 1. The rearrangement $\Theta^b \to \Theta^c$ has acted as a discrete increase in the supermodular order on the bivariate measure of character variables. Compared with when it does not apply, the partial order relation $\hat{G}_2(\theta_1, \theta_2) \overset{sm}{\geq} \hat{G}_1(\theta_1, \theta_2)$ will tend to concentrate market production toward firm 2 in this example since it is generally assumed that a firm's production will increase in factor use.

Example 4.3. Suppose that the jth firm's supermodular firm-level production $h(x_{1,j}, \ldots, x_{r,j} : \theta_{1,j}, \ldots, \theta_{s,j})$ is transformed to $h[g_1(x_{1,j}), \ldots, g_r(x_{r,j}) : g_{r+1}(\theta_{1,j}), \ldots, g_{r+s}(\theta_{s,j})]$ where each $g_m(\cdot)$, $m = 1, 2, \ldots, r+s$, is monotone in the same direction. Then $h[\cdot]$ remains supermodular in its original arguments so that Theorem 4.2 continues to apply. Since the property 'monotone' is ordinal, this example demonstrates the ordinal nature of the approach taken in this paper. Returning to Definition 2.1, if each of the w vectors is transformed by the univariate function $g_m(\cdot)$ where all w such functions are monotone in the same direction, then order among arrays is preserved.

5. Conclusion

The path taken in our theory of efficient resource allocation across firms in an industry 'builds' asymmetries into a firm's production technology through the use of linear functionals. The end has been to draw inferences concerning the factor allocations a social planner would make. While the underlying algebraic structure of the choice set available to the social planner differs from that countenanced by the micro-agent in the theory of optimization on a lattice, the similarities in other regards are marked. Both frameworks model decisions in an ordinal manner. Partly as a consequence, in both frameworks the typical regularity assumptions of differentiability, convexity, and even continuity may be dispensed with. Supermodular functions play an important role in both approaches, although in neither case are supermodular functions the most general functions for which the respective theory applies. Most importantly, both are, in one form or another, applications of separating hyperplane results. It would seem then that the analyst seeking an integrated theory of optimal economic choices in equilibrium might first turn to convex analysis.

Unfortunately, the theory of MAI functions has not developed far beyond the initial inquiry of Hollander, Proschan, and Sethuraman (1977) and the later multivariate extension due to Boland and Proschan (1988). It is the opinion of the authors that a systematic development of classes of functions with built-in structural asymmetries will shed light on a variety of economic problems. In the international trade literature, for example, it would seem that comparative advantage may be represented as an asymmetry in the available technologies. The theory of comparative advantage, however, involves trade in two or more goods. A remarkable feature of our single market model is that we could ignore price effects in output markets. This is due, at least in part, to the fact that our theory relates nothing about cardinal measures of factor use since factors may flow out of the studied output market. While our results remain valid in an efficient general equilibrium, an inquiry into the consequences of technical asymmetries for relative factor and product prices in general equilibrium would necessitate a more robust framework in which to

model the asymmetries. Output price effects would also have to be accommodated in a single market where firms have some market power. Perhaps the most immediate extension of this ordinal approach would involve the treatment of production externalities. In particular, how do production externalities affect the divergence between efficient equilibria and market-supported equilibria?

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Appendix

Proof of Lemma 2.1. We will demonstrate the forward implication first. The definition of multivariate arrangement order in the sense of $\overset{a}{\leq}$ requires that, when comparing a pair of coordinates across every vector in an array, they can only be transposed if the transposition increases their alignment with an arbitrarily chosen reference 'base' vector. In particular, let the base vector be $\vec{x}_i^b$. Then an increase in the multivariate $\overset{a}{\leq}$ order requires that, when comparing a pair of coordinates across vector $\vec{x}_j^b$, the pair can only be transposed if the transposition increases the vector's alignment with vector $\vec{x}_i^b$. Because $\vec{x}_i^b$ and $\vec{x}_j^b$ are distinct vectors chosen arbitrarily among $\vec{x}_k^b,\ k \in \Omega_w$, the forward implication must hold.

Now suppose that $\{\vec{x}_i^b, \vec{x}_j^b\} \overset{a}{\leq} \{\vec{x}_i^c, \vec{x}_j^c\}\ \forall\,(i,j) \in \Omega_w \times \Omega_w,\ i \neq j$ does not imply $\{\vec{x}_1^b, \vec{x}_2^b, \ldots, \vec{x}_w^b\} \overset{a}{\leq} \{\vec{x}_1^c, \vec{x}_2^c, \ldots, \vec{x}_w^c\}$. Then there would exist some $(i,j) \in \Omega_w \times \Omega_w,\ i \neq j$, and some array of vectors such that $\{\vec{x}_i^b, \vec{x}_j^b\} \overset{a}{\leq} \{\vec{x}_i^c, \vec{x}_j^c\}$ does not adhere when $\{\vec{x}_1^b, \vec{x}_2^b, \ldots, \vec{x}_w^b\} \overset{a}{\leq} \{\vec{x}_1^c, \vec{x}_2^c, \ldots, \vec{x}_w^c\}$ is satisfied. But the 'every' part of the definition ensures that there cannot exist such a pair $(i,j) \in \Omega_w \times \Omega_w,\ i \neq j$, when $\{\vec{x}_1^b, \vec{x}_2^b, \ldots, \vec{x}_w^b\} \overset{a}{\leq} \{\vec{x}_1^c, \vec{x}_2^c, \ldots, \vec{x}_w^c\}$ is satisfied. □

Proof of Theorem 3.1. The line of approach is the extension of a symmetry argument that has been used in the probability order literature by, for example, Shanthikumar and Yao (1991) and elsewhere in economics and finance (see Kijima and Ohnishi 1996, or Lapan and Hennessy 2001). The proof proceeds in three steps. The first clarifies the relevance of Theorem 2.1 to the analysis. The second proves that $G_j(\theta) \overset{lr}{\geq} G_k(\theta)$ implies $\vec{x}(j) \geq \vec{x}(k)$ for all MAI kernels. The third proves that if $G_j(\theta) \overset{lr}{\geq} G_k(\theta)$ is not true then there exists an MAI kernel such that the strong set order on factor use $\vec{x}(j) \geq \vec{x}(k)$ is not true.

<u>Step 1</u>: We seek a concept of total order, $\overset{*}{\geq}$, in the sense that if $G_j(\theta) \overset{*}{\geq} G_k(\theta)$, then efficiency requires that $\vec{x}(j) \geq \vec{x}(k)$. Because the kernel is MAI and the character array is a vector, we can readily appeal to the finding in Theorem 2.1 for intuition. That is, if $\theta_{1,j} \geq \theta_{1,k}$ then, in the parametric case of Theorem 2.1, under efficiency it must be that $\vec{x}(j) \geq \vec{x}(k)$.

Alternatively, if $\theta_{1,j} \le \theta_{1,k}$ then, in the parametric case, it must be that $\vec{x}(j) \le \vec{x}(k)$. In either case, and ignoring firms other than the jth and k^{th}, the efficient allocation will be maximally arranged. Independence (i.e., Assumption 3.1) in measure allows us to consider just two firms. The steps to follow will show that $\overset{lr}{\ge}$ is precisely the order property on a pair of positive, finite, univariate measures such that the measures act as if they were a point in $\mathbb{R}^2$.

<u>Step 2</u>: Now consider $H[X:G(\Theta)]$, written as $H[X:G_1(\theta_{1,1})\times G_2(\theta_{1,2})\times\ldots\times G_j(\theta_{1,j})\times\ldots\times G_k(\theta_{1,k})\times\ldots\times G_n(\theta_{1,n})]$. Interchange the jth and k^{th} evaluations in the character vector to obtain $H[X:G_1(\theta_{1,1})\times G_2(\theta_{1,2})\times\ldots\times G_j(\theta_{1,k})\times\ldots\times G_k(\theta_{1,j})\times\ldots\times G_n(\theta_{1,n})]$. To abbreviate, write the latter as $H[X:G(\Theta_\tau)]$ where the subscripted τ is understood to represent the transposition of parameter evaluations. Using the fact that variables of integration are just placeholders, it is readily shown that the difference between integrals can be written as

$$\begin{aligned} H[X:G(\Theta_\tau)] - H[X:G(\Theta)] &= \int_{\theta_{1,k}\ge\theta_{1,j}} F(X:\Theta)\,dG(\Theta_\tau) + \int_{\theta_{1,k}<\theta_{1,j}} F(X:\Theta)\,dG(\Theta_\tau) \\ &\quad - \int_{\theta_{1,k}\ge\theta_{1,j}} F(X:\Theta)\,dG(\Theta) - \int_{\theta_{1,k}<\theta_{1,j}} F(X:\Theta)\,dG(\Theta) \\ &= \int_{\theta_{1,k}\ge\theta_{1,j}} [F(X:\Theta) - F(X:\Theta_\tau)]\,[dG(\Theta_\tau) - dG(\Theta)] \end{aligned} \tag{A.1}$$

$$= \int_{\theta_{1,k}\ge\theta_{1,j}} [F(X:\Theta) - F(X:\Theta_\tau)]\,[g_j(\theta_{1,k})g_k(\theta_{1,j}) - g_j(\theta_{1,j})g_k(\theta_{1,k})] \prod_{\substack{t=1,\\ t\ne j, t\ne k}}^{n} g_t(\theta_{1,t}) \prod_{t=1}^{n} d\theta_{1,t}.$$

where the continuous differentiability of the finite measures allows us to double count the evaluation $\theta_{1,j} = \theta_{1,k}$. From step A, $F(X:\Theta) \in \text{MAI}$ ensures that the first part of the integrand after the final equality is negative when $\vec{x}(j) \ge \vec{x}(k)$. From Definition 3.2, the second part of that integrand is positive when $G_j(\theta) \overset{lr}{\ge} G_k(\theta)$. Then $H[X:G(\Theta_\tau)] \le H[X:G(\Theta)]$. Alternatively, observe that the rows in the market array may be interchanged so that what matters is how a firm's decision vector matches its character vector. Therefore, if (A.1) is true then we may write

$H[X_\tau : G(\Theta)] \le H[X : G(\Theta)]$ where X_τ represents allocation array X but where the jth and k^{th} firms interchange their decision vectors. Thus, with $G_j(\theta) \overset{lr}{\ge} G_k(\theta)$, efficiency requires that $\vec{x}(j) \ge \vec{x}(k)$.

<u>Step 3</u>: Suppose that $G_j(\theta) \overset{lr}{\ge} G_k(\theta)$ is not true on an interval of strictly positive Lebesgue measure, L, satisfying $\theta_{1,j} \ge \theta_{1,k}$. We will proceed by constructing a violation on a particular extreme point of the cone of MAI functions. In particular, let $F(X:\Theta) = 1$ on interval $T[\hat{\theta}_{1,j}, \Delta\theta_{1,j}, \hat{\theta}_{1,k}, \Delta\theta_{1,k}] = \{(\theta_{1,j}, \theta_{1,k}) : \hat{\theta}_{1,j} \le \theta_{1,j} \le \hat{\theta}_{1,j} + \Delta\theta_{1,j}, \hat{\theta}_{1,k} - \Delta\theta_{1,k} \le \theta_{1,k} \le \hat{\theta}_{1,k}, \Delta\theta_{1,j} > 0,$ $\Delta\theta_{1,k} > 0, \hat{\theta}_{1,j} > \hat{\theta}_{1,k}\}$ $\forall\, \theta_t \in [\theta_l, \theta^u],\ t \notin \{i,j\},\ \forall\, \vec{x}(j) \ge \vec{x}(k)$ and let $F(X:\Theta) = 0$ otherwise. Here, the set bounds are chosen such that $L = X \times [\hat{\theta}_{1,j}, \hat{\theta}_{1,j} + \Delta\theta_{1,j}] \times [\hat{\theta}_{1,k} - \Delta\theta_{1,k}, \hat{\theta}_{1,k}] \times [\theta_l, \theta^u]^{s-2}$. By construction, $F(X:\Theta) \in \text{MAI}$. Now, in the context of (A.1), take $\text{Lim}_{\Delta\theta_{1,j} \to 0, \Delta\theta_{1,k} \to 0} F(X:\Theta)$ to obtain

$$H[X : G(\Theta_\tau)] \;-\; H[X : G(\Theta)] \;=\; [g_j(\hat{\theta}_{1,k}) g_k(\hat{\theta}_{1,j}) \;-\; g_j(\hat{\theta}_{1,j}) g_k(\hat{\theta}_{1,k})] \prod_{\substack{t=1,\\ t \ne j, t \ne k}}^{n} G_t(\theta^u) \tag{A.2}$$

when $\vec{x}(j) \ge \vec{x}(k)$. Clearly, while this $F(X:\Theta)$ is MAI the difference in expression (A.1) is strictly positive when $\vec{x}(j) \ge \vec{x}(k)$. Then any allocation other than those satisfying $\vec{x}(j) \ge \vec{x}(k)$ will be supported in the market. Thus, $G_j(\theta) \overset{lr}{\ge} G_k(\theta)$ is also necessary for the allocation inference to be valid for all MAI functions. □

Proof of Theorem 4.1. Again, independence allows us to confine the analysis to two firms. We seek conditions under which we can apply Theorem 2.1, but where the character parameters are functions rather than scalars. We seek a partial ordering of $G_j(\theta)$ with respect to $G_k(\theta)$ such that, if the measures are so ordered, the expression

$$G_k(\theta^u) \int h[\vec{x}(j) : \theta]\, dG_j(\theta) \;+\; G_j(\theta^u) \int h[\vec{x}(k) : \theta]\, dG_k(\theta) \tag{A.3}$$

cannot be improved upon through re-allocations of factors across firms. Upon division, we require that

$$\int h[\vec{x}(j):\theta]\, d\hat{G}_j(\theta) \;+\; \int h[\vec{x}(k):\theta]\, d\hat{G}_k(\theta) \tag{A.4}$$

cannot be improved upon.

The linear functionals can be re-parameterized so that supermodularity is preserved in a functional sense. Write

$$V[\vec{x}(j):\gamma_j] \;=\; \int h[\vec{x}(j):\theta]\, d\hat{G}_j(\theta), \tag{A.5}$$

where $\gamma \in \mathbb{R}$ indexes, in ascending order, a chain in the dominance order of interest. Thus, by definition, $\hat{G}_j(\theta) \overset{P-\vec{p}}{\geq} \hat{G}_k(\theta)$ if and only if $\gamma_j \geq \gamma_k$. We know from Definition 4.4, the fact that monotonicity is preserved under integration with respect to a positive measure, and use of an interchange in the order of differentiation, that $V[\vec{x}(j):\gamma_j] - V[\vec{x}(j):\gamma_k]$ is monotone non-decreasing in each factor whenever $\gamma_j \geq \gamma_k$. Applying Theorem 2.1, we have the true assertion: when $\gamma_j \geq \gamma_k$ then it must be that $\vec{x}(j) \geq \vec{x}(k)$ if (A.4) cannot be improved upon. As usual, the converse can be demonstrated by picking out extreme points on the cone of functions. □

Proof of Theorem 4.2. From the conditions in the Theorem, we know that $dh(\cdot)/dx_{i,j}$ is supermodular in $\vec{\theta}(j)$ for each $i \in \Omega_r$. From Definition 4.5, if the supermodular partial order is parameterized by scalar γ so that $\gamma_u \geq \gamma_v$ whenever the function partial ordering $\hat{G}_u(\theta_1, \ldots, \theta_s) \overset{sm}{\geq} \hat{G}_v(\theta_1, \ldots, \theta_s)$ is true then $V[\vec{x}(u):\gamma_u] = \int h(x_{1,u}, \ldots, x_{r,u}:\theta_1, \ldots, \theta_s)\, dG_u(\theta_1, \ldots, \theta_s)$ is supermodular in $(x_1, \ldots, x_r, \gamma_u)$. The result then follows from applying Theorem 2.1. □