A Toolbox for the Numerical Study of Linear Rational Expectation Models

P. Marcelo Oviedo*
Iowa State University

First Draft: August 2001
This Draft: December 2003

*I am grateful to Huberto M. Ennis, Paul L. Fackler, and Douglas K. Pearce for their comments. All remaining errors are mine. Further comments are welcome; write to oviedo@iastate.edu
A Toolbox for the Numerical Study of Linear Rational Expectation Models

Abstract

The use of numerical methods to study dynamic economic systems has increased considerably during the last three decades. The simple pioneering work by ... sparked off a new .... By simplifying the computational tasks and by providing step-by-step explanations of the required procedures, this paper and the accompanying ‘toolbox’ of Matlab functions guides a researcher who has almost no experience in computational work to resolve and study his own dynamic model. To illustrate the use of the toolbox functions, the paper works thoroughly out the open- and closed-economy version of the neoclassical growth model.

JEL classification numbers:

P. Marcelo Oviedo
Department of Economics
Iowa State University
279 Heady Hall
Ames, IA 50011
oviedo@iastate.edu
1 Introduction

This paper accompanies and explain how to use the homonymous toolbox to study linear difference rational-expectation macroeconomic models. Although non macroeconomic models could also be studied using the toolbox, the characteristics of its output make it more suitable for dynamic macroeconomic models. The toolbox is a set of Matlab functions. These functions and this paper should help a researcher who has almost no experience in computational work to resolve and study his own model. The simplification of the computational tasks required to study a dynamic model is one distinguishing characteristic of the toolbox, vis–vis other computational algorithms available everywhere.\footnote{Some of the functions in the toolbox were borrow from these other sources although their documentation has been changed here. See for example http://www.ssc.uwo.ca/economics/faculty/klein/main.htm} The provision of step-by-step explanations of the procedures a researcher should follow to study his own model is the second distinguishing characteristic and goal of this work. Briefly, once the user has outlined his model, the toolbox accompanying this paper helps him to log-linearize the model equations, as well as to obtain impulse response functions; obtain sample and population moments conditions of the model variables; and simulate the model under an exogenously given sequence of shocks.

For a small although shrinking fraction of economists, the use of numerical methods to solve and analyze economics models looks odd. Much of the resistance to adopt numerical methods in economics arise from the ‘barrier to start’ meant ..... 

Why numerical methods?

Linear dynamic rational expectation models arise from the linearization of dynamic models around particular stationary points, which commonly is the non-stochastic steady state of the model. There exist several reasons to work with the linear version of a model instead of its original formulation. The most important one is related to the curse of dimensionality to which other solution methods, like value function iteration or those based on a collocation strategy, are subject to. The computer cost of these other methods become significant. Contrarily, linear models can be solved virtually in a couple of seconds. This advantage could be shadowed by a lower accuracy of the results. However, it has been shown that linear models produce very accurate results when the shocks being considered are
not too large.\(^2\)

To provide the reader with two examples of the use of the toolbox, the open- and closed-economy versions of the neoclassical growth model are discussed throughout the paper. The organization of the paper is as follows. The second section shows the type of models than can be studied with the toolbox. The third section describes the two example models. In section four, these two models are calibrated and log linearized. Section five discusses the theoretical solution of the type of models considered in this paper. Section six offers detailed explanations on how the toolbox is used to study the example models. The log-linearization of the models in the examples as well as the solution to these models are in appendices A to NN.

## 2 Models to be Considered

In order to establish a convention on how to classify the variables in the models under study, four types of model-variables are identified here. First, predetermined or backward looking variables. Second, nonpredetermined or forward looking variables. Third, innovations to the backward looking variables. Fourth, flow or additional variables. The meaning of this convention is explained next.

The value of a predetermined variable at time \(t\) is given by the evolution of the system in the previous period and the innovation (third type) that takes place at \(t\). Thus, the value of a predetermined variable at time \(t\) is given by the decisions taken at \(t-1\), the evolution of the system between \(t-1\) and \(t\), and the innovations observed at \(t\). Commonly, backward looking variables correspond to the state variables in a dynamic programming framework. Throughout the paper (and also in the accompanying toolbox), we denote predetermined variables at time \(t\) as \(x_1(t)\).

Within the set of predetermined variables, we distinguish between endogenous and exogenous variables. While the value of an endogenous predetermined variable is affected by economic decisions, exogenous predetermined variables do not depend on any other variable in the model but the innovations to their own processes. The stock of capital and the amount of foreign assets are typical examples of endogenous backward looking variables in an open economy, while a productivity shock is an

\(^2\)See for example ?, ?, and ?.
example of an *exogenous* backward looking variable. The value of both types of backward looking variables at time \( t \) are included in \( x_1(t) \). Without loss of generality, we assume that there are \( n_1 \) backward looking variables.

A forward looking variable is a variable whose \( t \)-value depends on actions taken at time \( t \). We group all forward looking variables in the vector \( x_2(t) \) whose size is equal to \( n_2 \). Appealing to the parallelism with the dynamic programming framework, policy variables like consumption or the accumulation of assets are examples of nonpredetermined variables.

Innovations to the backward looking variables typically are iid shocks hitting some of the state variables in \( x_1 \). For example, the international interest rate or the production-function productivity shock might be modelled as an exogenous autoregressive process subject to period-by-period innovations. Let \( \epsilon(t) \) denote the vector of innovations to the variables in the model. At a high level of generality, we might consider that both forward and backward looking variables could be subject to innovations. Thus, \( \epsilon(t) \) is a vector of length equal to \( n_1 + n_2 \). In most models, however, most of the components of \( \epsilon(t) \) are equal to zero because some predetermined variables (overall, the endogenous ones) and most, if not all, of the nonpredetermined variables are not subject to any kind of shock.

Finally, the fourth set of variables are called flow or additional variables. They are variables not explicitly included in the model but whose behavior is of interest to the researcher and their dynamics depend on other variables in the model. For example, the trade balance may not be a variable in the model but its dynamic behavior can be computed from those of output, consumption and investment.

With this classification of the variables in hand, we can state the general class of models for which the toolbox accompanying this paper is suitable. The specification of a macroeconomic model usually involves three sets of equations, namely: a set of optimality conditions, a set of market clearing conditions and resource constraints, and a set of equations specifying the evolution of some exogenous variables. Examples of these equations are, respectively, the equalization of the marginal rate of substitution of labor for consumption to the real wage rate; the equalization of output to the sum of investment and consumption; and an autoregressive process describing the productivity-shock dynamics.

After log-linearizing the described sets of equations and using the symbol “\(^{\text{\textsuperscript{\%}}}\)” over a variable to express the percentage deviation of its value with respect to the
steady state value, the log-linear version of the model is:

\[ AE_t \hat{x}(t + 1) = B \hat{x}(t) \]  

(1)

where \( \hat{x}(t) \) is a vector defined by \( \hat{x}(t) \equiv (\hat{x}_1(t), \hat{x}_2(t))' \). Inasmuch as the model has \( n_1 \) predetermined variables and \( n_2 \) non-predetermined variables, the size of \( \hat{x}(t) \) is \( n \), where \( n = n_1 + n_2 \). \( A \) and \( B \) are matrices of coefficients, both of size \( n \times n \). In most models, the matrix \( A \) is singular. This happens when at least one equation in the model does not involve any variable dated at time \( t + 1 \). An example is the equation describing optimal labor decisions where the marginal rate of substitution of labor for consumption is equal to the wage rate and none of the variables in the equation is a future-dated variable.

The actual evolution of the system is also affected by the innovations to the \( n \) variables in \( \hat{x} \):³

\[ A \hat{x}(t + 1) = B \hat{x}(t) + \epsilon(t + 1) \]

When only the backward looking variables are subject to innovations, a partition of the vector \( \epsilon(t + 1) \) permits writing the system as:

\[
A \begin{pmatrix}
\hat{x}_1(t + 1) \\
\hat{x}_2(t + 1)
\end{pmatrix} = B \begin{pmatrix}
\hat{x}_1(t) \\
\hat{x}_2(t)
\end{pmatrix} + \begin{pmatrix}
\epsilon_1(t + 1) \\
0_{n_2 \times 1}
\end{pmatrix}
\]

where \( \epsilon_1(t + 1) \) is an \( n_1 \)-vector. Furthermore, when only the exogenous backward-looking variables are subject to innovations, some components of \( \epsilon_1(t + 1) \) are also equal to zero.

Let \( x_3(t) \) be the vector of additional or flow variables at time \( t \). Without loss of generality, assume that there are \( n_3 \) flow variables. The value of these variables depends on the value taken by the variables in \( x_1 \) and \( x_2 \), both at time \( t \) and \( t + 1 \). Log-linearizing the set of equations defining the flow variables:

\[ \hat{x}_3(t) = C \hat{x}(t + 1) + D \hat{x}(t) \]  

(2)

Here, \( \hat{x}_3 \) is an \( n_3 \)-vector and \( C \) and \( D \) are matrices of dimension \( n_3 \times n \).

³Observe that \( A \hat{x}(t + 1) = AE_t \hat{x}(t + 1) + \epsilon(t + 1) \).
3 Two Model: The Open- and Closed-Economy Neoclassical Growth Model

This section presents two models that fit into the class of models discussed above. One is the standard neoclassical growth model of a closed economy and the other is the open-economy version of that model. The models are calibrated and log-linearized in the next section and then solved and analyzed in section 6 and in the Appendix B.

3.1 An Open Economy Facing a Positive-Slopped Supply of Funds

Consider the problem of a central planner of a small open economy that faces a positively sloped supply of external financing. The function depicting the supply of funds is not derived from an optimal contract but it is an ad-hoc one.

The central planner starts with a given stock of capital, $k_0$ and internationally issued bonds, $b_0$, and wants to maximize the expected lifetime utility of the representative agent choosing optimal (infinite) sequences of consumption $c$, labor supply $n$, capital, and bonds. The central planner problem is summarized as follows:

$$\max_{\{c_t, n_t, k_{t+1}, b_{t+1}\}_{t=0}^\infty} \left[ E_0 \sum_{t=0}^\infty \beta^t \frac{(c_t^{\gamma}(1-n_t)^{1-\gamma})^{1-\sigma}}{1-\sigma} \right]$$

subject to the following constraints for $t = 0, \ldots, \infty$:

\begin{align*}
e^{zt} A k_t^\alpha n_t^{1-\alpha} + r_t b_t &\geq c_t + i_t + b_{t+1} - b_t & (4a) \\
 r_t - r_t^* &\geq \omega(b_t - b_t) & (4b) \\
i_t &\geq k_{t+1} - k_t(1 - \delta) & (4c) \\
z_{t+1} &\geq \rho^r z_t + \epsilon_{t+1} & (4d) \\
r_{t+1}^* &\geq (1 - \rho^r) r^* + \rho^r r_t^* + \epsilon_{t+1} & (4e) \\
(b_0, k_0) &\text{ given } & (4f)
\end{align*}

In eq. (3), the Cobb-Douglas utility index indicates that the representative agent derives utility from consumption and leisure, $1 - n$; the parameter $\beta$ is the in-
tertemporal discount factor; $\gamma$ ....; and $\sigma$ is the negative of the elasticity of marginal utility.

The flow budget constraint in eq. (4a) says that the sum of consumption, investment, $i_t$, and the accumulation of international financial assets, $b_{t+1} - b_t$, must not be greater than the sum of production and the net financial income. The production function is Cobb-Douglas; $\alpha$ is the capital share in total output; $z_t$ is a productivity shock, and $A$ scales the total factor productivity. The interest rate on the bonds issued by this economy, $r_t$, evolves according to eq. (4b). There, $r^*_t$ is the international interest rate on debt issued by countries whose outstanding stock of debt is equal to $b$, which is in turn equal to the steady-state value of the country’s debt. Notice that when $b_t$ is falling below $b$, the country risk premium $r_t - r^*_t$ is positive and vice versa. The parameter $\omega$ indicates the sensitivity of the country spread with respect to the gap $b - b_t$.

The investment equation (4c) has no adjustment costs and $\delta$ stands for the depreciation rate. Eqs. (4d) and (4e) are the forcing processes of the productivity shock and the international interest rate. The unconditional mean of the productivity shock is equal to zero, whereas the unconditional mean of the interest rate $r^*_t$ is equal to $r^*$. The parameter $\rho^j$, $j = z, r$, measures the autocorrelation implicit in the forcing process of $j_t$.

After substituting $i_t$ and $r_t$ from eqs. (4c) and (4b) into eq. (4a) respectively, the first order conditions of the problem, as of time $t$ ($t = 0, ..., \infty$), are:

$$\frac{1 - \gamma}{\gamma} \frac{c_t}{1 - n_t} = (1 - \alpha)A e^{z_t} k_t^{\alpha} n_t^{1-\alpha}$$  \hspace{1cm} (5a)

$$MUC_t = \beta E_t \left[ MUC_{t+1} \left( 1 + \alpha A e^{z_t} k_{t+1}^{\alpha} n_{t+1}^{1-\alpha} - \delta \right) \right]$$  \hspace{1cm} (5b)

$$MUC_t = \beta E_t \left[ MUC_{t+1} \left( 1 + r^*_t - \omega (b_{t+1} - b) \right) \right]$$  \hspace{1cm} (5c)

$$A e^{z_t} k_t^{\alpha} n_t^{1-\alpha} + [r^*_t - \omega (b_t - b)] b_t = c_t + k_{t+1} - k_t (1 - \delta) + b_{t+1} - b_t$$  \hspace{1cm} (5d)

where $MUC_t \equiv \gamma c_t^{\gamma (1-\sigma) - 1} (1 - n_t)^{(1-\gamma)(1-\sigma)}$ is the marginal utility of consumption at time $t$. The interpretation of these conditions is as follows. Eq. (5a) makes the marginal rate of substitution of leisure for consumption equal to the marginal productivity of labor. Eq. (5b) shows that the economy must accumulate capital

---

4 When the country is a net debtor in international financial markets, $b_t$ is negative

5 Notice that if $\omega$ is equal to 0, $r_t = r^*_t$, and there is no country spread over $r^*_t$. 

8
until the marginal cost in utility terms (lhs) is equal to the marginal expected benefit, in utility terms, of the future additional consumption (rhs). A similar interpretation fits for eq. (5c) although the latter refers to savings made in international bonds. Eq. (5d) is the budget constraint showed above and the equality arises from no-satiation. An additional limiting condition that must be satisfied by this problem is:

$$\lim_{t \to \infty} \beta^t E_0 \left[ u_c(t) (k_t + b_t) \right] = 0$$

Four additional or flow variables are considered, namely: output, $y_t$; investment, $i_t$; the trade balance, $tb_t$; and the domestic interest rate, $r_t$. They are defined by⁶:

$$y_t = e^z_t Ak_t^{1-\alpha} n_t^\alpha$$ (6a)

$$i_t = k_{t+1} - k_t (1 - \delta)$$ (6b)

$$tb_t = e^z_t k_t^{1-\alpha} - c_t - i_t$$ (6c)

$$r_t = r_t^* - \omega (b_t - b)$$ (6d)

The equilibrium and solution to this model are infinite sequences from $t = 0$ to $\infty$, one for each of the following six variables: $c_t$, $n_t$, $k_{t+1}$, $b_{t+1}$, $r_t^*$, and $z_t$, such that the first order conditions above hold for every time period $t$.⁷ We have to use a system of six equations to find the solution. These equations are given by the (four) first order conditions (5) and the equations describing the temporal evolution of the interest rates and productivity factor, eqs. (4d) and (4e).

### 3.2 ‘Closing’ the Open Economy

What distinguishes a closed from an open economy is the impossibility of the former to borrow or lend in international financial markets. This is the same as saying that $b_t$ is always equal to zero. The central planner of the closed economy wants to maximize the same expected utility as in eq. (3), but now only choosing sequences

---

⁶After the substitution made before deriving the first-order conditions, $i_t$ and $r_t$ are no longer variables of the system of equations and this is the reason they are included among the flow variables.

⁷If the problem above were expressed in a recursive form, it can be said that for given values of the predetermined variables, the model solution gives the solution values for the seven variables above.
\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$. The maximization is now subject to the following constraints:

\[
e^{z_t} \tilde{A}_t k_t^\alpha n_t^{1-\alpha} \geq c_t + i_t \quad (7a)
\]

\[
i_t = k_{t+1} - k_t (1 - \delta) \quad (7b)
\]

\[
z_{t+1} = \rho z_t + \epsilon_{t+1} \quad (7c)
\]

\[
(k_0, z_0) \quad \text{given} \quad (7d)
\]

Here, the first order conditions are:

\[
\frac{1 - \gamma}{\gamma} \frac{c_t}{1 - n_t} = (1 - \alpha) e^{z_t} \tilde{A}_t k_t^\alpha n_t^{-\alpha} \quad (8a)
\]

\[
MUC_t = \beta E_t \left[ MUC_{t+1} \left( 1 + \alpha e^{z_{t+1}} \tilde{A}_t^{\alpha-1} n_{t+1}^{1-\alpha} - \delta \right) \right] \quad (8b)
\]

\[
e^{z_t} \tilde{A}_t k_t^\alpha n_t^{1-\alpha} - X = c_t + k_{t+1} - k_t (1 - \delta) \quad (8c)
\]

And the limiting condition is:

\[
\lim_{t \to \infty} \beta^t E_0 [MUC_t k_t] = 0
\]

The flow variables considered here are output and investment as they were defined in eqs. (6a) and (6b).

## 4 Variable and Equation Sorting, Calibration and Log-Linearization of the Example Models

### 4.1 Sorting Variables and Equations

Table 4.1 proceeds with the categorization of variables of the two models above according to the criteria shown in section 2.

In the open economy model, at time $t = 0$, the economy starts with a stock of capital, a stock of outstanding international assets (see eq. (4f)), and there is a draw of $\epsilon(t)$. These are the four predetermined variables of the model and their value do not depend on any other variable in the model (see eqs. (4d) and (4e)). At $t = 0$, the central planner chooses the optimal value of $n_0$ and $c_0$, as well as the
Table 1: Sorting of Variables according to the Criteria of Section 2

<table>
<thead>
<tr>
<th></th>
<th>Predetermined Variables</th>
<th>Non-Predetermined Variables</th>
<th>Flow Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>$x_1(t)$</td>
<td>$x_2(t)$</td>
<td>$x_3(t)$</td>
</tr>
<tr>
<td>Open Economy</td>
<td>$k_t$, $b_t$</td>
<td>$z_t$, $r_t^*$</td>
<td>$y_t$, $i_t$, $tb_t$, $r_t$</td>
</tr>
<tr>
<td>Closed Economy</td>
<td>$k_t$</td>
<td>$z_t$</td>
<td></td>
</tr>
</tbody>
</table>

value of the predetermined endogenous variables $b_1$ and $k_1$. Therefore, at $t = 1$ the value of all predetermined variables is given by the decisions taken at $t = 0$ and the next draw of innovations to the exogenous predetermined variables. The same reasoning is valid for the closed economy, with the only difference that the closed economy does not trade in international-asset markets.

As for the equation sorting in the open economy, eqs. (5a) to (5c) are the optimality conditions of the model; Eq. (5d) is the market clearing condition for goods; and eqs. (4d) and (4e) are the two forcing processes of the model. In the closed economy model, eqs. (8a) and (8b) are the optimality conditions; eq. (8c) is the budget constraint or the good market clearing condition, and the forcing processes (??).

4.2 Calibration and Log-Linearization

The non-stochastic steady-state version of the model equations forms a system of equations used to solve for a number of parameter and variable values. In each case, deciding upon which parameter and variable values are taken from the data and which ones are arising from the equilibrium conditions of the model depends, among other things, on the available statistical information of the modelled economy.

Recalling that $b = b^{ss}$, the non-stochastic steady state of the open-economy model is characterized by the following six equations:

$$\frac{1 - \gamma}{\gamma} \frac{c^{ss}}{1 - n^{ss}} = (1 - \alpha) \frac{y^{ss}}{n^{ss}}$$

(9a)

This is the state where the system would rest in the long run if the innovations to the exogenous backward-looking variables are set equal to zero in every time period. All variables in the system are said to remain at their steady-state value. Therefore, for any variable $x$, $x_t = x_{t+1} = x^{ss}$, where $x^{ss}$ is the steady-state value of the variable.
1 = \beta \left( 1 + \alpha \frac{y^{ss}}{k^{ss}} - \delta \right) \quad (9b)
1 = \beta (1 + r^{ss}) \quad (9c)
y^{ss} + r^{ss}b^{ss} = c^{ss} + k^{ss} \delta \quad (9d)
z = \rho^r z \quad (9e)
r^* = (1 - \rho^r)r^* + \rho z r^* \quad (9f)

where \( y \) is the steady-state value of output (see eq. (6a)). For \( \rho^r \neq 1 \), eq. (4d) implies that \( z = 0 \). Eq. (9f) is valid for any \( r^* \) so its value will be endogenously determined. We then have four equations remaining (9a) to (9d) to solve for a combination of four variables and parameters. Specifically, we choose to solve these equations to find the value of \( k, b, \beta, \) and \( \gamma \). Setting the value of output equal to one, we start with the following assumptions: \( c/y = 0.75; \ n = 0.2; \ r^* = 0.06; \ \alpha = 0.4; \) and \( \delta = 0.02 \). We then obtain that \( k = 11.5361; \ b = -1.3138; \ \beta = 0.9855; \) and \( \gamma = 0.2381 \). Plugging these results into the production function, we obtain that: \( A = 0.9875 \).

Since the parameters \( \omega, \sigma, \rho^\delta, \) and \( \rho^r \) do not enter into the system of eqs. (9a)-(9d), we can set their value exogenously. We set \( \rho^\delta = 0.87, \ \rho^r = 0.95, \ \omega = 0.0025, \) and \( \sigma = 2 \). Furthermore, as there are no mathematical constraints on the statistical properties of the innovations to interest-rate and productivity shocks these properties can be freely chosen. Notwithstanding this, the central assumption here is that the innovations to the exogenous predetermined variables are zero-mean Gaussian processes.

The non-stochastic steady-state version of the closed-economy model is characterized by the following equations:

\[
\frac{1 - \gamma}{\gamma} \frac{c^{ss}}{1 - n^{ss}} = (1 - \alpha) \frac{y^{ss}}{n^{ss}} \quad (10a)
\]
\[
1 = \beta \left( 1 + \alpha \frac{y^{ss}}{k^{ss}} - \delta \right) \quad (10b)
\]
\[
y^{ss} = c^{ss} + k^{ss} \delta \quad (10c)
\]

To facilitate the comparisons between the two versions of the neoclassical growth model above, the parameter values of the closed-economy model are the same
as those of the open economy. Namely, $\alpha = 0.4$, $\beta = 0.9855$, $\gamma = 0.2381$, and $\delta = 0.02$, and $\sigma = 2$. Output is normalized to be equal to one. Eq. (10b) solves for the same value of the $k/y$ as in the open economy; eq. (10b) indicates that $c/y = 0.7687$ and eq. (10a) gives $n = 0.1961$.\(^9\) The statistical properties of the productivity shocks in the closed economy will mimic those of the open economy to facilitate the comparison between both models. Therefore, $\rho_z = 0.95$.

### 4.3 Log-Linearization of the Models

The toolbox accompanying this paper carries out the log-linearization of any dynamic model, once the model equations are written in a Matlab function to be specified later. However, the equivalence between the paper-and-pencil and the numerical version of the log-linearization of the models is going to be illustrated here using the optimality equation (5a). The exercise is completed in Appendix A, where all the equations in the open- and closed-economy models are log-linearized. There, it can be seen how the coefficients arising from the log-linearization parallels the matrices of coefficients $A$ to $D$ in equations (1) and (2).

Starting with the $lhs$ of equation (5a), totally differentiating and evaluating the changes around the steady state of the model give:

$$
\frac{1 - \gamma}{\gamma} \frac{1}{1 - n} dc_t + \frac{1 - \gamma}{\gamma} \frac{c}{(1 - n)^2} dn_t
$$

For a variable $x$, define $\hat{x}_t = dx_t/x_{ss}$; this permits writing the expression above as:

$$
\frac{1 - \gamma}{\gamma} \frac{c}{1 - n} \hat{c}_t + \frac{1 - \gamma}{\gamma} \frac{cn}{(1 - n)^2} \hat{n}_t
$$

Since all but the “hat” variables have numerical values, the linear version of the $lhs$ of eq. (5a) is:

$$(3.2)(0.9375)\hat{c}_t + (3.2)(0.2344)\hat{n}_t = 3\hat{c}_t + 0.75\hat{n}_t$$

In order to proceed with the $rhs$ of eq. (5a), it is convenient to define: $\zeta_t \equiv e^{zt}$.\(^{10}\)

---

\(^9\)The absence of debt in the closed economy implies a higher wealth vis-à-vis the open economy. Therefore consumption and leisure, which are normal goods, are higher in the closed economy.

\(^{10}\)Notice that $d\zeta_t = e^z dz_t$. Also, when $z = 0$, $\zeta = 1$ and $\zeta_t = dz_t$. 

13
In the first step, totally differentiate to obtain:

\[(1 - \alpha)A^{k^\alpha}n^{-\alpha}d\zeta_t + (1 - \alpha)\alpha \zeta A^{k^{\alpha-1}}n^{-\alpha}dk_t + (1 - \alpha)(-\alpha)\zeta A^{k^n}n^{-\alpha-1}dn_t\]

By converting differentials to relative changes, the second gives:

\[(1 - \alpha)\zeta A^{k^\alpha}n^{-\alpha}\hat{\zeta}_t + (1 - \alpha)\alpha \zeta A^{k^\alpha}n^{-\alpha}\hat{k}_t + (1 - \alpha)(-\alpha)\zeta A^{k^n}n^{-\alpha}\hat{n}_t\]

Evaluating the expression above at the steady state, the numerical version of the rhs of eq. (5a) becomes:

\[1.2\hat{k}_t + 3\hat{\zeta}_t - 1.2\hat{n}_t\]

Combining both sides of the equation gives the log-linear version of eq. (5a),

\[-1.2\hat{k}_t - 3\hat{\zeta}_t + 3\hat{c}_t + 1.95\hat{n}_t = 0\]

As shown in Appendix A (on pag. 28), the first row of matrix \(B\) is equal to:

\([-1.200, 0, -3.000, 0, 3.000, 1.950]\); recall that \(\hat{x}_t\), is \(\hat{x}_t = [\hat{k}_t, \hat{b}_t, \hat{\zeta}_t, \hat{r}_t, \hat{c}_t, \hat{n}_t]\).

5 The Theoretical Solution of a Linear Model

A theoretical solution for models like those shown above may be found applying Klein’s (1999) method. A solution can be stated as a pair of functions, one providing the law of motion of the state (or backward looking) variables and another giving the optimal policy rule (the optimal decision on forward looking variables). The first maps the state space into itself and the second maps the state space into the set of optimal policy functions. Therefore, we are solving for the recursive representation of the stable solution to a system of linear difference equations.

Klein’s method is based on the Schur decomposition of the square matrices \(A\) and \(B\) in eq. (1). This decomposition gives the square complex matrices \(Q, S, T,\) and \(Z\) such that\(^{11}\)

\[A = QSZ^H \quad \text{and} \quad B = QTZ^H\]

\(^{11}\)If the reader does not feel comfortable with the statement above, notice that it takes just a Matlab call to get the Schur decomposition of two matrices. Given two square matrices \(A\) and \(B\), through the sentence \([S,T,Q,Z]=qz(A,B)\). Matlab returns the matrices \(S, T, Q,\) and \(Z\). In fact, the decomposition that Matlab makes is such that \(A = Q^HST^H\).
where \( Z^H \) denotes the transpose of the complex conjugate of \( Z \). \(^{12}\) \( Q \) and \( Z \) are unitary matrices, that is \( Q^H Q = Z^H Z = I \), and \( S \) and \( T \) are upper triangular.

The generalized eigenvalues of the matrices \( A \) and \( B \) are equal to the \( i \)th diagonal element of \( T \) divided by the \( i \)th diagonal element of \( S \). When the matrix \( A \) is singular, some of the generalized eigenvalues are infinite. Let us define as stable generalized eigenvalues those that are less than one. The unstable are those larger or equal to one in absolute value, including infinite values.

The decomposition can be reordered without altering the result. The reordering made here is such that the block of stable generalized eigenvalues come first.

Starting with the vector of variables \( \hat{x} \), define the auxiliary variable \( \hat{y} \) as \( \hat{y} = Z^H \hat{x} \), so

\[
\hat{y}(t) = \begin{pmatrix} \hat{y}^s(t) \\ \hat{y}^u(t) \end{pmatrix} = Z^H \begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{pmatrix}
\] (11)

By virtue of the Schur decomposition, premultiplying both sides of (1) by \( Q^H \) gives:

\[
SE_t Z^H \hat{x}(t+1) = T Z^H \hat{x}(t)
\]

which, employing the auxiliary variables defined in (11), becomes:

\[
S E_t \hat{y} (t+1) = T \hat{y}(t)
\]

A partition of the matrices \( S \) and \( T \) conformably with \( y^s \) and \( y^u \), is:

\[
\begin{pmatrix} S_{x1x1} & S_{x1x2} \\ 0 & S_{x2x2} \end{pmatrix} E_t \begin{pmatrix} \hat{y}^s(t+1) \\ \hat{y}^u(t+1) \end{pmatrix} = \begin{pmatrix} T_{x1x1} & T_{x1x2} \\ 0 & T_{x2x2} \end{pmatrix} \begin{pmatrix} \hat{y}^s(t) \\ \hat{y}^u(t) \end{pmatrix}
\] (12)

Notice that the second difference equation contains the unstable roots due to the reordering of the eigenvalues. Therefore, a stable solution for that equation requires \( \hat{y}^u(t) = 0 \) for all \( t \), which implies the existence of a linear dependency among the components of the vector \( \hat{y}^u(t) \). The remaining equations in (12) can be written as:

\[
E_t \hat{y}^s(t+1) = S^{-1}_{x1x1} T_{x1x2} \hat{y}^s(t)
\] (13)

where it can be shown that \( S^{-1}_{x1x1} \) exists. This is because \( S \) is a triangular matrix that in turn was reordered in such a way that none of its elements in the diagonal

\(^{12}\)explain
are zero (otherwise a unstable generalized eigenvalue would have arisen and its correspondent block could not be part of the upper right block of $S$).\(^{13}\)

Multiplying eq. (11) by $Z$ gives $ZZ^H x(t) = \dot{x}(t) = Z \hat{y}(t)$ because $ZZ^H = I$.

Thus, making a partition of $Z$ conformably with those of $\dot{x}$’s and $\hat{y}$’s:

$$
\begin{pmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{pmatrix} = 
\begin{pmatrix}
Z_{x1y_s} & Z_{x1y_u} \\
Z_{x2y_s} & Z_{x2y_u}
\end{pmatrix}
\begin{pmatrix}
\dot{y}^s(t) \\
\dot{y}^a(t)
\end{pmatrix} = 
\begin{pmatrix}
Z_{x1y_s} \\
Z_{x2y_u}
\end{pmatrix} \hat{y}^s(t) 
$$

(14)

The second equality arises because $\dot{y}^a(t) = 0$. The first of the equations implicit in (14) says that $\dot{x}_1(t) = Z_{x1y_s} \hat{y}^s(t)$. Recalling that the value of $\dot{x}_1(0)$ is given as a component of the initial conditions of the problem at hand, one can solve for $\hat{y}^s(0)$:

$$
\hat{y}^s(0) = Z_{x1y_s}^{-1} \dot{x}_1(0)
$$

provided that $Z_{x1y_s}$ is invertible. It can be shown that the invertibility is assured when the number of predetermined variables (rows in $Z_{x1y_s}$) equals the number of stable roots (columns in $Z_{x1y_s}$). Why???

Some of the backward looking variables are affected by innovations to the exogenous processes and some others are not. For instance, while the next period stock of financial assets may affected by changes in the interest rate, the stock of capital will be equal to the existent stock plus the net investment in the current period. Hence, we may write $\dot{x}_1(t + 1) = E_t \dot{x}_1(t + 1) + \epsilon_1(t + 1)$, with some of the components of $\epsilon_1(t + 1)$ equal to zero. On the other hand, after observing that (14) implies that:

$$
\dot{x}_1(t) = Z_{x1y_s} \hat{y}^s(t)
$$

$\dot{x}_1(t + 1) = E_t \dot{x}_1(t + 1) + \epsilon_1(t + 1)$ can be written as:

$$
Z_{x1y_s} [\hat{y}^s(t + 1) - E_t \hat{y}^s(t + 1)] = \epsilon_1(t + 1)
$$

So:

$$
\hat{y}^s(t + 1) = E_t \hat{y}^s(t + 1) + Z_{x1y_s}^{-1} \epsilon_1(t + 1)
$$

This is not an explicit solution for $\hat{y}^s(t + 1)$ because its expected value as of time $t$ is also on the right hand side of the equation. However, expression (13) may then

\(^{13}\)Hence the determinant of a triangular matrix is equal to the product of the elements in the diagonal, which are all non zero in the matrix $S$. 

16
be used to give:

\[ \hat{y}^s(t + 1) = S_{x_1}^{-1} T_{x_1} \hat{y}^s(t) + Z_{x_1}^{-1} \epsilon_1(t + 1) \]  

(15)

It remains to get the solution in terms of the original variables in \( \hat{x}_1(t) \), something that is straightforward if one appeals to the definition of the auxiliary variable \( \hat{y}(t) \) and the partition of \( Z \) made above (see equations (11) and (14)):

\[ \hat{x}_1(t + 1) = Z_{x_1} \hat{y}^s(t + 1) = Z_{x_1} S_{x_1}^{-1} T_{x_1} Z_{x_1}^{-1} \hat{x}_1(t) + \epsilon_1(t + 1) \]  

(16)

Now, in order to solve for \( \hat{x}_2(t) \), notice that (14) implies that:

\[ \hat{x}_2(t) = Z_{x_2} \hat{y}^s(t) \]

and again, since \( \hat{x}_1(t) = Z_{x_1} \hat{y}^s(t) \),

\[ \hat{x}_2(t) = Z_{x_2} Z_{x_1}^{-1} \hat{x}_1(t) \]  

(17)

To establish a direct link between these results and the Matlab functions in the toolbox, define

\[ p \equiv Z_{x_1} S_{x_1}^{-1} T_{x_1} Z_{x_1}^{-1} \]

(18)

\[ f \equiv Z_{x_2} Z_{x_1}^{-1} \]

(19)

Thus, (18) and (19) completely describe the evolution of the system once the initial conditions and the shocks hitting the economy are specified. Particularly, \( p \) represents the state transition function which governs the evolution of the state variables. Likewise \( f \) represents the policy function or decision rule and it maps the state of the economy into the decisions regarding the forward looking variables.

To see the dynamics implied by \( p \) and \( f \), remember that the initial conditions are specified in \( \hat{x}_1(0) \). Then \( f \) gives the value of \( \hat{x}_2(0) \) and \( p \) gives the value of the backward looking variables at \( t + 1 \) for a particular shock \( \epsilon_1(1) \). Thus, new predetermined values for the variables in \( x_1 \) are stated for the next period. At \( t=1 \) \( \hat{x}_2(1) \) can be obtained using \( f \) and \( \hat{x}_1(2) \) using \( p \), and so on.

The value of the additional variables is straightforward to get at this point. Making partitions of the matrices \( C \) and \( D \) and the vector \( x \) at times \( t \) and \( t + 1 \)
in eq. (2),

\[ \hat{x}_3(t) = \left( \begin{array}{c|c} C_1 & C_2 \\ \hline \end{array} \right) \left( \begin{array}{c} \hat{x}_1(t + 1) \\ \hat{x}_2(t + 1) \end{array} \right) + \left( \begin{array}{c|c} D_1 & D_2 \\ \hline \end{array} \right) \left( \begin{array}{c} \hat{x}_1(t) \\ \hat{x}_2(t) \end{array} \right) \] (20)

where both \( C_1 \) and \( D_1 \) are of dimension \( n_3 \times n_1 \) while both \( C_2 \) and \( D_2 \) are of dimension \( n_3 \times n_2 \). Notice that the elements of \( C_2 \) are all zeros in most of the cases. Also recalling that \( \hat{x}_1(t + 1) = p \hat{x}_1(t) \) and that \( \hat{x}_2(t) = f \hat{x}_1(t) \), (20) can be re-written as:

\[ \hat{x}_3(t) = (C_1 p + C_2 f p + D_1 + D_2 f) \hat{x}_1(t) + C_1 \epsilon_1(t + 1) \] (21)

The following notation is used in the toolbox:

\[ g \equiv C_1 p + D_1 + D_2 f \] (22)
\[ h \equiv C_1 + C_2 f p \] (23)

6 The Toolbox of Matlab Functions

This section shows how the Matlab functions in the toolbox can be used to solve a dynamic model. Explanations are illustrated solving and analyzing the models presented in section 3.\(^\text{14}\) The Matlab functions in the toolbox can be grouped into three subsets, which are coordinated by the function \texttt{Control}. The first subset of functions performs the log-linearization. The second subset provides the solution to the system, that is the two fundamental equations summarized in (18) and (19). Additionally, the set of coefficients in (22) and (23) which are used to follow up the evolution of the flow variables are obtained. The functions in this subset are the core of the toolbox. The third subset of Matlab functions, provides the researcher with the tools for plotting impulse response functions, computing business cycles statistics, and simulating the economy under a particular sequence of innovations to the exogenous predetermined variables. The next table shows the functions included in each of the sets above. The functions that should be modified by the researcher with information on his model are underlined.

\(^{14}\)It might be worth to read this section keeping an eye on the Matlab functions in the toolbox. Some but not all these functions were included in Appendix.
Table 2: Matlab functions in the toolbox

<table>
<thead>
<tr>
<th>Log Linearization</th>
<th>Core Functions</th>
<th>Output Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>st_st.m*</td>
<td>rec_repres.m</td>
<td>procedures.m</td>
</tr>
<tr>
<td>call_lin.m</td>
<td>qzswitch.m</td>
<td>NameVars.m*</td>
</tr>
<tr>
<td>eqns.m*</td>
<td>reorder.m</td>
<td>impres.m</td>
</tr>
<tr>
<td>flows.m*</td>
<td></td>
<td>actual_shocks.m</td>
</tr>
<tr>
<td></td>
<td></td>
<td>tableBCstats.m</td>
</tr>
<tr>
<td></td>
<td></td>
<td>graph.m</td>
</tr>
<tr>
<td></td>
<td></td>
<td>moments1.m</td>
</tr>
<tr>
<td></td>
<td></td>
<td>moments2.m</td>
</tr>
<tr>
<td></td>
<td></td>
<td>prtfigs.m</td>
</tr>
<tr>
<td></td>
<td></td>
<td>shokcs.wk1*</td>
</tr>
</tbody>
</table>

An asterisk after the file extension indicates that the function has to be modified to provide information regarding the researcher’s own model.

In order to run the functions in the toolbox the researcher must save all the files that accompany these notes and whose names coincide with the functions in the Table, in a directory. They are NN files.

In designing the toolbox we preferred to minimize the number of function calls as well as the steps involved in solving a model. Notwithstanding, there are several elements of the toolbox that are model-specific and hate to be modified by the researcher according to the specifications of his model. We are explaining here what he has to modify in the files above. We have limited the number of function calls to just one, the function CONTROL. There, the user has to indicate details of the model under study and the desired procedures. This minimization of function calls has the advantage of avoiding the research to keep track of the order in which several functions should be called.

The researcher should provide the following in-file inputs to CONTROL.m following the instructions contained in the function:

1. **Number of each type of variable.** The sorting of variables commented before should be declared here. The researcher should indicate the number of backward looking variables (including both the endogenous and the exogenous), \( n_1 \); the number of forward looking variables, \( n_2 \); the number of exogenous variables among the backward looking ones, \( n_e \); and the number of additional or flow variables, \( n_3 \).

2. **Analysis to be Performed.** DesProc is a binary \((1 \times 3)\) vector, with
1’s indicating a desired procedure and 0’s otherwise. For example, setting $DesProc=(1,0,0)$ the toolbox returns the impulse response functions of his model; setting just the second component equal to one the toolbox returns the population moment statistics; and setting just the third component equal to one the toolbox performs the simulation with actual shocks.

3. **Shocks Hitting the Economy.** Vector $\text{shock}$ indicates what shock or shocks are hitting the economy. Shock’s size is $(1 \times n_e)$ and its ordering matters. The first element of $\text{shock}$ corresponds to the first exogenous state and so on. It is also a binary vector and ones are for variables being shocked.

4. **Statistical Properties of the Shocks.** $\text{StdDev}$ is the variance covariance matrix of the innovations to the exogenous predetermined variables in the model. It is a square matrix of size $n_e$. Attention should be pay here to the order in which the standard deviations are entered. Again, the upper-left element of $\text{StdDev}$ is the variance of the innovations to the first exogenous state variable, and so on.

5. **Location of the Key Variable.** A row vector $\text{kvar}$ is used to compute the statistics of the economy, either the population moment conditions or the sample moment conditions. $\text{kvar}$ is a row vector of size 2. It indicates the position of the variable against which the researcher would like to compare the evolution of all variables in his model. In most macroeconomic models, output is the key variable since the relative volatilities and correlations of all other variables are computed with respect to that of the output. The specification is make indicating the type of variable and the position in the respective subset. The first column takes values from 1 to 3; 1 is for backward looking variables, 2 for forward looking variables, and 3 for additional or flow variables. The second column of $\text{kvar}$ indicates the position of the variable in the respective subset. Thus, if the key variable of a model is the second flow variable, $\text{kvar}=[3,2]$.

6. **Files.** Finally, $\text{CONTROL}$ asks the user to indicate the name of a directory. This is where results are going to be stored and/or where the functions must read some information.

When simulations with actual shocks are requested, the user should previously
provide that sequence of shocks in a .wk1 file. The spreadsheet in that file has to be filled in the following way. There will be a number of columns equal to the number of exogenous variables and the columns are ordered according to the order these variables have in the log-linearization. The upper-left cell with data is $B2$; the second column of innovations starts at $C2$ and so on.

### 6.1 Log-linearizing the Model

Here we explain the steps involved in log-linearizing a dynamic macroeconomic model using the toolbox. Of the three functions in this subset, `st_st.m` and `eqns.m` are mostly model specific and several parts should be rewritten by the researcher according to his own model. `st_st.m` takes the value of parameters and variables that are known and return the value of all parameters and variables in the model. The user may want to check the consistency of the value of parameters and variables. He can write all the equations in his model to see whether or not they are satisfied at the non-stochastic steady state. `st_st.m` is called without any input and its output is a vector $X$ which contains the value of the variables at the non-stochastic steady state, and $\text{par}$, a vector containing all parameter values.

The actual file `st_st.m` in the toolbox corresponds to the example of the open economy developed above. The file `st_st2.m`, along with all `NameVars2.m`, `eqns2.m`, and `flows.m` are for the close economy example. Make a permutation of file name to run the example of a close economy.

The role of the function `call_lin.m` is to call twice the function `eqns.m`, which has the model equations, and twice the function `flows.m`, which has the equations defining the flow variables. `call_lin.m` is not model specific and should not be modified by the user.

Inside `eqns.m` the user has to write the equations of his own model. The inputs are the vectors $X$ and $\text{par}$ obtained using `st_st.m` and two additional arguments whose functions consist in telling the function whether the matrix $A$ or $B$ is required, whether the log-linearization or a differentiation of the system is required. On the other hand, the function `flows.m` also has to be written for each model. When writing his model equations (in both `eqns.m` and `flows.m`), the user first has to read the parameters and variables in his model where it is specified.

---

15The permutation of names is such that when the programs call the function “eqns.m” for example, it should be calling the file that corresponds to the example under study.
Notice that the command \texttt{repmat} used in this function has converted the vector \(X\) into a matrix where its columns have been repeated. The user should be aware of the places in which the variables enter into vector \(X\) because all computations are made assuming the researcher keeps an specific ordering. Concretely, the vectors of variables \(x(t)\) and \(x(t+1)\) must be ordered such that endogenous states come first, then exogenous states, and finally all the remaining variables. Also, it is convenient to order the model equations in the following way. First, write the dynamic Euler equations; next, other dynamic equations with those governing the dynamic of the exogenous state variables last. And then all the remaining equations.

### 6.2 The Core Functions

The function \texttt{CONTROL} thus receives all the information about the model and the desired procedures. \texttt{CONTROL} will call, first, the files necessary to compute \(p\) and \(f\) [see eqs. (18) and (19)] and also \(g\) and \(h\) [see eq. (22) and (23)]; then it calls the functions necessary to cast the requested procedures.

The former step is completed calling the functions \texttt{rec_repres}, which in turn calls \texttt{reorder}, \texttt{qzswitch} and the Matlab built-in function \texttt{qz}.\footnote{\texttt{qzswitch} is a function written by Christopher Sims and the toolbox’s version of it has minor changes in its notation and documentation.} \texttt{rec_repres’s} output are the functions \(p\) to \(h\). \texttt{reorder} checks, at pairs and starting from the top, whether or not the generalized eigenvalues are arranged in an ascending order. If they are, then it checks the next pair and so on. If they are not, it then calls \texttt{qzswitchc}, which in turns completes the desired ordering.

### 6.3 The Procedure Functions

\texttt{Procedures} is a function that reads what the user wants to do once the model has been solved and then it calls the functions required accordingly.

\texttt{NameVars}. A file of the toolbox is called \texttt{NameVars} and it must be modified for each model. Following the instructions there, the user must write a name for each of the variables in his model. These names are employed as headings of graphs and tables. The order in which the variable names are ingressed should be the same used in the log-linearization. Then, start writing the name of all backward looking variables, then those of all forward looking variables and finally those of the flow variables.
or additional variables.

After computing the matrices \( p, f, g, \) and \( h \) \texttt{CONTROL.m} can carry out any of the three computation, namely the impulse response functions (calling \texttt{impres.m}), obtain the population moment conditions (calling \texttt{moments1.m}) and the dynamic exercise under actual shocks (calling \texttt{actual_shocks.m}). The main input of these three functions are the matrices \( p \) to \( h \). After executing \texttt{actual_shocks}, \texttt{moments2} compute the sample moment conditions of the variables. Next, we review what each of the mentioned functions does.

The toolbox employ the functions \texttt{rec_repres.m}, \texttt{qzswitch} and \texttt{reorder} to obtain these four matrices. They are then passed to the function \texttt{procedures}, which is going to control the output desired by the researcher.

\texttt{impres.m} computes the impulse response functions arising from shocking one or more of the exogenous variables in the system. It is assumed that the impulse is equal to 1% with respect to the steady state value of the variable being shocked. Briefly, what \texttt{impres} does is to iterate \( \hat{x}_1(t+1) = p\hat{x}_1(t) \) for the state variables, \( \hat{x}_2(t) = f\hat{x}_1(t) \) for the control variables, and \( \hat{x}_3(t) = g\hat{x}_1(t) \) for the additional variables.

Fixing \( \text{DesProc} = [100] \) in the function \texttt{Control}, the researcher obtains a matrix \( X \) of impulse response functions. There is a column per each variable and a row per time period. The order of the columns of \( X \) have first the backward looking variables, then the forward looking variables, and last the flow variables. A graph per each of the variables of the model is produced and saved in the file indicated in the function \texttt{graph} (see line 17).

An important note follows here. In order the user can save the files in the indicated directory, the utility \texttt{PRTFIGS} in Fackler and Miranda (2001) should be in your directory. As it is explained there you must look for a Matlab file called \texttt{startupsav.m} and add the following line there: \texttt{set(0,'DefaultAxesFontName','B')}.

Without this modification the user may obtain an error message and not get the figures.

\texttt{moments1.m} is the function used to calculate the population moment conditions. It reads the variance covariance matrix created in \texttt{CONTROL} along with the functions \( p,f, g, \) and \( h \) and gives the standard deviation (or the absolute volatility) of the variables, along with the relative volatility (measured with respect to
that of the key variable) and several correlations. The result is a table that will be saved in your directory called TableBCstats.out. Each time the researcher requests population moment conditions or to simulate an specific sequence of shocks, the mentioned file is overwritten. The file TableBCstats.out can be open with the Matlab editor.

Setting $DesProc = [010]$, the toolbox calculates the population moment conditions of the variables. Now $X$ is a structure containing those moment conditions. A file containing a table of results is saved in the file specified in `tableBCstats` (see line 22). To obtain the results in numerical form, write $X.moment$, where moment can be any of the following arguments. $SD$: standard deviations; $RV$: ratio of the standard deviation of each of the variables to the standard deviation of the key variable; $Corr$: correlation between each of the variables and the key variable; $Autocorr$: correlation between each of the variables dated at $t + 1$ and the key variable at $t$; $AutoCorr$: autocorrelation of each variable.

**actual shocks** reads the shocks from the directory and then computes the same statistics as with `moments1.m`, but now they are moments of a sample and not population moments. The statistics are also reported in the file TableBCstats.out and remember that the file is modified each time the user run either `actual_shocks` or `moments`. User provided shocks should be saved in a file with extension `.wk1` and named `shocks`.

When the researcher sets $DesProc = [001]$ the researcher obtains simulations and model sample moments arising from the sequence of actual shocks provided by him. The result $X$ returned by the function `Control` has two components. One, $X.Data$, is a matrix similar to the matrix of impulse response described above. The other, $X.Moments$, corresponds to the numerical values of the sample moments conditions. $X.Data$ has a column per variable and row per observation. $X.Moment$ contains the same moments as above and the calling is now $X.Moments.moment$ where moment is one of the arguments described in the precedent paragraph.
7 The Toolbox at Work

In this section, we show how the toolbox can be used to solve and study a dynamic macroeconomic model of the type described in section 2. We do so using the models of section 3. Since the log-linearization of these models was shown in subsections 3.1 and 3.2, we concentrate here on the utility of the Core and Output Functions (see Table).

With the information provided by the researcher, the file Control calls the function rec_repres and four matrices are returned: a) The transition function summarized by matrix $p$ which gives the expected value of the predetermined variables one period forward (see eq. (18); b) The policy function that prescribes the optimal decisions regarding the forward looking variables. This function is summarized by matrix $f$ (see eq. (??eq:f); c) The two matrices $g$ and $h$ in eqs. (22) and (23) to track the flow variables’ dynamics.

In the open economy example, these four matrices are:

$$
p = \begin{pmatrix}
0.654 & 0.009 & 1.226 & -0.266 \\
0.000 & 0.000 & 0.950 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.870
\end{pmatrix}
$$

$$
f = \begin{pmatrix}
0.198 & -0.011 & 0.542 & -0.068 \\
0.311 & 0.017 & 0.705 & 0.105
\end{pmatrix}
$$

$$
g = \begin{pmatrix}
0.586 & 0.010 & 1.423 & 0.063 \\
-3.765 & 0.099 & 14.143 & -3.068 \\
4.203 & -0.080 & -13.127 & 3.182 \\
0.000 & 0.003 & 0.000 & 0.015
\end{pmatrix}
$$

$$
h = \begin{pmatrix}
0.000 & 0.000 & 0.000 & 0.000 \\
7.540 & 0.099 & 14.143 & -3.068 \\
-7.540 & -0.099 & -14.143 & 3.068 \\
0.000 & 0.000 & 0.000 & 0.000
\end{pmatrix}
$$
In the close economy example, the matrices \( f, p, g, \) and \( h \) are the following.

\[
f = \begin{pmatrix}
0.544 & 0.491 \\
-0.221 & 0.783
\end{pmatrix}
\]

\[
p = \begin{pmatrix}
0.966 & 0.096 \\
0.000 & 0.950
\end{pmatrix}
\]

\[
g = \begin{pmatrix}
0.267 & 1.470 \\
-0.141 & 1.102
\end{pmatrix}
\]

\[
h = \begin{pmatrix}
0.000 & 0.000 \\
11.536 & 0.000
\end{pmatrix}
\]

Before showing the results of the procedures performed with the functions in the toolbox, we are going to discuss the ordering of equations and variables. Equations in both \texttt{eqns.m} and \texttt{flows.m} were ordered as they were introduced in the text. In our \texttt{eqns.m} of the open economy example, we wrote first the marginal rate of substitution of labor for consumption, then the optimal condition for the accumulation of capital; the the optimal condition for international assets accumulation; and finally the budget constraint. The ordering of the variables in the open economy example is: \( k, b, \epsilon^z, \epsilon^r, r, c, n, y, i, tb, \) and \( r \).

### 7.1 The Open Economy

The following impulse response functions follow a 1% increase in the international interest rate.
To calculate the population moment conditions we have first set the standard deviation of the interest rate equal to zero and the standard deviation of the innovations to productivity shocks so that the standard deviation of output is equal to 4%. Under this condition, the population moment conditions are:

Table 3: Second Moments of the Variables in the Open Economy: Productivity Shocks

<table>
<thead>
<tr>
<th>Variable</th>
<th>St.Dev. (percent)</th>
<th>Relative Volatil.</th>
<th>Correlation with $y_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>3.488</td>
<td>0.96</td>
<td>0.994 0.951 0.905 0.861 0.819</td>
</tr>
<tr>
<td>b</td>
<td>93.65</td>
<td>25.90</td>
<td>0.630 0.995 0.989 0.984 0.979</td>
</tr>
<tr>
<td>z</td>
<td>0.803</td>
<td>0.22</td>
<td>0.776 0.950 0.903 0.857 0.815</td>
</tr>
<tr>
<td>$r^*$</td>
<td>0.000</td>
<td>0.00</td>
<td>-0.422 1.000 1.000 0.999 0.999</td>
</tr>
<tr>
<td>c</td>
<td>1.130</td>
<td>0.31</td>
<td>0.239 0.991 0.973 0.957 0.941</td>
</tr>
<tr>
<td>n</td>
<td>2.817</td>
<td>0.78</td>
<td>0.950 0.980 0.948 0.917 0.888</td>
</tr>
<tr>
<td>y</td>
<td>3.616</td>
<td>1.00</td>
<td>1.000 0.973 0.927 0.883 0.841</td>
</tr>
<tr>
<td>i</td>
<td>53.99</td>
<td>14.93</td>
<td>0.011 -0.021 -0.020 -0.019 -0.018</td>
</tr>
<tr>
<td>tb</td>
<td>668.9</td>
<td>185.0</td>
<td>0.254 0.014 0.014 0.014 0.014</td>
</tr>
<tr>
<td>r</td>
<td>0.419</td>
<td>0.12</td>
<td>0.630 0.995 0.989 0.984 0.979</td>
</tr>
</tbody>
</table>

Then, we add a shock to the international interest rate. We assume that the international interest rate has a standard deviation equal to 2%. The results are:

Table 4: Second Moments of the Variables in the Open Economy: Productivity and Interest-Rate Shocks

<table>
<thead>
<tr>
<th>Variable</th>
<th>St.Dev. (percent)</th>
<th>Relative Volatil.</th>
<th>Correlation with $y_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>3.785</td>
<td>1.01</td>
<td>0.984 0.938 0.882 0.829 0.780</td>
</tr>
<tr>
<td>b</td>
<td>94.703</td>
<td>25.34</td>
<td>0.638 0.992 0.985 0.978 0.971</td>
</tr>
<tr>
<td>z</td>
<td>0.803</td>
<td>0.21</td>
<td>0.751 0.950 0.903 0.857 0.815</td>
</tr>
<tr>
<td>$r^*$</td>
<td>1.521</td>
<td>0.41</td>
<td>-0.209 0.870 0.757 0.659 0.573</td>
</tr>
<tr>
<td>c</td>
<td>1.153</td>
<td>0.31</td>
<td>0.271 0.989 0.968 0.948 0.930</td>
</tr>
<tr>
<td>n</td>
<td>2.880</td>
<td>0.77</td>
<td>0.951 0.971 0.936 0.903 0.872</td>
</tr>
<tr>
<td>y</td>
<td>3.737</td>
<td>1.00</td>
<td>1.000 0.965 0.914 0.867 0.822</td>
</tr>
<tr>
<td>i</td>
<td>56.822</td>
<td>17.62</td>
<td>-0.030 -0.035 -0.031 -0.028 -0.026</td>
</tr>
<tr>
<td>tb</td>
<td>814.937</td>
<td>218.10</td>
<td>0.252 -0.011 -0.009 -0.006 -0.004</td>
</tr>
<tr>
<td>r</td>
<td>1.534</td>
<td>0.41</td>
<td>-0.031 0.871 0.768 0.678 0.600</td>
</tr>
</tbody>
</table>
7.2 The Close Economy

To compare the close to the open economy, we compute the business cycle statistics of the close economy under the same type of productivity shock that hit the open economy. The results are:

<table>
<thead>
<tr>
<th>Variable</th>
<th>St.Dev. (percent)</th>
<th>Relative Volatil.</th>
<th>Correlation key(t) with x( )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>t</td>
</tr>
<tr>
<td>k</td>
<td>3.324</td>
<td>0.991</td>
<td>0.768 0.794 0.817 0.836 0.853</td>
</tr>
<tr>
<td>z</td>
<td>1.859</td>
<td>0.554</td>
<td>0.978 0.929 0.883 0.839 0.797</td>
</tr>
<tr>
<td>c</td>
<td>2.478</td>
<td>0.739</td>
<td>0.920 0.922 0.921 0.919 0.916</td>
</tr>
<tr>
<td>n</td>
<td>1.157</td>
<td>0.345</td>
<td>0.743 0.665 0.592 0.524 0.460</td>
</tr>
<tr>
<td>y</td>
<td>3.355</td>
<td>1.000</td>
<td>1.000 0.967 0.935 0.904 0.875</td>
</tr>
<tr>
<td>i</td>
<td>1.797</td>
<td>0.536</td>
<td>0.915 0.852 0.793 0.738 0.686</td>
</tr>
</tbody>
</table>

Appendix A: Matrices of Coefficients of the Log-Linear Version of the Models and Model Solutions

Now using the toolbox to log-linearize the complete model as we will be explained in section (6), we obtain the following matrices $A$ and $B$:

$$A_{oe} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.008 & 0 & 0.014 & 0 & 0.499 & 0.085 \\
0 & 0 & 0 & 0 & 0.006 & 0.499 & 0.077 & 11.536 & -1.314 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.000 & 0 & 0 & 0 & 0 & 0 & 0.015 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$B = \begin{pmatrix}
-1.200 & 0 & -3.000 & 0 & 3.000 & 1.950 \\
0 & 0 & 0 & 0 & 0.499 & 0.077 \\
0 & 0 & 0 & 0 & 0.499 & 0.077 \\
11.705 & -1.333 & 1.000 & 0.019 & 0.750 & 0.600 \\
0 & 0 & 0.950 & 0 & 0 & 0 & 0 & 0.013 & 0 & 0 & 0 & 0
\end{pmatrix}$$

The linearization of the flow variables produces the following two matrices of coefficients:

$$C = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 50 & 0 & 0 & 0 & 0 & 0 & -598.389 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$D = \begin{pmatrix}
0.400 & 0 & 1.000 & 0 & 0 & 0.600 \\
-49.000 & 0 & 0 & 0 & 0 & 0 \\
507.170 & 0 & 51.871 & 0 & -38.903 & 31.123 \\
0 & 0.004 & 0 & 1.000 & 0 & 0 & 0
\end{pmatrix}$$

Using the toolbox to log-linearize the close-economy model, we obtain the fol-
lowing four matrices of coefficients:

$$A = \begin{pmatrix}
0.000 & 0.000 & 0.000 & 0.000 \\
0.008 & 0.014 & 0.499 & 0.085 \\
11.536 & 0.000 & 0.000 & 0.000 \\
0.000 & 1.000 & 0.000 & 0.000
\end{pmatrix}$$

$$B = \begin{pmatrix}
-1.200 & 0 & -3.000 & 0 & 3.000 & 1.950 \\
0 & 0 & 0 & 0 & 0.499 & 0.077 \\
0 & 0 & 0 & 0 & 0.499 & 0.077 \\
11.705 & -1.333 & 1.000 & 0.019 & 0.750 & 0.600 \\
0 & 0 & 0.950 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.013 & 0 & 0
\end{pmatrix}$$

$$C = ()$$

$$D = ()$$

References