Optimal Income Taxes: An Example

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Abstract

We develop an algorithm for computing optimal taxes in an open economy extension of the standard one-sector growth model. Typically in the literature on optimal taxation, the first period tax rate is arbitrarily assumed. We construct an example where the first period tax rate is endogenously determined. In addition to computing the exact time path of taxes, the example brings forth some interesting policy implications for developing countries.

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1 Introduction

Governments in developing countries often borrow internationally for financing public sector investment projects that are usually undertaken in the private sector in richer countries. Some common examples are: large infrastructure projects in power, telecommunications, irrigation, and transportation sectors\(^1\). Moreover, a majority of poor countries also routinely seek bilateral as well as multi-lateral loans for the developing human capital, especially for providing educational and health services\(^2\). The standard arguments in favor of the government role are: (a) market failure (b) ‘strategic’ importance of certain sectors, and (c) credit constraints and lack of access to international capital markets for the private sector. As these debts are repaid, in large part, through revenues obtained by taxing the private sector, there is a natural economic question to ask: are these borrowings (and repayments) efficient?

While the past and the ongoing debate has primarily focused on the efficiency aspects of public-sector investments; in this paper we completely abstract from this issue by assuming that all investments are implicitly made in the private sector. The governments only facilitate transfer of foreign capital to private sector by sovereign borrowing, and repay it back through tax revenues. Then we ask the following question: suppose there is a sudden rise in productivity, and the private sector is unable to access international capital markets to increase its capital use to an efficient level. What is the optimal sequence of proportional transfers and taxes that maximize a country’s welfare\(^3\)?

Having set our main question, then, the purpose of this paper is twofold: (1) to develop a framework and to characterize the solution to a small-economy optimal taxation problem with capital accumulation, where only the government can borrow from abroad, and (2) to develop a computational algorithm for solving the corresponding dynamic programming problem, which can be used for exact computations of model variables, and facilitate welfare evaluation, under various parametrization of preferences, technology, and tax structures.

Our main finding is that although proportional income taxes distort efficient allocations by depressing capital use, and hence the steady-state

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\(^1\)In some countries, ‘heavy’ industries such as steel, petroleum, coal are directly under the public sector

\(^2\)Even for industrialized economies, Lambertini (1998) argues that budget deficits have widened with increase in government borrowings, which were required to finance growing human capital investment needs as a result of a skill-based technological change.

\(^3\)We assume that the government levies only proportional income taxes, which is usually the practice.
level of output and consumption, the government may find it optimal to borrow in the event of a positive productivity shock, and then levy a sequence of taxes that has a rising time-path. Further, although consumption is smoother relative to a closed economy case, the time-path is upward sloping. As anticipated, our welfare rankings under different institutional constraints have the following order: (1) an open economy where private sector can directly borrow abroad, (2) a small economy with government borrowings and optimal taxation (3) a small economy with government borrowings and a constant tax-rate policy, and (4) a closed economy where even the government is prohibited from borrowing internationally. One surprising result is that even though an optimal transfer-tax policy is welfare superior to a constant-rate tax policy the difference is almost negligible.

To address this problem, we use a perfect foresight small partially-open economy model with optimizing representative households, where only the government can access international financial markets at a given world interest rate. In the event of a positive productivity shock, the government optimization problem boils down to balancing between two conflicting objectives. An initial positive transfer by borrowing from abroad would raise welfare through smoother consumption for the private sector. On the other hand, a higher foreign debt implies levying higher taxes in future, thus depressing future capital use, and hence a lower steady-state output, consumption, and welfare. Thus, the choice essentially entails a trade-off between consumption smoothening and production efficiency.

Our framework builds on the primal approach to optimal taxation\textsuperscript{4} within the public finance tradition. This approach characterizes the set of allocations that can be implemented as a competitive equilibrium with distorting taxes by two simple conditions: a resource constraint and an implementability constraint. The implementability constraint is the household budget constraint in which the first-order conditions are used to substitute out for tax policies. Thus both constraints depend only on allocations. This characterization implies that optimal allocations are solutions to a simple programming problem. In literature, this optimal tax problem is referred as the \textit{Ramsey problem} and to the solutions and the associated policies as the \textit{Ramsey allocations and policies}.

However, we clearly differ in two respects with the literature. To make this problem interesting, it is standard to assume an arbitrary tax rate on capital in the first period; as otherwise the government will tax all first period capital income, given its inelastic supply. In our framework, it is the first period transfer that \textit{begins} the sequence of optimal taxes,

\textsuperscript{4}See, for example, Atkinson and Stiglitz (1980), Lucas and Stokey (1983), and Chari et al. (1991).
thus making this problem more meaningful and complete. The second difference is that while in the optimal taxation literature the markets are decentralized, so as to allow unequal tax rates on labor and capital, in our model production and consumption choices are made by the same representative household and, therefore, all income is taxed at the same rate. It turns out that with an inelastic supply of capital available to the government, through foreign borrowing, the tax rates on capital are zero right from the beginning. The government borrows from abroad and makes an initial positive transfer in the first period. Thereafter, it repays its loans by taxing labor income only. In appendix (5.1), we present a model within the traditional framework with labor leisure choice, where these results are analyzed. In the main text of our paper, we differ in our approach for mainly two reasons: first, we feel that an income tax is more often the practice as against taxes on labor income only. Second, as shown in the appendix, taxes on labor income are constant right from the second period, when labor leisure choice is explicitly modeled. From a computational perspective, we feel that the other alternative, where optimal tax rates only asymptotically converge to a steady state, is more interesting.

Throughout, we assume that the government effectively has access to commitment technology. As is well known, without such a technology, there are time inconsistency problems, so the equilibrium outcomes with commitment are not necessarily sustainable without commitment. Economies with commitment technologies can be interpreted in two ways. One is that the government can simply commit to its future actions. The other is that the government has no access to a commitment technology, but the commitment outcomes are sustained by reputational mechanisms\(^5\). We also restrict our attention to only proportional taxes\(^6\).

In section 2, we present our model and characterize optimal policies. Section 3 outlines a computational algorithm and presents numerical results of a parameterized simulation. Section 4 offers some remarks. In appendix, we present an alternative model with labor leisure choice and with unequal tax rates on labor and capital income.

\(^5\)See, for example, Chari et al. (1989), Chari and Kehoe (1990,1993), and Stokey (1991) for analyses of optimal policy in environments without commitment.

\(^6\)It is easy to see that the other theoretical alternative of lump-sum taxes will lead to the same equilibrium outcome as in a small open economy with no borrowing restrictions.
2 Model

Consider a small economy with an optimizing representative household, whose preferences over a single consumption good is given by

$$U_t = \sum_{t}^{\infty} \beta^t u(c_t); \quad u' > 0, \text{ and } u'' < 0$$  \hspace{1cm} (1)

where $u(.)$ is the one-period utility function. The household owns a technology for producing the consumption good described as

$$y_t = f(k_t; \theta); \quad f' > 0, \text{ and } f'' < 0$$  \hspace{1cm} (2)

where $y_t$ is the output on date $t$, $k$ is the capital stock and $\theta$ is a productivity parameter. The household is restricted from borrowing from the rest of the world. However, the government can access international capital markets at a given world interest rate $r$, and transfer resources to the household. Further, it repays its foreign borrowings by taxing the household through a proportionally levied income tax. The household budget constraint on date $t$ is

$$c_t + k_{t+1} = y_t(1 - \tau_t)$$  \hspace{1cm} (3)

where $\tau$ is the income tax rate levied by the government. The household maximizes its utility (1), subject to technology (2) and budget constraint (3). The first order condition is given by

$$u'(c_t) = \beta(1 - \tau_{t+1})f'(k_{t+1})u'(c_{t+1})$$  \hspace{1cm} (4)

The first order condition 4) has a standard interpretation: the household equates its marginal rate of substitution with its effective after-tax marginal rate of transformation.

The government date $t$ budget constraint is given by

$$b_{t+1} + \tau_t y_t = (1 + r)b_t + g_t$$  \hspace{1cm} (5)

where $g$ is an exogenous level of government spending that does not provide any utility to the household.

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7For simplicity we assume that output includes depreciated capital.

8Essentially this is equivalent to imposing a constraint that both labor and capital income be taxed at the same rate. With this restriction, we are able to abstract from households’ labor leisure choice. The simplification is made only for expositional convenience. In appendix we work out an example with a representative consumer and a firm where we address this issue.

9For notational simplicity, we suppress $\theta$ as an argument in the production function, as it remains constant throughout after its initial increase on date $t = 0$.

10In our computations, we set $g = 0 \forall t$.  


Suppose the economy is in its steady state and on date \( t = 0 \) there is an unanticipated positive productivity shock i.e. an increase in the value of parameter \( \theta \). The household will like to increase its capital employment, but can not borrow from the rest of the world. However, the government can borrow and transfer resources to the household and repay it in future through revenues obtained from income taxes. Our problem is to characterize this optimal sequence of tax (transfer) rates \( \{\tau\}_{0}^{\infty} \) that maximizes household welfare.

2.1 The Ramsey Problem

From the household first order condition (4), we can substitute for tax rates in its budget constraint (3) to obtain

\[
c_t + k_{t+1} = \frac{u'(c_{t-1}) f(k_t)}{\beta u'(c_t) f'(k_t)} ; \quad t \geq 1
\]

(6)

In the literature on optimal taxation, equation (6) is known as the implementability constraint. Given a policy, i.e. a sequence of proportional tax rates, allocations in a competitive equilibrium have to necessarily satisfy this constraint. Notice, however, that this constraint only applies from date \( t = 1 \) onwards. Household’s budget constraint on date \( t = 0 \) remains

\[
c_0 + k_1 = (1 - \tau_0)f(k_0)
\]

(7)

where \( \tau_0 \) is the income tax rate levied on date 0. Let us merge household and government budget constraints, (3) and (5), to obtain economy’s resource constraint

\[
b_{t+1} + f(k_t) = (1 + r)b_t + c_t + g_t + k_{t+1}
\]

(8)

Given initial foreign debt \( b_0 \) on date \( t = 0 \), using (8) economy’s intertemporal resource constraint can be written as

\[
b_0 = \sum_{t=0}^{\infty} \frac{f(k_t) - c_t - g_t - k_{t+1}}{(1 + r)^{t+1}}
\]

(9)

A Ramsey allocation is a sequence of allocations \( \{c_t, k_{t+1}\}_{0}^{\infty} \), given \( k_0 \) and \( b_0 \), which maximizes (1) subject to implementability constraint (6) and intertemporal resource constraint (9). Once the solution for optimal allocations is obtained, it is straightforward to decentralize this equilibrium through Ramsey policies, which is a sequence of optimal tax rates \( \{\tau_t\}_{0}^{\infty} \) obtained from

\[
\tau_0 = 1 - \frac{c_0 + k_1}{f(k_0)}
\]

(10)
\[
\tau_t = 1 - \frac{u'(c_{t-1})}{\beta f'(k_t)u'(c_t)} ; \quad t \geq 1
\] (11)

2.1.1 Ramsey Allocations and Policies

To solve for Ramsey allocations, using (1), (6), and (9), we form a Lagrangian

\[
\mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=1}^{\infty} \beta^t \mu_t \left( \frac{u'(c_{t-1})}{\beta u'(c_t)} f'(k_t) - c_t - k_{t+1} \right) + \phi \left( -b_0 + \sum_{t=0}^{\infty} \frac{f(k_t) - c_t - g_t - k_{t+1}}{(1+r)^{t+1}} \right)
\] (12)

where \( \mu_t \) is the current (date \( t \)) Lagrangian multiplier for the implementability constraint and \( \phi \) is the Lagrangian multiplier for the intertemporal resource constraint. For date \( t = 0 \), the first order conditions are

\[
c_0 : \quad \frac{u'(c_0)}{\beta u'(c_1)} f'(k_1) = \frac{\phi}{1+r}
\]

\[
k_1 : \quad \beta \mu_1 \frac{u'(c_0)}{\beta u'(c_1)} \frac{\partial}{\partial k_1} \left( \frac{f(k_1)}{f'(k_1)} \right) = \frac{\phi}{1+r} - \frac{\phi}{(1+r)^2 f'(k_1)}
\] (13)

For any other date \( t \geq 1 \)

\[
c_t : \quad \beta^t u'(c_t) = \beta^t \mu_t + \beta^t \mu_t u''(c_t) \frac{u'(c_{t-1})}{\beta (u'(c_t)^2)} f'(k_t)
\]

\[
-\beta^{t+1} \mu_{t+1} \beta^t u'(c_{t+1}) f'(k_{t+1}) + \frac{\phi}{(1+r)^{t+1}}
\] (14)

\[
k_{t+1} : \quad \beta^{t+1} \mu_{t+1} \frac{u'(c_{t+1})}{\beta u'(c_{t+1})} \frac{\partial}{\partial k_{t+1}} \left( \frac{f(k_{t+1})}{f'(k_{t+1})} \right) = \frac{\phi}{(1+r)^{t+1}}
\]

Given \( k_0 \), Ramsey allocations are obtained as a solution to the above first order conditions (13), (14), (15), the implementability constraint (6), and the resource constraint (9). Once the allocations \( \{c_t, k_{t+1}\}_0^\infty \) are obtained, Ramsey policy \( \{\tau_t\}_0^\infty \) is obtained from (10) and (11).

2.2 Welfare without Borrowing Constraints

As a benchmark, let us first consider an economy where private sector is not barred to access international capital markets. Let the initial
foreign debt (held by the private sector) \( b_0 = 0 \). As before, suppose on date \( t = 0 \) there is a positive productivity shock. In an unconstrained economy, where individuals can directly lend and borrow in international markets, the steady state is achieved on date \( t = 0 \) itself (we assume \( \beta = \frac{1}{1+r} \)). Also, in this steady state, the capital stock adjusts to its efficient level

\[
\tilde{k} = f^{-1}(1 + r) 
\]  

Further, the stock of foreign debt is

\[
\tilde{b} = \frac{1}{1 + r} \left( f(\tilde{k}) - f(k_0) \right) 
\]  

Finally, the consumption (steady-state) is

\[
\tilde{c} = f(\tilde{k}) - \tilde{k} - r \tilde{b} 
\]  

On the other extreme, with no foreign borrowings by either the private sector or the government, the economy adjusts to its new steady state asymptotically, as in a one-sector closed-economy growth model. In this case steady-state capital stock and consumption are

\[
\hat{k} = f^{-1}(1 + r) 
\]  

and

\[
\hat{c} = f(\hat{k}) - \hat{k} 
\]  

Although steady-state consumption in an economy with no borrowings (18) is higher than with open markets (20), the latter obtains a higher utility value as in (1). Denoting the post-shock value of (1) on date 0 as \( \tilde{W} \), thus

\[
\tilde{W} > \hat{W} 
\]

It is obvious that welfare under an optimal income tax policy (denote \( W^{opt} \)) will be at least as much as that in the completely closed economy case. Moreover, no policy can improve upon the welfare obtained under perfect capital mobility. Thus

\[
\tilde{W} > W^{opt} > \hat{W} 
\]  

### 2.3 Constant Tax-Rate Policies

For comparison with the benchmark economy and its extreme counterpart, let us again assume that the economy begins date \( t = 0 \) with no foreign debt, i.e. \( b_0 = 0 \). Further, suppose that the economy was in a steady state before the productivity shock on date \( t = 0 \).
One natural candidate for a welfare improving\textsuperscript{11} policy may be thought of as a constant tax rate policy, since one may conjecture that such a policy will be optimal for removing intertemporal distortions and, thus, ensure consumption smoothening. In this section, we impose further restrictions on the government in regard to the type of policies that it may be able to implement. Specifically, we assume that, for political economy considerations\textsuperscript{12}, it may only be able to implement constant tax-rate policies.

An optimal tax policy that has an additional constraint, that the tax-rate shall remain constant on dates \( t \geq 1 \), can be formally characterized by adding another sequence of constraints to our Lagrangian (12) for the Ramsey problem

\[
\frac{u'(c_{t-1})}{\beta u'(c_t)f'(k_t)} = (1 - \tau)
\]

where \( \tau \) is time invariant constant tax rate levied from date \( t \geq 1 \). It is obvious that the welfare under an additional constraint will be lower than the one obtained without it as in (12).

There is a more intuitive interpretation of a constant tax-rate policy. It may be thought of as a closed economy with a permanently lower than the actual productivity, but with a higher initial capital stock. Here, the government only makes an initial positive transfer to the household on date \( t = 0 \) and, then, from date \( t \geq 1 \) onwards it taxes the household income \((f(k_t))\) at a constant rate \( \tau \). To see this, we look at the household date 0 income

\[
f(k_0) + b_1 = (1 - \tau)f(k_0^c)
\]

where \( b_1 \) is the government transfer (its foreign borrowing) to the household and \( k_0^c \) is the implicit capital stock with the household, if all of its income came from production only. Thus, the household problem is maximizing its utility (1) subject to a modified budget constraint

\[
c_t + k_{t+1} = (1 - \tau)f(k_t)
\]

with initial capital stock \( k_0^c \) as given. The factor \( 1 - \tau \) implicitly denotes a lower productivity. Notice that the initial capital stock \( k_0^c \), in turn, depends on the future capital time-path \((k_t = k(\tau, t))\) as

\[
(1 - \tau)f(k_0^c) = f(k_0) + \sum_{t=1}^{\infty} \frac{\tau f(k_t)}{(1 + r)^t}
\]

\textsuperscript{11}Relative to the case where the government sits idle and the economy grows as in a standard closed-economy one-sector growth model.

\textsuperscript{12}Although we don’t model these issues, one may argue that the governments may find it difficult to convince the public about the optimality of a time-varying tax rate and announce them in advance. The other reason could be that an incumbent government thinks that it may be hard for its successor to change tax rates.
There is a continuum of equilibria associated with each $\tau \in [0, \bar{\tau}]$, where $\bar{\tau} (< 1)$ is the tax rate for which the steady state is obtained on date $t = 0$. Any tax rate higher than $\bar{\tau}$ is clearly dominated by $\bar{\tau}$.\footnote{A formal proof is left to the reader!}

To see this, imagine a tax rate $\tau = 0$. Obviously, in this case the (closed) economy travels over time on a saddle path to its steady state equilibrium. Now, increase the tax rate by an infinitesimally small amount, so that $\tau = \epsilon$. This distortionary tax lowers both steady-state consumption and capital stock. Remember, however, that an equivalent value of all future tax revenues is transferred to the household on date 0, through government borrowing from abroad. Thus, the corresponding implicit capital stock (as obtained from (25)) is increased. Relative to a closed economy, where the dynamic system begins at the given $k_0$, the new system begins closer (at a higher initial capital stock(implicit)) to its new steady-state equilibrium (at a lower level of steady-state consumption and capital stock). Increasing the tax rate further, in this manner, brings the initial point on the saddle path closer to its steady-state equilibrium. Finally, the tax rate reaches a value $\bar{\tau}$, where the economy directly jumps on the steady state on date $t = 0$.\footnote{It is easy to see that a tax rate higher than $\bar{\tau}$ will further depress the steady state equilibrium point. Thus, for any $\tau > \bar{\tau}$ the implicit capital stock ($k_0$) will be smaller than $\bar{k}$, the implicit initial capital stock obtainable with $\tau = \bar{\tau}$. Since all such equilibria are strictly dominated (in welfare) by $\bar{\tau}$, we restrict our attention to $\tau \in [0, \bar{\tau}]$.}

One natural candidate for a welfare improving\footnote{Relative to the case where the government sits idle and the economy grows as in a standard closed-economy one-sector growth model.} policy may be thought of as a constant tax rate policy that brings the economy immediately to a new steady state, since one may conjecture that such a policy will be optimal for removing intertemporal distortions and, thus, ensure smooth consumption.\footnote{It should be obvious that no Ramsey allocation (with or without an additional constant tax-rate constraint) can arrive at a steady state instantly. In appendix (5.1), we provide a proof.} At this constant tax-rate policy, where $\tau = \bar{\tau}$, the allocations are such that the steady state is instantly achieved, i.e. $c_t = \bar{c}$ $\forall$ $t \geq 0$ and $k_t = \bar{k}$ $\forall$ $t \geq 1$. The government borrows $\bar{b}$ on date 0 and, thereafter, has the following budget constraint

$$\tau f(\bar{k}) = r\bar{b} \quad t \geq 1$$

Then, it must be the case that on date $t = 0$

$$\bar{c} + \bar{k} = f(k_0) + \bar{b}$$

At $t = 1$

$$\bar{c} + \bar{k} = (1 - \tau) f(\bar{k})$$

$$= \bar{b}$$
Comparing (27) and (28), it is obvious that the implicit capital stock 
\( k_0^e = \bar{k} \), with effectively lower productivity (by a factor \( 1 - \tau \)) from date 
\( t = 0 \). Further, at an optimum

\[
\beta f'(\bar{k})(1 - \tau) = 1 \tag{29}
\]

The new (steady-state) capital stock is uniquely determined from (27),
(28), (26), and (29) as a solution to\(^{17}\)

\[
f(\bar{k}) \left( \frac{1}{\beta f'(k)} - \frac{1}{1 + r} \right) = \frac{r}{1 + r} f(k_0) \tag{30}
\]

Finally, post-shock consumption (steady-state) is obtained from (27),
where \( \bar{k} \) is obtained from (30), and steady-state foreign debt is given by

\[
\bar{b} = \frac{1}{1 + r} \left( f(\bar{k}) - f(k_0) \right) \tag{31}
\]

Notice that although a constant tax rate \( \bar{\tau} \) perfectly achieves con-
sumption smoothening, it distorts production capital allocation. The
steady-state capital stock employed is permanently lower than that with
open capital markets or than that under completely closed economy.
Moreover, clearly, this policy is not an optimal Ramsey policy, even
when solved with an additional constant tax-rate constraint\(^{18}\). Hence,

\[
\bar{W} > W^{opt} > \bar{W} \tag{32}
\]

However, we are unable to rank welfare under \( \bar{\tau} \) (\( \bar{W} \)) against that ob-
tained in a closed economy (\( \hat{W} \)) since one can not obtain closed form
solutions for a closed economy case. For this purpose, we turn to nu-
merical computations. For our parameterization, we find that

\[
\bar{W} > W^{opt} > \bar{W} > \hat{W} \tag{33}
\]

3 An Algorithm for Computing Optimal Tax Se-
quence

We compute the sequence of optimal taxes and evaluate welfare gains
through the following six steps.

1. First, we find the steady state of the system of equations given by
the planner’s constraints and date \( t (\geq 1) \) first-order-conditions,

\(^{17}\)Obviously, for a solution to (30), we require \( \beta f'(\bar{k}) < 1 + r \).
\(^{18}\)The proof is quite straightforward.
which are being repeated below for convenience

\[ f(k_{ss}) = r b_{ss} + c_{ss} + k_{ss} - k_{ss}(1 - \delta) + g_{ss} \]

\[ c_{ss} + k_{ss} = \frac{f(k_{ss})}{\beta (f'(k))_{k=k_{ss}}} \]

\[ u'(c_{ss}) = \frac{\phi}{(1 + r)} + \mu_{ss} + (1 - \beta) \mu_{ss} \frac{u''(c_{ss})}{\beta u'(c_{ss}) (f'(k))_{k=k_{ss}}} \]

\[ f'(k_{ss}) = 1 + r + (1 + r)^2 \mu_{ss} \left( 1 - \left( \frac{\partial}{\partial k} \left( \frac{f(k)}{f'(k)} \right) \right)_{k=k_{ss}} \right) \] (34)

Since (34) has five unknowns \( c_{ss}, k_{ss}, b_{ss}, \mu_{ss}, \) and \( \phi \) in four equations, a value for \( b_{ss} \) is assumed. Of course, the steady-state level of debt depends on the initial net worth of the agents. After the complete sequence of allocations \( \{c_t, k_{t+1}, b_t\}^\infty_0 \) are computed (as discussed in following paragraphs), we check if the final debt level is commensurate with the initial debt holdings \( (b_0) \) and capital stock \( (k_0) \) by backward calculation using resource constraint (8).

2. Having computed the steady-state allocations, we first begin with date 0 first-order-conditions (13)

\[ u'(c_0) + \beta \mu_1 \frac{u''(c_0)}{\beta u'(c_1) f'(k_1)} = \frac{\phi}{1 + r} \]

\[ \beta \mu_1 \frac{u'(c_0)}{\beta u'(c_1)} \frac{\partial}{\partial k_1} \left( \frac{f(k_1)}{f'(k_1)} \right) = \frac{\phi}{1 + r} - \frac{\phi}{(1 + r)^2} \left( \frac{f'(k_1)}{f'(k_0)} \right) \] (35)

Notice from (35) that \( \frac{\mu_1}{u'(c_1)} \) can be treated as one variable. As \( \phi \) is known (from steady-state computations), we have two equations in three variables: \( c_0, \frac{\mu_1}{u'(c_1)}, \) and \( k_1 \). We begin with a guess for \( c_0 \). With this guess, values for \( c_0 \) and \( \frac{\mu_1}{u'(c_1)} \) (let = \( \psi_1 \)) are computed and we proceed to the next steps.

3. Now, we look at date \( t \) implementability constraint (6) and first-order-conditions (14) and (15), which is repeated for convenience

\[ c_t + k_{t+1} = \frac{u'(c_{t-1}) f(k_t)}{\beta u'(c_t) f'(k_t)} \] (36)

\[ c_t: \beta^t u'(c_t) = \beta^t \mu_t + \beta^t \mu_1 u''(c_t) \frac{u'(c_{t-1}) f(k_t)}{\beta (u'(c_t))^2 f'(k_t)} - \beta^{t+1} \mu_{t+1} \frac{u''(c_t) f(k_{t+1})}{\beta u'(c_{t+1}) f'(k_{t+1})} + \frac{\phi}{(1 + r)^{t+1}} \] (37)
Again, observe that $\mu_{t+1} u_0(c_t)$ can be treated as one variable. Having obtained $c_0, k_1,$ and $\mu_1 u_0(c_1)$ from date 0 computations, date 1 equations are: above (37), (38), and

\[ c_1 = \frac{\psi_1}{\mu_1} \]  

(39)

where $\psi_1$ is obtained in the previous step. Now, we have four unknowns $\mu_1, c_1, k_2,$ and $\frac{\mu_2}{u(c_2)}$ in four equations: (37), (38), and (39). With date 1 allocations and $\frac{\mu_2}{u(c_2)}$ computed, we proceed to compute date 2 allocations and $\frac{\mu_3}{u(c_3)}$. We iterate in this manner for a number of periods $T$, until convergence is achieved\(^{19}\).

4. After the iterations converge, the consumption is matched with the steady-state value obtained in the first step. If they don’t, the initial guess for $c_0$ is corrected by a simple linear factor of the difference. This is repeated, until the iterations converge to the steady-state allocations.

5. Finally, we use the sequence of allocations $\{c_t, k_{t+1}\}_0^T$, and $b_{ss}(\text{which we assume } = b_T)$ as obtained from step 1, to compute the entire debt sequence $\{b_t\}_0^T$ backwards

\[ b_t = \frac{b_{t+1} + f(k_t) - c_t - g_t - k_{t+1} + k_t(1-\delta)}{1+r} \]  

(40)

Given initial capital stock $k_0$, if the value for $b_0$ obtained in step 5 matches with the one given, we stop. Otherwise, the guess for $b_{ss}$ is updated, again using a linear rule, and steps 1 – 5 is repeated, until convergence is achieved.

6. After steps 1 – 6 converge, the optimal sequence of taxes $\{\tau_t\}_0^T$ is obtained from

\[ \tau_t = 1 - \frac{c_t + k_{t+1}}{f(k_t)} \]  

(41)

\(^{19}\)Usually, convergence is achieved within 20 iterations. Therefore, we keep $T = 20$. 

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### 3.1 Simulation Results

We parametrize our model for

$$ u(c_t) = \ln c_t $$  \hspace{1cm} (42)

and

$$ f(k_t) = \theta (1 + Ak_t)^\alpha ; \quad \theta = 1.5, \; \alpha = 0.4 $$  \hspace{1cm} (43)

We simulate our model with the economy’s old productivity parameter $A = 10$. After the shock on date $t = 0$, $A = 11$. Below in Figure 1, we present our results.

**Figure 1. Impulse Response With Optimal Taxation**

Figure 1, panel IV exhibits the optimal tax sequence, which follows a rising path before reaching its steady state level. To encourage higher savings and capital accumulation, the optimality requires that initially taxes be at a lower level relative to steady state. This implies a lower level of date 0 borrowing by the government, and hence a lower level of debt service in future. A lower level of debt service in steady state, in turn, implies a lower steady-state tax rate, which implies a relatively more efficient production level. An observation of the time paths for consumption, capital stock, and output illustrates that although all of them monotonically rise with time, the variation is very small. This variation is negligible when compared with the time paths for consumption and capital stock in a closed economy (where even the government is unable
to access international markets). This is starkly exhibited in Figure 2

![Figure 2: Impulse Response Without Debt and Taxes](image)

As expected, in a closed economy, both consumption and capital stock are higher in steady state. Although distortionary taxes depress steady-state capital stock and output, they are able to smoothen consumption over time through the government borrowing today in return for future payments through income taxes. From Figure 2 it appears that the economy instantly reaches its steady state under government borrowings and optimal income taxation. It gives an impression that perhaps a constant tax-rate from date 1, which brings the economy to the steady state instantly, may do at least equally well. To have a rough idea, we illustrate this point by the following comparison

<table>
<thead>
<tr>
<th>Economy</th>
<th>Consumption</th>
<th>Capital</th>
<th>Tax Rate(%)</th>
<th>Welfare (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Open</td>
<td>3.252027</td>
<td>1.91873</td>
<td>–</td>
<td>−2.6 * 10&lt;sup&gt;−6&lt;/sup&gt;</td>
</tr>
<tr>
<td>Constant Tax</td>
<td>3.252026</td>
<td>1.916149</td>
<td>0.07712</td>
<td>2.1 * 10&lt;sup&gt;−9&lt;/sup&gt;</td>
</tr>
<tr>
<td>Optimal Tax</td>
<td>3.252028</td>
<td>1.91615</td>
<td>0.077089</td>
<td>–</td>
</tr>
<tr>
<td>Closed</td>
<td>3.2561</td>
<td>1.91873</td>
<td>0</td>
<td>1.3 * 10&lt;sup&gt;−3&lt;/sup&gt;</td>
</tr>
</tbody>
</table>

Table 1 presents steady state values of some of the key variables of our model. We find that steady state tax rates under optimal taxation are
almost of the same order as under constant rate taxation that achieves steady state instantly. All welfare gains are relative to the welfare with optimal taxes. As expected, all of these alternatives are inferior to an economy with no borrowing constraints. Next, government borrowing with optimal taxes is welfare improving both relative to a closed economy as well as relative to an economy with constant taxes. However, it is clear that the gain relative to constant taxes is negligible. Moreover, as all these gains are in percentages, the results show that welfare gains within alternatives are very small and, thus, cast serious doubt on the usefulness of the discussion of alternative regimes.

4 Remarks

We end this paper by discussing some of the shortcomings of our work. The results show that welfare gains, obtained by switching from one regime to another, are too small to warrant any serious consideration of these alternatives. We feel that a fully calibrated model should be simulated before reaching this conclusion. As an excuse, our focus has mainly been on developing a computational algorithm and this important detail has been inadvertently ignored.

Our results also show that a constant tax rate that brings steady state instantly does almost as well as the one with optimal taxes. Again, one will have to be more careful with preferences while calibrating the model. As an example, if the household has linear preferences, there may be initial conditions where the household can wait to consume later and no government intervention will be required. On the other hand, with Leontif preferences constant tax rate policy ($\bar{\tau}$) will turn out to be the optimal policy. In addition, more work is needed to characterize an optimal constant tax rate policy as we have mainly focused on the one that brings steady state in the first period.

Finally, our framework is limited to analyzing issues under perfect foresight. Modeling uncertainty in future world interest rates, technological changes, implementability of future policies, and removal of borrowing restrictions are some interesting issues that require a stochastic general equilibrium framework. Developing such a framework will be clearly more challenging - both analytically as well as computational. All of these are left for future research.
5 Appendix

5.1 A Model with Labor

5.1.1 The Household

The household maximizes its lifetime utility

$$U_t = \sum_{t}^{\infty} \beta^t u(c_t, l_t)$$

(44)

where the utility function has the standard properties. The maximization problem is subject to the budget constraint

$$c_t + k_{t+1} = (1 - \tau_l)w_t l_t + (1 - \tau_k)\rho_t k_t \quad t \geq 1$$

(45)

where $\tau_l$ and $\tau_k$ are tax rates on labor and capital respectively. Further, $w$ and $\rho$ are existing market wage and rental rates respectively. However, after a positive productivity shock, the government makes a positive transfer $b_1$ on date 0. Here, to make the problem interesting we assume that the government does not tax the labor income on date 0. Otherwise, all the transfer will be made proportional to the labor income since capital stock is fixed. The date 0 budget constraint is

$$c_0 + k_1 = w_0 l_0 + \rho_0 k_0 + b_1$$

(46)

The first order conditions are

$$\frac{-u_l(c_t, l_t)}{u_c(c_t, l_t)} = (1 - \tau_l)w_t$$

(47)

and

$$u_c(c_t, l_t) = \beta \rho_{t+1}(1 - \tau_{kt+1})u_c(c_{t+1}, l_{t+1})$$

(48)

Combining (45), (47), and (48), we obtain

$$u_c(c_{t-1}, l_{t-1})k_t = \beta u_c(c_t, l_t)k_{t+1} + \beta (u_c(c_t, l_t) c_t + u_l(c_t, l_t) l_t)$$

(49)

Further, using transversality condition $\lim_{t \to \infty} \beta^t u_c(c_t, l_t)k_{t+1} = 0$, we can rewrite (49) as

$$u_c(c_0, l_0)k_1 = \sum_{t=1}^{\infty} \beta^t (u_c(c_t, l_t) c_t + u_l(c_t, l_t) l_t)$$

(50)

Finally, (45) can be combined with (50) to yield

$$u_c(c_0, l_0)(\rho_0 k_0 + b_1) = \sum_{t=0}^{\infty} \beta^t (u_c(c_t, l_t) c_t + u_l(c_t, l_t) l_t)$$

(51)

Thus, any competitive equilibrium allocation will have to satisfy (51).

\[\text{\textsuperscript{20}}\text{For simplicity, we have assumed that capital depreciates completely.}\]
5.1.2 The Firm
The representative firm maximizes profits
\[ \pi_t = f(k_t, l_t) - w_t l_t - \rho_t k_t \]  
(52)
The first order conditions are
\[ w_t = f_t(k_t, l_t) \]  
(53)
and
\[ \rho_t = f_t(k_t, l_t) \]  
(54)
We assume constant returns to scale. Hence, equilibrium profits are zero.

5.1.3 The Government
The government budget constraint is
\[ (1 + r)b_t + g_t = b_{t+1} + \tau_{lt}w_t l_t + \tau_{kt}\rho_t k_t \]  
(55)
Using the fact that the technology is homogeneous of degree one \( f(k, l) = f_1 k + f_2 l \), the economy’s combined intertemporal resource constraint is
\[ b_1 = \sum_{t=1}^{\infty} \frac{f(k_t, l_t) - c_t - g_t - k_{t+1}}{(1 + r)^{t+1}} \]  
(56)
Where \( b_1 \) is the government foreign borrowings on date 0.

5.1.4 The Ramsey Problem
Ramsey allocations are solution to the problem of maximizing household welfare (44) subject to the implementability constraint (51), the intertemporal resource constraint (56), and the date 0 budget constraint.
We form a Lagrangian
\[ L_t = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) + \phi \left( -b_1 + \sum_{t=1}^{\infty} \frac{f(k_t, l_t) - c_t - g_t - k_{t+1}}{(1 + r)^{t+1}} \right) \]
\[ + \lambda \sum_{t=0}^{\infty} \beta^t (u_{cc}(c_t, l_t) c_t + u_{ct}(c_t, l_t) l_t) + \]
\[ -\lambda u_{cc}(c_0, l_0)(f_k(k_0, l_0)k_0 + b_1) + \nu(f(k_0, l_0) + b_1 - c_0 - k_1) \]  
(57)
We define
\[ W(c_t, l_t) = u(c_t, l_t) + \lambda (u_{cc}(c_t, l_t) c_t + u_{ct}(c_t, l_t) l_t) \]
Then, for date 0, the first order conditions are
\[ b_1 : u_{cc}(c_0, l_0) = \frac{\phi - \nu}{\lambda} \]
\[ k_1 : \frac{\phi}{(1 + r)^2} f_k(k_1, l_1) = \nu \]
\[ c_0 : W_c(c_0, l_0) = \lambda u_{cc}(c_0, l_0)(f_k(k_0, l_0)k_0 + b_1) + \nu \]  
(58)
For date $t \geq 1$

\[
W_c(c_t, l_t) = \frac{\phi}{1 + r} \\
W_l(c_t, l_t) = f_l(k_t, l_t) \\
k_{t+1} = f_k(k_{t+1}, l_{t+1}) = (1 + r)
\]

Clearly, these conditions (59) imply that the optimal policy achieves steady state from date 1 onwards. Moreover, the labor tax rates are also constant from date $t \geq 1$

\[
\tau_{lt} = \bar{\tau}_l
\]

Further as $\rho = 1 + r$, the household first order condition (48) implies that for all dates $t \geq 2$

\[
\tau_{kt} = 0
\]

This is our version of the famous result in the optimal taxation literature due to Chamley (1986) that the optimal tax on capital in a steady state is zero. It comes as a surprise that why does the economy not achieve steady state from date 0 onwards. To see this, notice that we have kept the tax rate on date 0 labor as zero. Since the government makes a positive transfer on date 0 an optimal tax policy may require a very high negative tax (transfer) on date 0 that may, through intratemporal substitution require a high (relative to date $t \geq 1$) level of date 0 consumption. Again, optimality will not smoothen consumption from date 0. An intertemporal distortion between date 0 and date 1 entails a non-zero tax on capital on the latter date. Hence, the capital taxes are zero only from date 2 onwards.

5.2 Can Ramsey Allocations Achieve Steady State on Date 0?

The answer is no. We prove by contradiction.

Suppose that an optimal allocation is such that the steady state is achieved instantly, i.e. $c_t = \bar{c} \ \forall \ t \geq 0$ and $k_t = \bar{k} \ \forall \ t \geq 1$. It implies, then

\[
\mu_t = \bar{\mu} \ \forall \ t
\]

First order condition w.r.t. consumption at date 0, equation (13), implies

\[
u'(\bar{c}) = \frac{\phi}{1 + r} - \beta \bar{\mu} \frac{u''(\bar{c})}{\beta u'(\bar{c}) f'(k)}
\]

But FOC w.r.t consumption at date $t$

\[
u'(\bar{c}) = \frac{\phi}{(1 + r)} - \beta \bar{\mu} \frac{u''(\bar{c})}{\beta u'(\bar{c}) f'(k)} +
\]
\[
\hat{\mu} + \frac{\hat{\mu} u''(\bar{c}) f'(\bar{k})}{\beta u'(\bar{c}) f'(\bar{k})} = 0 \quad (64)
\]

But equations (63) and (64) can be equivalent if and only if

\[
\frac{\hat{\mu} u''(\bar{c}) f'(\bar{k})}{\beta u'(\bar{c}) f'(\bar{k})} = 0
\]

Which implies that either

\[
\frac{u''(\bar{c}) f'(\bar{k})}{\beta u'(\bar{c}) f'(\bar{k})} = -1 \quad (66)
\]

or

\[
\hat{\mu} = 0 \quad (67)
\]

But (66) will hold along with (27) and (30) for only one value of initial capital stock \(k_0\). Further, if (67) holds, then from (13) and (15), we have

\[
f'(\bar{k}) = 1 + r \quad (68)
\]

which means that optimal taxes are zero from date \(t = 2\) onwards. But, this is not possible as Ponzi games are ruled out in equilibrium. Hence, we conclude that no optimal taxation policy makes the economy arrive at the steady state instantly, i.e. on date \(t = 0\).

References


