Dynamic Costly State Verification

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Summary. I study a model of dynamic risk sharing with costly state verification (CSV). In the model, a risk neutral agent enters an infinitely repeated relationship with a risk averse agent. In each period, the risk averse agent receives a random income which is observed only by himself, unless the risk neutral agent engages in costly monitoring. I provide a set of characterizations for the optimal contract, and I show that CSV has interesting effects on the long run distribution of the agents’ expected utilities.

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1 Introduction

The model of costly state verification (CSV) (Townsend 1979) has been used as a vehicle for studying a variety of issues related to investment contracting, banking, and macroeconomics. ¹ A typical interpretation of the model is that it describes a financial lending process where the borrower’s income is not directly observed by the lender, unless the lender incurs a cost to monitor.

In this paper, I extend the model of CSV to a fully dynamic setting. Specifically, I study a problem of optimal dynamic contracting between a risk neutral agent (the lender) and a risk averse agent (the borrower). In each period, the risk averse agent receives a random income which is observed directly only by himself. The risk neutral agent can observe the risk averse agent’s income realization by incurring a fixed monitoring cost. Like Townsend, I focus on deterministic monitoring strategies.

My model thus stands between the literature of CSV and the literature of (non-CSV) dynamic contracting. ² I will focus on two sets of questions. First, what is the structure of the optimal contract? In particular, what is the risk neutral agent’s optimal monitoring strategy, and how does it depend on the parameters and the dynamics of the model? When comparisons are relevant, how do the answers to these questions differ between the dynamic and the static CSV models? Second, what are the effects of CSV on the long run distribution of the agents’ consumption and utility? The second question is of interest partly because the existing non-CSV dynamic contracting literature has focused a lot on the long run implications of dynamic contracts. ³

I proceed through a series of environments that are progressively more technically demanding. I first consider the case of CARA utility function and two income levels. I then consider the case of CARA with more than two income levels. Finally, I study the model with non-CARA utilities. The characterizations of the optimal contract that I can obtain will be fairly complete for the first case, they are much less complete for the latter cases. In quite a few situations I will have to rely on numerical computation for insights.

In the case of CARA and two income levels, there is a critical level of the fixed monitoring cost, \( \gamma^* \), below which monitoring is optimal and above which monitoring is not optimal. When monitoring is optimal, first-best risk sharing is obtained. When monitoring is not optimal, the optimal provision

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of intertemporal incentives implies that on average, and relative to the first-
best allocation, the risk neutral agent’s consumption must be postponed,
while the risk averse agent’s consumption must be moved up. Thus, as
interest rate rises, it is more expensive for the risk neutral agent to finance
the risk averse agent’s relatively earlier consumption, making monitoring
more efficient relative to no-monitoring. In other words, $\gamma^*$ decreases as
interest rate increases.

This has interesting implications, especially for macroeconomists who
seek to study CSV in dynamic general equilibrium settings. This implies
that, depending on the cost of monitoring, a small change in interest rates
can have profound effects on the form of the optimal contract and on the
long run distribution of wealth between the agents. Specifically, a small
increase in interest rates can shift the optimal contract from monitoring to
no-monitoring, making the risk neutral agent better off while the risk averse
agent worse off in the long run.

In the case of CARA with two income levels, I show that the effects
of a change in risk aversion on the optimal monitoring policy is different
in the dynamic model than in the static model. In the static model, the
cut-off monitoring cost $\gamma^*$ is strictly increasing in the coefficient of absolute
risk aversion, $\rho$. In the dynamic model, $\gamma^*$ is increasing in $\rho$ only if $\rho$ is
sufficiently small, and is decreasing in $\rho$ if $\rho$ is sufficiently large; moreover, $\gamma^*$
goes to zero as $\rho$ goes to infinity. In the static model, the risk averse agent’s
consumption variability required by incentive compatibility is constant in
$\rho$. As risk aversion increases, this fixed amount of consumption variability
becomes increasingly costly to the risk neutral agent. Thus, an increase
in $\rho$ strengthens the case for monitoring. In the dynamic model, there are
two forces working in different directions. On the one hand, an increase in
$\rho$ implies that any fixed amount of consumption variability becomes more
costly, strengthening the case for monitoring. On the other hand, an increase
in risk aversion reduces the amount of consumption variability that is needed
for truth-telling in the absence of monitoring, due to a change in the risk
averse agent’s attitude towards intertemporal substitutions. This weakens
the case for monitoring. The former force dominates when $\rho$ is sufficiently
small, and the latter dominates for $\rho$ sufficiently large.

This non-monotonic relationship between monitoring and absolute risk
aversion implies that when the risk averse agent’s utility function takes a
non-CARA form, the optimal monitoring policy should depend critically on
and not be monotonic in the risk averse agent’s expected utility (wealth).
This, in turn, can generate interesting life cycle dynamics and have implications
for the limiting distribution of the risk averse agent’s consumption
and expected utility. In a numerical example with log utility, I find that
monitoring is optimal only when the risk averse agent’s expected utility falls between two cut-offs. Monitoring is not optimal when the risk averse
agent’s expected utility is sufficiently low (he is sufficiently risk averse) or
when the risk averse agent’s expected utility is sufficiently high (he is not
sufficiently risk averse). I show that this generates interesting dynamics and has important implications for the limiting distribution of the risk averse agent’s consumption and expected utility.


The model is set up in Section 2. The optimal contract under CARA is characterized in Sections 3. Section 4 considers non-CARA utilities. Section 5 concludes the paper.

2 Model

Time is discrete and lasts forever: $t = 0, 1, 2, \ldots$, where $t = 0$ is the ex ante contracting period. There is one consumption good and there are two agents: a risk neutral agent and a risk averse agent. The risk neutral agent has the following preferences: $E_0 \sum_{i=0}^{\infty} \alpha^{-i} (c_i - \gamma_i)$, where $c_i$ is consumption, $\gamma_i (\geq 0)$ is monitoring effort, and $\alpha \in [0, 1)$ is his discount factor. The risk averse agent’s preferences are given by $E_0 \sum_{i=0}^{\infty} \beta^{-i} v(c_i)$, where $c_i$ is consumption, $\beta \in [0, 1)$ is his discount factor. The risk averse agent’s period utility function $v$ is strictly concave and continuously differentiable.

In each period $t \geq 1$, the risk averse agent receives a random income of $\theta^t$. Assume $\theta^t$ takes values from a finite set $\Theta \equiv \{\theta_1, \theta_2, \ldots, \theta_N\}$, where $\theta_1 < \theta_2 < \ldots < \theta_N$, and $\theta^t$ is i.i.d. across periods. Let $\pi_i \in (0, 1)$ be the probability with which $\theta^t$ takes the value of $\theta_i$: $\text{Prob}\{\theta^t = \theta_i\} = \pi_i$, $i = 1, 2, \ldots, N$, and $\sum_i \pi_i = 1$.

Assume that the probability distribution of the risk averse agent’s random income is common knowledge between the two agents. However, each period, the realization of the risk averse agent’s random income can be costlessly observed only by the risk averse agent himself. The risk neutral agent can observe the risk averse agent’s current income in any period, but must incur a fixed cost $\gamma \geq 0$ in order to do so, where $\gamma$ is interpreted as the cost of information acquisition. As in Townsend (1979), I restrict attention in this paper to deterministic monitoring (verification) strategies, where the risk neutral agent monitors the risk averse agent’s report of income with probability one or zero. 4 To insulate the model from any dynamics other
than those associated with the CSV problem, I assume that the consumption good cannot be stored across periods, and the risk averse agent has no outside saving or borrowing opportunities.

Note that if $\gamma = 0$ then there is complete information. On the other hand, if $\gamma = \infty$ or state verification is infeasible, then my model resembles the models of Green (1987), and Thomas and Worrall (1990), the key difference being that I allow asymmetric discounting between the agents. This difference is important. It allows me to study for example the effects of a change in the interest rate that the risk neutral agent faces on the risk sharing dynamics and the optimal monitoring policy.

3 The Exponential Utility Function

Throughout this section, I assume the risk averse agent’s period utility function takes the CARA form: $v(c) = -\exp(-c)$, $c \in \mathbb{R}$.

3.1 The First-Best Contract

In the case of complete information where $\gamma = 0$, The optimal allocation is first-best and can be written: $\sigma^{fb} \equiv \{c^{fb}_t\}_{t=1}^{\infty}$, where $c^{fb}_t$ is the risk averse agent’s consumption in period $t$. The first-best allocation solves

$$\max_{\{c_t\}} \sum_{t=1}^{\infty} \alpha^{t-1} (\bar{\theta} - c_t) \text{ subject to } \sum_{t=1}^{\infty} \beta^{t-1} \left[-\exp(-c_t)\right] = w_0,$$

where $w_0$ is the promised ex ante lifetime utility of the risk averse agent.

It is straightforward to show that

$$c^{fb}_t = -t \log(\alpha) - \log(-w_0) + \log(1 - \alpha).$$

So $c^{fb}_t$ depends linearly on $t$, and in a deterministic fashion: it increases in $t$ if $\alpha < \beta$, decreases in $t$ if $\alpha > \beta$, and it is constant in $t$ if $\alpha = \beta$. Let $w^{fb}_{t-1}$ denote the risk averse agent’s expected utility at the beginning of period $t$. Then it is easy to show that the law of motion for the risk averse agent’s expected utility takes the following form:

$$w^{fb}_t = \left(\frac{\alpha}{\beta}\right) w^{fb}_{t-1} = \left(\frac{\alpha}{\beta}\right)^t w_0. \quad (1)$$

Thus over time the risk averse agent’s expected utility also follows a deterministic path: $w^{fb}_t$ is strictly decreasing and converges to $-\infty$ if $\alpha > \beta$, is increasing and converges to $0$ if $\alpha < \beta$, and remains constant if $\alpha = \beta$. It also follows that

$$-\exp[-c^{fb}_t] = (1 - \alpha) w^{fb}_{t-1} = (1 - \alpha) \frac{\beta}{\alpha} w^{fb}_t. \quad (2)$$

This equation characterizes the intertemporal structure of the risk averse agent’s consumption. The risk averse agent’s current period utility is a

constant fraction $(1 - \alpha)\beta$ of his lifetime utility from next period onward, where the constant fraction is increasing in $\beta$ but decreasing in $\alpha$. An increase in $\alpha$ (or $\beta$) indicates that the risk neutral (averse) agent is relatively more patient and hence it is optimal to have his consumption postponed.

### 3.2 Costly State Verification

I now return to the case of costly state verification, i.e., the case where $\gamma > 0$. I start by describing the extensive form of a contract.

Let $h^t$ denote a history of all events that have taken place up to and including period $t$, $t \geq 0$, where $h^0 = \emptyset$, $h^t = \{h_1, h_2, \ldots, h_t\}$, $t \geq 1$, and $h_t = (\vartheta_t, \theta_t, s_t)$, where $\vartheta_t$ denotes the risk averse agent’s report of income in period $t$; $s_t$ is an indicator: $s_t = 1$, if monitoring occurred in period $t$, and $s_t = 0$ otherwise; and lastly, $\theta_t$ denotes the risk-neutral agent’s information regarding the realization of $\vartheta^t$ at the end of the period: $\theta_t = \theta_i$ if $s_t = 1$, and $\theta_t = \tilde{\theta}_i$ if $s_t = 0$.

As in Townsend (1979), I assume that the risk averse agent will not misrepresent if he is indifferent between truth-telling and misrepresentation. Thus conditional on the risk-averse agent submitting for state verification, I have $\hat{h}_t = \theta_i$. Thus, without loss of generality, I can rewrite $h_t$ as $h_t = (\theta_t, 1)$ if $s_t = 1$, and $h_t = (\tilde{\theta}_i, 0)$ if $s_t = 0$.

A contract $\sigma$ is a sequence of functions that map from histories to current-period verification strategies and payments: $\sigma = \{S_t(h^{t-1}), M_t(h^t)\}_{t=1}^{\infty}$ where $S_t(h^{t-1}) \subseteq \Theta$ is the risk neutral agent’s verification strategy in period $t$, which is the set of (reported) states in which verification takes place, and $M_t(h^t)$ is payment from the risk-averse agent to the risk-neutral agent in period $t$. Note that there are $2^N$ possible verification strategies for each period. Let $S$ denote the set of all such verification strategies, and for each $S_t \in S$, let $\mu(S_t)$ denote the probability with which verification takes place, given strategy $S_t$. That is, $\mu(S_t) = \sum_{\theta \in S_t} \pi_i$. For example, if $N = 2$, then there are four possible verification strategies: (i) $S_t = \emptyset$, (ii) $S_t = \{\theta_1\}$, (iii) $S_t = \{\theta_2\}$, (iv) $S_t = \{\theta_1, \theta_2\}$, and $\mu(S_t) = 0, \pi_1, \pi_2, 1$ respectively.

Following Green (1987) and Spear and Srivastava (1987), the extensive form contract $\sigma$ can be represented recursively by using the risk averse agent’s expected utility $w$ as a state variable. Specifically, the recursive form of $\sigma$ takes the form $\{S(w), \{M(\theta_i, w), W(\theta_i, w)\}_{i=1}^{N}\}$, where $S$ is the risk neutral agent’s monitoring policy, $M$ is the payment scheme, and $W$ is the law of motion for the state variable $w$. It is then straightforward to show that the optimal recursive contract is a solution to the following Bellman equation: for all $w \in (-\infty, 0)$,

$$
U^*(w) = \sup_{S(w), \{M(\tilde{\theta}_i, w), W(\tilde{\theta}_i, w)\}_{i=1}^{N}} \left( \sum_{i=1}^{N} \pi_i [M(\theta_i, w) + \alpha W(\theta_i, w)] - \mu[S(w)] \right) 
$$

subject to

$$
S(w) \subseteq \Theta; M(\theta_i, w) \in R, W(\theta_i, w) \in (-\infty, 0), \forall \theta_i \in \Theta,
$$
\[ v[-\theta_i + M(\theta_i, w)] + \beta W(\theta_i, w) \geq v[-\theta_j + M(\theta_j, w)] + \beta W(\theta_j, w), \]
\[
\forall \theta_i \in \Theta, \forall \theta_j \not\in S(w), \quad (5)
\]
\[
\sum_{i=1}^{N} \pi_i \{ v[-\theta_i + M(\theta_i, w)] + \beta W(\theta_i, w) \} = w. \quad (6)
\]

Here, (5) is the incentive constraint, (6) is the “promise-keeping” constraint. The term \(\mu[S(w)]\gamma\) represents the risk averse agent’s expected cost of monitoring in the current period, given that his monitoring policy is \(S(w)\).

Constraint (5) imposes that, given any history (i.e., for any \(w\)), and for any current-period realization of the risk averse agent’s random income, the risk averse agent have no incentives to misrepresent. Suppose (5) is violated for some \(\theta_i \in \Theta\), and \(\theta_j \not\in S(w)\). Then whenever the risk averse agent receives \(\theta_i\), he will have an incentive to report \(\theta_j\) and will not be caught. Remember if the risk averse agent’s income is \(\theta_i\), he never have incentives to report any \(\theta_j(\neq \theta_i)\) which lies in the verification region, knowing he would be caught in a state verification, and his true income would then be revealed anyway. Thus, as is clear from (5), monitoring reduces the number of constraints that an incentive compatible contract must satisfy. \(^5\)

The above functional equation can be solved almost analytically. Let \(S \subseteq \Theta\) and let
\[
\phi(S) \equiv \max_{(m_i, w_i)} \sum_{i=1}^{N} \pi_i \left[ \log m_i + \frac{\alpha}{1-\alpha} \log w_i \right] \quad (7)
\]
subject to \(m_i > 0, w_i > 0, \forall i\), and
\[
\exp(-\theta_i)m_i + \beta w_i \leq \exp(-\theta_j)m_j + \beta w_j, \forall \theta_i \in \Theta, \forall \theta_j \not\in S, \quad (8)
\]
\[
\sum_{i=1}^{N} \pi_i \{ \exp(-\theta_i)m_i + \beta w_i \} = 1. \quad (9)
\]

**Proposition 1** Let \(\{S^*(w), M^*(\theta_i, w), W^*(\theta_i, w)\}\) denote an optimal contract. Then \(S^*(w) \in \arg\max_{S \subseteq \Theta} [\phi(S) - \mu(S)]\gamma\), \(M^*(\theta_i, w) = \log(m_i^*); \log(-w_i), W^*(\theta_i, w) = w_i^*w\), where \(\{m_i^*, w_i^*\}\) maximizes the right hand side of (7) for \(S = S^*\).

Notice that \(S^*(w)\) is constant in \(w\). In the remainder of the paper I will use \(S^*\) for \(S^*(w)\). Let \(C^*(\theta_i, w) \equiv \theta_i - M^*(\theta_i, w)\) denote the optimal

\(^5\)Notice that constraint (5) covers the special cases where \(\gamma = 0\) or \(\gamma = \infty\). Specifically, \(\gamma = 0\) implies it is optimal to set \(S(w) = \Theta\) for all \(w\), so constraint (5) is never binding. In the case where \(\gamma = \infty\), \(S(w) = \emptyset\), for all \(w\). Also note that for \(\beta = 0\), (5) implies, as in standard static models of costly state verification such as Townsend (1979), that for all \(\theta_i, \theta_j \not\in S(w), M(\theta_i, w) = M(\theta_j, w)\). Thus, without monitoring, no risk sharing is feasible.
consumption scheme for the risk averse agent. Proposition 1 shows that $M^*(\theta, w)$ is monotone decreasing in $w$, and both $C^*(\theta, w)$ and $W^*(\theta, w)$ are monotone increasing in $w$.

Another monotonicity property of the incentive compatible contract that is easy to obtain is: If $\theta_i, \theta_j \notin S^*$, then $\theta_i < \theta_j$ implies $M^*(\theta_i, w) < M^*(\theta_j, w)$ and $W^*(\theta_i, w) < W^*(\theta_j, w)$.

Finally, note that the first best allocation can also be represented recursively, using the risk averse agent’s expected utility as a state variable. Let $\{M_{th}(\theta, w), W_{th}(\theta, w)\}$ denote the first-best contract.

\section{Two Income Levels}

In this section I characterize the optimal contract under the assumption that there are only two possible income levels, that is, $N = 2$.

\textbf{Proposition 2} There exists $\gamma^* > 0$ such that the optimal verification strategy is given by: $S^* = \{\theta_1\}$ if $\gamma \leq \gamma^*$ and $S^* = \emptyset$ otherwise; where

$$
\gamma^* \equiv \frac{\phi(\{\theta_1\}) - \phi(\emptyset)}{\pi_1}
$$

where

$$
\phi(\{\theta_1\}) \equiv \bar{\theta} + \log(1 - \alpha) + \frac{\alpha}{1 - \alpha} \log\left(\frac{\alpha}{\beta}\right),
$$

and

$$
\phi(\emptyset) \equiv \max_{\{m, w\}_{i=1,2}} \sum_{i=1,2} \pi_i \left[ \log m_i + \frac{\alpha}{1 - \alpha} \log w_i \right]
$$

subject to $m_i > 0, w_i > 0, (9)$, and

$$
\exp(-\theta_2)m_2 + \beta w_2 = \exp(-\theta_2)m_1 + \beta w_1.
$$

Thus, it is optimal either to engage in no state verification [$S^* = \emptyset$], or to verify the state when reported income is low [$S^* = \{\theta_1\}$]. Suppose $S^* = \{\theta_1\}$, then first best risk sharing is attained. Proposition 2 is an extension of a parallel result in Smith and Wang (1998) for the static and two period models of CSV.

A corollary of Proposition 2, and an extension of a parallel result in Wang and Smith (1998) for the case of static CSV, is: changes in the values of $\theta_1$ and $\theta_2$ affect the level of the critical monitoring cost $\gamma^*$ only through changes in the value of $(\theta_2 - \theta_1)$. Moreover, $\gamma^*$ is strictly increasing in $(\theta_2 - \theta_1)$. This is easy to prove. I need only use the following transformation in the problem that defines $\phi(\emptyset)$: $\tilde{m}_i \equiv \exp(-\theta_i)m_i$. The detailed proof is left for the reader.

Therefore an increase in $(\theta_2 - \theta_1)$ strengthens the case for monitoring. This is intuitive. Everything else equal, an increase in $(\theta_2 - \theta_1)$ makes truth telling more difficult to achieve, for now conditional on $\theta^1 = \theta_2$, the
temptation to report \( \theta_1 \) is greater. Mathematically, this means that all else equal, a higher value of \((\theta_2 - \theta_1)\) makes the incentive constraint tighter. \(^6\) Hence without monitoring, a greater amount of distortion in the risk averse agent’s consumption is needed for truth-telling.

I now move on to provide a characterization for the dynamics of the optimal contract under \( S^* = \emptyset \). Although this characterization is not directly about the optimal monitoring policy, it provides some of the underlying intuitions for an important result that will be contained in my next proposition.

**Proposition 3** Suppose \( S^* = \emptyset \). Then for all \( w \), \( C^*(\theta_2, w) > C^*(\theta_1, w) \), \( W^*(\theta_2, w) > W^*(\theta_1, w) \), and

\[
\begin{align*}
  v[C^*(\theta_1, w)] &> (1 - \alpha) \frac{\beta}{\alpha} W^*(\theta_1, w), \\
  v[C^*(\theta_2, w)] &= (1 - \alpha) \frac{\beta}{\alpha} W^*(\theta_2, w), \\
  \sum \pi_i W^*(\theta_i, w) &< \frac{\alpha}{\beta} w.
\end{align*}
\]

Thus when \( S^* = \emptyset \), in order to provide incentives for truth-telling, two kinds of distortions occur in the risk-averse agent’s consumption scheme relative to the first-best consumption scheme. There is an intratemporal distortion across states of current income, there is also an intertemporal consumption distortion across the current and the future periods. Notice that the intertemporal distortion occurs only in the low income state. Since the risk-averse agent never has an incentive to report \( \theta_2 \) when his income is \( \theta_1 \), no intertemporal incentives are needed in state \( \theta_2 \).

Proposition 3 is not entirely new. Specifically, equation (12) is an extension of a similar result in Green (1987) and Thomas and Worrall (1990) who deal with the case of \( \alpha = \beta \), although my proof of (12) (see Appendix) is much simpler and perhaps more intuitive.

Proposition 3 indicates that monitoring has long-run implications for the agents’ expected utilities. Under full risk sharing (which is attained when the monitoring cost is sufficiently small), equations (1) and (2), when written recursively, imply \( \sum \pi_i W_{\beta}(\theta_i, w) = \frac{\alpha}{\beta} w \) where \( W_{\beta}(\theta_i, w) \) is the first-best law of motion of the risk averse agent’s expected utility. Thus equation (12) shows that with no monitoring, on average the optimal contract pushes the expected utility of the risk averse agent down and that of the risk neutral agent up, relative to the case of monitoring. I will return to the long run implications of monitoring for the agents’ expected utilities in the later sections. The long-run implications of dynamic contracts have been a preoccupation of the dynamic contracting literature (Phelan (1998)).

\(^6\)With the transformation \( \tilde{m}_i = e^{-\delta_i} m_i \), the incentive constraint that is binding becomes \( \tilde{m}_2 + \beta \tilde{w}_2 \leq e^{-\delta_2 - \delta_1} \tilde{m}_1 + \beta \tilde{w}_1 \).
Compared to the static CSV model, one advantage of the dynamic model is that it allows me to consider the effects of the agents’ attitudes toward future consumption on the structure and dynamics of the optimal contract. Proposition 4 provides a result about the relationship between monitoring and the agents’ discount factors that has no parallel in the static CSV model.

**Proposition 4** (i) $\frac{\partial \gamma^*}{\partial \alpha} < 0$, and $\gamma^* \to 0$ as $\alpha \to 1$; (ii) $\frac{\partial \gamma^*}{\partial \beta} = 0$; (iii) Suppose $\alpha = \beta = \eta$, then $\frac{\partial \gamma^*}{\partial \eta} < 0$.

Proposition 4 says that as the risk neutral agent becomes more patient, $\gamma^*$ decreases monotonically. In other words, everything else equal, a higher discount factor of the risk neutral agent makes monitoring less likely to be optimal. Moreover, fixing the monitoring cost $\gamma$, a sufficiently high $\alpha$ can always make monitoring non-optimal. The story goes as follows. Remember that with no monitoring, the optimal contract implies that, relative to first-best, the risk neutral agent must shift resources from the future to the present to finance the risk averse agent’s current consumption. Now the cost of doing this is lower when $\alpha$ is higher. Put differently, a higher $\alpha$ implies intertemporal incentives for truth telling are more readily available, monitoring is thus relatively more expensive, and so $\gamma^*$ should be lower.

Proposition 4 shows that $\gamma^*$ is invariant in $\beta$. Given the CARA utility function of the risk averse agent, the value function is in the log of $w$, and that the discount factor $\beta$ always enters the equations multiplicatively with $w$, the effects of a variation in the risk averse agent’s discount factor are canceled out. However, intuitively, one might expect $\partial \gamma^*/\partial \beta > 0$ to hold. As $\beta$ increases, future consumption is more valuable for the risk averse agent, relative to current consumption. This in turn means that in order to persuade the risk averse agent to give up any specified amount of future consumption, the risk neutral agent must pay a higher cost in terms of current consumption. That is, as $\beta$ increases, intertemporal incentives are more expensive, or, monitoring is relatively more efficient in achieving incentive compatibility, $\gamma^*$ should thus be higher.

The results I have obtained so far continue to hold if the risk averse agent’s utility function takes the more general form $v(c) = -\exp(-\rho c)$, where $\rho > 0$ is the coefficient of CARA. Now the question I want to ask is: how would a change in the value of $\rho$ affect the structure of the optimal contract? In particular, how does the cutoff monitoring cost $\gamma^*$ depend on the value of $\rho$?

**Proposition 5** In the one period model, $\frac{\partial \gamma^*}{\partial \rho} > 0$. In the dynamic model, (i) $\frac{\partial \gamma^*}{\partial \rho} > 0$ for $\rho$ sufficiently small, (ii) $\frac{\partial \gamma^*}{\partial \rho} < 0$ for $\rho$ sufficiently large, (iii) $\gamma^* \to 0$ as $\rho \to \infty$.

Thus there is a monotonic relationship between $\gamma^*$ and $\rho$ in the static CSV model, but not in the dynamic model. Moreover, in the dynamic
model, the optimal demand for monitoring falls to zero as risk aversion becomes infinitely great. Why?

In the static model, suppose there is no monitoring. Then incentive compatibility requires $M(\theta_1, w) = M(\theta_2, w)$, and hence $C(\theta_2, w) - C(\theta_1, w) = \theta_2 - \theta_1$, where $C(\theta_i, w)$ denotes the risk averse agent’s consumption in state $\theta_i$. This equation implies the risk averse agent’s consumption variability is constant as $\rho$ increases. However, as risk aversion increases, this fixed amount of consumption variability becomes increasingly costly to the risk neutral agent, holding constant the risk averse agent’s expected utility. In other words, the benefits of consumption smoothing, which can be provided by monitoring, increase as $\rho$ increases. Thus an increase in $\rho$ strengthens the case for monitoring.

In the dynamic model, there are two mechanisms that work in different directions. On the one hand, as in the single period model, an increase in $\rho$ strengthens the case for monitoring, because consumption variability is more costly. On the other hand, in the dynamic model, because of the new intertemporal dimension along which incentives can be created, the total utility variability that is needed for incentive compatibility is bounded, which in turn implies that as $\rho$ increases, the amount of consumption variability needed for truth-telling decreases, weakening the case for monitoring. Note that this second mechanism is completely shut down in the static model.

To elaborate on how the second mechanism works, imagine a simple two-period setting where the first period environment is the same as the stage environment in my fully dynamic model, but the risk averse agent has no income in the second period. The agents consume in both periods, and there is no discounting across periods. With no monitoring, the optimization problem is

$$U_{nm}(w) \equiv \max_{v_1, v_2, w_1, w_2} \bar{\theta} + \frac{1}{\rho} \sum \pi_i [\log(v_i) + \log(w_i)]$$

subject to

$$v_1 + w_1 \geq \exp[-\rho(\theta_1 - \theta_2)]v_2 + w_2, \quad (13)$$
$$v_2 + w_2 \geq \exp[-\rho(\theta_2 - \theta_1)]v_1 + w_1, \quad (14)$$
$$\pi_1(v_1 + w_1) + \pi_2(v_2 + w_2) = w, \quad (15)$$

where $v_i$ ($w_i$) is the risk averse agent’s utility in the first (second) period given that his first period income is $\theta_i$. In stead of tackling the above problem directly, consider the following auxiliary problem:

$$U_{\text{nm}}(w) \equiv \max_{v_1, v_2, w_1, w_2} \bar{\theta} + \frac{1}{\rho} \sum \pi_i [\log(v_i) + \log(w_i)]$$

subject to (13), (14), (15), and $v_1 = v_2$. Note that by imposing $v_1 = v_2$ we have ruled out utility inequality across states in period 1 as an incentive device. That is, the contract must rely entirely on intertemporal arrangements to obtain incentive compatibility. As is straightforward to show, in
this problem, at the optimum, constraint (13) is not binding, it holds that \( w_2 = v_2 \) and
\[
K(\rho)v_1 = w_1 = K(\rho)w_2, \tag{16}
\]
where \( K(\rho) \equiv 2 - \exp[-\rho(\theta_2 - \theta_1)] \). Here the first equation of (16) characterizes the intertemporal inequality in the risk averse agent’s utility that is required for incentive compatibility by the optimal contract, and the second equation characterizes the intratemporal inequality in the risk averse agent’s period 2 utility. Notice that \( K'(\rho) > 0 \), so higher risk aversion implies higher utility inequality across states and periods. Moreover, \( K(\rho) \) is bounded between 1 and 2. Thus total utility variability required for incentive compatibility is bounded as \( \rho \) increases. Note that in the single period model, \( -e^{-\rho C(\tilde{z},w)} = e^{\rho(\tilde{z} - \tilde{w})} \left[ -e^{-\rho C(\tilde{z},w)} \right] \), where \( e^{\rho(\tilde{z} - \tilde{w})} \to \infty \) as \( \rho \to \infty \), so utility variability is unbounded as \( \rho \) increases.

That utility variability in the optimal dynamic contract is bounded is essential for obtaining our result. I have
\[
\tilde{c}_2 - c_1 = \tilde{c}_1 - c_1 = c_2 - c_1 = \frac{\log[K(\rho)]}{\rho},
\]
where \( -\exp(-\rho c_i) = w_i \) and \( -\exp(-\rho \tilde{c}_i) = v_i \). The expression \( \frac{\log[K(\rho)]}{\rho} \) measures in a sense the minimum consumption variability required for truth-telling. Clearly, \( \log[K(\rho)]/\rho \) is increasing in \( \rho \) for \( \rho \) sufficiently small, and decreasing in \( \rho \) for \( \rho \) sufficiently large. Moreover, given \( K(\rho) \to 2 \) as \( \rho \to \infty \), \( \log[K(\rho)]/\rho \) converges to zero as \( \rho \to \infty \), implying that the optimal contract converges to the first best as risk aversion goes to infinity.

To show that \( \gamma^* \to 0 \) as \( \rho \to \infty \) formally, rewrite \( \bar{U}_{nm}(w) \):
\[
\bar{U}_{nm}(w) = \max_{w_1,w_2} \left[ \pi_1 \log(-w_1) + (1 + \pi_2) \log(-w_2) \right]
\]
subject to (16) and \( \pi_1 w_1 + (1 + \pi_2) w_2 = w \). Obviously,
\[
\bar{C}_{nm}(w) = U_{lb}(w) + \frac{1}{\rho}Z(\rho)
\]
where \( U_{lb}(w) \) is the risk neutral agent’s value function in the case of complete information, \( U_{lb}(w) = \bar{\theta} + 2\log(-w) \), and
\[
Z(\rho) \equiv -\max_{z_1,z_2} \left[ \pi_1 \log(z_1) + (1 + \pi_2) \log(z_2) \right]
\]
subject to \( \pi_1 z_1 + (1 + \pi_2) z_2 = 1 \), and \( z_1 = K(\rho) z_2 \). But \( K(\rho) \to 2 \) as \( \rho \to \infty \), and hence \( Z(\rho) \) converges to a finite number as \( \rho \to \infty \), and so
\[
\pi_1 \gamma^* = U_{lb}(w) - U_{nm}(w) \leq U_{lb}(w) - \bar{C}_{nm}(w) = \frac{1}{\rho}Z(\rho) \to 0, \text{ as } \rho \to \infty.
\]

I provide a numerical example to further illustrate the relationship between \( \rho \) and \( \gamma^* \). Let \( \theta_1 = 1.0, \theta_2 = 2.0, \pi_1 = \pi_2 = 0.5, \alpha = \beta = 0.90 \). Figure 1 plots \( \gamma^* \) as a function of \( \rho \).
3.3.1 Interpretations

Propositions 2, 4, and 5 show that the structure of the optimal contract is sensitive to the model's parameters. Suppose the risk neutral agent has access to a credit market where he can lend and borrow at a constant rate of interest equal to $r$ and hence his discount factor $\alpha$ is determined by $\alpha \equiv \frac{1}{1+r}$. Then Propositions 2 and 4 imply that, depending on the monitoring cost $\gamma$, small changes in interest rates can have profound effects on the long-run distributions of the agents' expected utilities. This, in turn, suggests that if dynamic CSV is to be taken seriously for studying distributions in the long run, then it is important that it be imbedded in a general equilibrium framework that endogenizes the interest rate. In addition, Proposition 4 indicates that models that use dynamic CSV as a vehicle for studying macroeconomic quantities should also treat risk aversion carefully.

One way to interpret monitoring is to think of it as an essential feature of bank loans (intermediated lending). This interpretation implies that Propositions 4 and 5 may be useful for thinking about the firm's choice between intermediated and un-intermediated lendings. For example, Proposition 4 implies that, depending on the monitoring cost, a small increase in interest rates can make borrowers to switch from borrowing directly from the credit market to intermediated bank loans.

A second way to think about Proposition 4 is to suppose $\alpha = \beta = \eta$, and think of $\eta$ as measuring the length of the credit relationship. Thus Proposition 4 asserts, in a sense, that a "shorter" credit relationship strengthens the case for monitoring. This prediction of our model seems consistent with the empirical evidence that bank loans have considerably shorter maturity than either private or public debt.

---

7 In practice, some business enterprises seek financing from financial intermediaries while others borrow directly from the credit market (e.g., commercial paper, corporate bond). A key distinction between the two financing mechanisms is that financial intermediaries often engage in extensive monitoring during the process of financing, whereas typical individual lenders do not monitor, or do so much less. A theoretical explanation for this distinction is that monitoring of private information is more efficient when it is delegated to a financial intermediary rather than when done repetitively by individual lenders (Diamond 1984). The idea that banks are delegated monitors is central to the models of financial intermediation based on costly state verification (e.g., Williamson 1986, 1987). Recent studies on the choice of the optimal financing mechanism by Diamond (1991) and Holmstrom and Tirole (1997) have also taken seriously the notion that bank financing is closely related to monitoring. In both papers, financial intermediaries are modeled as monitors who can detect the borrower's choosing a bad project.

8 Does this have anything to say with respect to the financial "disintermediation" that occurred in the U.S. during the early 90's when interest rates were historically low?

9 Consider the following interpretation of the common discount factor $\eta$. Suppose the relationship between the two agents terminates exogenously with probability $(1-\eta)$ at the end of each period. That is, suppose $\eta$ is the probability with which the lending and borrowing relationship will survive into the next period. We can then compute the expected length of the contract as follows $L \equiv \sum_{t=1}^{\infty} t \eta^{t-1} (1-\eta) = \frac{1}{1-\eta}$. Obviously then $L$ increases as $\eta$ increases.

10 James (1987) reports that the longest bank loan is 12 years, less than the median
Proposition 4 also implies that the optimal contract without monitoring is more efficient relative to the optimal contract with monitoring the more frequently the random income of the risk averse agent is realized. Thus for example our model predicts that lending to agricultural farmers, which is often associated with long lags between output realizations, is more likely to be provided by financial intermediaries (or government agencies) rather than by individual lenders directly.

Proposition 5 may be useful for thinking about a country’s financial development cycle. Assuming that a country’s economic development results in a decline in absolute risk aversion among the risk taking entrepreneurs, either because of a wealth effect, \(^{11}\) or because markets become less incomplete as the economy develops and hence entrepreneurs have more access to external risk sharing mechanisms. Our model predicts that both the very poor and very rich economies should rely more on direct financing whereas those in between should rely mainly on financial intermediation.

3.4 N Income Levels

In this section, I seek to generalize the analysis for the case of \(N = 2\) to the case of \(N \geq 3\).

Consider \(\phi(S^*)\). Let \(\theta_i \in S^*\). Notice that \(m_i\) and \(w_i\) never appear on the right hand side of the incentive constraints. Thus by the first order conditions with respect to \(m_i\) and \(w_i\), I have \(e^{-\theta_i} m_i^* = (1-\alpha) \beta \alpha w_i^*\), which, in turn, implies

\[
-\exp[-C^*(\theta_i, w)] = (1-\alpha) \beta \alpha W^*(\theta_i, w), \forall \theta_i \in S^*, \forall w. 
\]

Note that with complete information, i.e., when \(\gamma = 0\), the above equation must hold for all \(\theta_i \in \Theta\). We restate the above equation in the following proposition.

**Proposition 6** At the optimum, in the states of current income where monitoring occurs, there is no intertemporal consumption distortion between the current period and the future periods.

The intuition is the following: given that monitoring occurs in state \(\theta_i\), it is not feasible for the agent in state \(\theta_j (j \neq i)\) to report \(\theta_i\), implying that the risk averse agent’s intertemporal consumption scheme in state \(\theta_i\) need not be distorted in order to prepare for any possible “invasions”. This is one benefit that state verification provides: it reduces intertemporal consumption distortion.

\(^{11}\) Assume the CARA utility function I adopt here provides a local approximation of the entrepreneur’s utility function which displays an inverse relationship between wealth and risk aversion.
Proposition 6 is in some sense a generalization of Proposition 2 which says that full risk sharing is attained when monitoring occurs in the low income inside and outside the monitoring region.

A corollary of Proposition 6 is the following.

**Corollary 1** Suppose \( \theta_i, \theta_j \in S^* \) and \( \theta_i < \theta_j \). Then \( C^*(\theta_i, w) \leq C^*(\theta_j, w) \), \( W^*(\theta_i, w) \leq W^*(\theta_j, w) \), and, \( M^*(\theta_i, w) < M^*(\theta_j, w) \).

Remember if \( \theta_i, \theta_j \not\in S^* \) and \( \theta_i < \theta_j \), then \( W^*(\theta_i, w) \leq W^*(\theta_j, w) \) and \( M^*(\theta_i, w) < M^*(\theta_j, w) \). I now move on to compare \( C \) and \( W \) for states of income inside and outside the monitoring region.

**Proposition 7** (i) Suppose for some \( \theta_i \) and some \( w, C^*(\theta_i, w) < C^*(\theta_j, w) \) for all \( j \neq i \). Then it must hold that \( \theta_i \not\in S^* \). (ii) Suppose for some \( \theta_i \) and some \( w, W^*(\theta_i, w) < W^*(\theta_j, w) \), \( \forall j \neq i \). Then \( \theta_i \not\in S^* \).

By Proposition 7 then, if \( C^*(\theta_i, w) = \min\{C^*(\theta_j, w) : \theta_j \not\in S^* \} \), then it also holds that \( C^*(\theta_i, w) = \min\{C^*(\theta_j, w) : \theta_j \in \Theta \} \). Thus, the lowest consumption of the risk averse agent always occurs outside the verification region. Similarly, suppose \( W^*(\theta_i, w) = \min\{C^*(\theta_j, w) : \theta_j \not\in S^* \} \). Then, given that \( W^*(\theta_i, w) < W^*(\theta_j, w) \) if \( \theta_i < \theta_j \) and \( \theta_j \not\in S^* \), I have: if \( \theta_i = \min\{\theta_j : \theta_j \not\in S^* \} \), then \( W^*(\theta_i, w) = \min\{W^*(\theta_j, w) : \theta_j \in \Theta \} \). So the risk averse agent’s lowest future utility occurs in the lowest state outside the verification region. A corollary of Proposition 7 is:

**Corollary 2** Suppose \( S^* = \{\theta_1, ..., \theta_k \} \subset \Theta \). Then

\[
\min\{C^*(\theta_j, w), j > k\} \leq C^*(\theta_1, w) \leq ... \leq C^*(\theta_k, w),
\]

\[
W^*(\theta_{k+1}, w) \leq W^*(\theta_1, w) \leq ... \leq W^*(\theta_k, w).
\]

That is, suppose the optimal monitoring strategy is “monotonic”. Then it is necessary that the optimal compensation schemes are not.

Let me use the case of \( N = 3 \) to offer some intuition for Proposition 7 and Corollary 2. Suppose \( S^* = \{\theta_1\} \). Then it must hold that \( W^*(\theta_2, w) < W^*(\theta_3, w) \). Suppose the optimal contract is such that \( W^*(\theta_1, w) < W^*(\theta_2, w) \), so the proposition is violated. Then increase \( W^*(\theta_1, w) \) by \( \delta > 0 \). This will only create more incentives for the agent in states \( \theta_2 \) and \( \theta_3 \) to report \( \theta_1 \), but since monitoring occurs in state \( \theta_1 \), the contract remains incentive compatible. Next, reduce both \( W^*(\theta_2, w) \) and \( W^*(\theta_3, w) \) by \( \Delta > 0 \). Again this provides more incentives for the agent in state \( \theta_2 \) or \( \theta_3 \) to report \( \theta_1 \) if monitoring were not to occur in state \( \theta_1 \). Since \( W^*(\theta_2, w) \) and \( W^*(\theta_3, w) \) are reduced by the same amount, overall there are no more or less incentives for the agent in \( \theta_2(\theta_3) \) to report \( \theta_3(\theta_2) \). In other words, the contract remains incentive compatible. Choose \( \delta \) and \( \Delta \) to satisfy the promise keeping constraint. Then the value of the risk neutral agent is increased because
the value function is strictly concave, a contradiction. The same intuition
can be employed to explain the non-monotonicity of the risk averse agent’s
consumption scheme.

A result parallel to Proposition 7 (i) and the first part of Corollary 2 can
be obtained for the case of static CSV.

Proposition 7 and Corollary 1 show that the nice monotonicity result
with respect to the optimal compensation scheme in the case of \( N = 2 \)
cannot be generalized. Proposition 7 and Corollary 1 can be compared to
Thomas and Worrall (1990) where at least the risk averse agent’s future
utility is strictly monotonic in income. Not surprisingly, monitoring complicates
the structure of the optimal contract: a higher income in the current
period does not necessarily make the risk averse agent “richer” from next
period on, it depends on whether the agent is monitored.

I now turn to examine the structure of the optimal monitoring strategy.
How does the optimal monitoring strategy depend on the model’s parameter
values? I begin by providing a linkage between \( S^* \) and \( \gamma \), holding other
aspects of the model constant. I use \( S^*(\gamma) \) to denote the optimal monitoring
strategy when the cost of monitoring is \( \gamma \).

**Proposition 8** Suppose \( S^*(\gamma_1) = S_1 \) and \( S^*(\gamma_2) = S_2 \). Then \( \mu(S_1) > \mu(S_2) \) implies \( \gamma_1 \leq \gamma_2 \), and \( \gamma_1 < \gamma_2 \) implies \( \mu(S_1) \geq \mu(S_2) \).

Proposition 8 says that a lower monitoring cost always implies more
monitoring, and vice versa, just like in the two income case. In the two
income case, the monotonicity relationship between monitoring and the cost
of monitoring is characterized by the cut-off monitoring cost \( \gamma^* \). In the \( N \)
income case, the counterpart of \( \gamma^* \) is a pair of cut-off levels \( \underline{\gamma} \) and \( \overline{\gamma} \) which
are characterized by the following propositions.

**Proposition 9** (i) Conditional on \( S = \{\theta_1, ..., \theta_{N-1}\} \), the optimal pay-
ment scheme and the law of motion of the state variable are first best. (ii)
There exists a unique \( \gamma > 0 \) such that \( S^* = \{\theta_1, ..., \theta_{N-1}\} \) and the optimal
allocation is first best if and only if \( \gamma \leq \underline{\gamma} \).

Thus other things equal, as long as the cost of monitoring is sufficiently
low, it is optimal to monitor heavily (in all income states except the highest)
and to obtain the first best risk sharing. Two corollaries follow immediately.
First, \( S^* \neq S \) if \( \mu(S) > \mu(\Theta - \{\theta_N\}) \). Second, \( S^* \neq \{\theta_1, ..., \theta_N\} \).

My next result shows that there is a critical level of the monitoring cost
\( \overline{\gamma} \) such that monitoring is too expensive to be used at all if \( \gamma > \overline{\gamma} \), but
otherwise it is sufficiently inexpensive to justify at least some monitoring.
That is, \( \overline{\gamma} \) is the highest price at which the risk neutral agent is ever willing
to pay for monitoring.

**Proposition 10** There exists \( \overline{\gamma} > 0 \) such that \( S^* = \emptyset \) if and only if
\( \gamma \geq \overline{\gamma} \). Moreover, (i) \( \overline{\gamma} \rightarrow 0 \), as \( |\theta_N - \theta_1| \rightarrow 0 \), and (ii) \( \overline{\gamma} \rightarrow 0 \), as \( \alpha \rightarrow 1 \).
Proposition 10(i) is an extension of the result in the case of $N = 2$ where $\gamma^*$ is an increasing function of $(\theta_2 - \theta_1)$. Proposition 10(ii) is a partial generalization of Proposition 4(i).

I now move on to consider the monotonicity property of monitoring with respect to income. Given Proposition 2, one might speculate that it is more efficient to monitor in lower rather than higher states of the risk averse agent’s current income. That is, either $S^* = \emptyset$, or $S^* = \{\theta_1, \ldots, \theta_n\}$ for some $\theta_n < \theta_N$. Our next proposition provides exactly such a result, but only under the assumption that the risk averse agent’s random income is uniformly distributed, i.e., $\pi_1 = \pi_2 = \cdots = \pi_N$.

**Proposition 11** Suppose that the risk averse agent’s random income $\theta^t$ is uniformly distributed on $\Theta$, then either $S^* = \emptyset$, or $S^* = \{\theta_1, \ldots, \theta_n\}$ for some $n < N$.

The intuition is that it requires more consumption distortion to prevent the risk averse agent from reporting untruthfully a lower rather than a higher income state in which monitoring does not occur. Now the assumption that $\pi_i$ is constant in $i$ is important. What it does is to hold the cost of monitoring constant while we evaluate the benefits associated with monitoring in one state relative to another.

Proposition 11 provides one sufficient condition for the optimal monitoring policy to be optimal. Obviously, this is a very incomplete result. What if the assumption of Proposition 11 is violated and, hence, when we evaluate the benefits of monitoring in one state relative to another, we must also evaluate the cost of monitoring in that state relative to the other? In other words, is the uniform distribution assumption necessary for the optimal monitoring strategy to be monotonic? I use a numerical example to provide an answer to these questions.

Consider the following parameterization of the model: $N = 3$, $\theta_1 = 1.0$, $\theta_2 = 2.0$, $\theta_3 = 3.0$, $\pi_1 = 0.5$, $\pi_2 = 0.2$, $\pi_3 = 0.3$, and $\alpha = \beta = 0.8$. In this example, there are three cut-off levels of the monitoring cost, $\gamma_1$, $\gamma_2$, and $\gamma_3$ such that for $0 \leq \gamma \leq \gamma_1$, $S^* = \{\theta_1, \theta_2\}$. For $\gamma_1 \leq \gamma \leq \gamma_2$, $S^* = \{\theta_1\}$. For $\gamma_2 \leq \gamma \leq \gamma_3$, $S^* = \{\theta_2\}$. And for $\gamma \geq \gamma_3$, $S^* = \emptyset$. In this example, for $\gamma_1 \leq \gamma \leq \gamma_2$, $C^*(\theta_2, w) < C^*(\theta_1, w) < C^*(\theta_3, w)$, and $W^*(\theta_2, w) > W^*(\theta_1, w) < W^*(\theta_3, w)$. (See Wang (2000) for more details for this example.)

In this example, neither the consumption nor the expected utility of the risk averse agent is monotonic in the risk averse agent’s current income. Second, it can be optimal to monitor a higher income state ($\theta_2$) but not a lower income state ($\theta_1$). Third, the uniform distribution function assumption is not necessary for obtaining a monotonic monitoring policy. In this example, $\gamma$ is an important parameter for determining whether the monitoring region is monotonic.

My main results in Section 3.3 are about the relationship between the
optimal monitoring strategy and two of the model's important parameters: \( \alpha \) and \( \rho \). Unfortunately, I am not able to obtain parallel analytical results for the case of \( N \) income levels. The difficulty lies in the complexity of the optimal monitoring strategy and compensation rules. So instead, I will use the current numerical example to show that it is at least hopeful that the comparative statics results can be extended to the case of \( N \) income levels.

In the same numerical example, hold \( \beta \) constant at 0.8 but vary the value \( \alpha \) to compute \( \Upsilon \) as a function of \( \alpha \). This function is shown in Figure 3 where \( \Upsilon \) decreases as \( \alpha \) increases. Remember in the two income case, we showed analytically \( \gamma^* \) is a decreasing function of \( \alpha \).

The non-monotonicity relationship between monitoring and risk aversion in the two income case also holds in the three income case. Figure 4 shows \( \Upsilon \) as a function of \( \rho \), the risk averse agent’s coefficient of absolute risk aversion. Compare Figure 4 with Figure 1.

4 Non-CARA Utility Functions

In this section, I study the case of non-CARA utility functions. The complexity of the case of CARA with \( N \) income levels suggests that the analysis of the non-CARA case will be even more difficult. Here, I will not attempt to provide a full set of characterizations for the optimal contract. Instead, I will focus more or less on just one key feature of the optimal contract that Proposition 5 suggests: the optimal monitoring policy is now a function of the risk averse agent’s expected utility. This feature of the optimal contract may lead to interesting life-cycle effects.

Assume the risk averse agent’s utility function \( v : R_+ \to R \) is strictly increasing and concave in \( c \). I also assume that the risk neutral agent faces the following period-by-period limited-liability/budget-balancing constraint:

\[
-\theta_0 \leq M_t((h^{t-1}, \theta_i)) \leq \theta_i, \quad \forall t, \theta_i, h^{t-1},
\]

where \( \theta_0 \geq 0 \) is the maximum resource to which the risk neutral agent has access in each period. If \( \theta_0 = \infty \) (this is equivalent to assuming that the risk neutral agent is never constrained or that he does not have to face a budget-balancing constraint), then the above constraint is simply a non-negativity constraint on the risk averse agent’s consumption. In this section, I will assume \( \alpha = \beta \) for simplicity.

A contract \( \sigma \) is feasible if, for all \( t \geq 1 \), all \( h^{t-1} \in H^{t-1} \), and all \( \theta_i \), the constraints \( S_t(h^{t-1}) \subseteq \Theta \) and (17) are satisfied. Obviously, any expected utility of the risk averse agent strictly below \( \underline{w} \) or above \( \overline{w} \) is not attainable by a feasible contract, where

\[
\underline{w} = \frac{1}{1-\beta} v(0), \quad \overline{w} = \frac{1}{1-\beta} \sum_i \pi_i v(\theta_0 + \theta_i).
\]
It is also useful to define $w_*$ as the expected utility of the risk averse agent if he makes a constant payment $\theta_1$ to the risk neutral agent each period, i.e.,

$$w_* = \frac{1}{1 - \beta} \sum_i \pi_i v(\theta_i - \theta_1) \in (\underline{w}, \overline{w}).$$

Clearly, $\underline{w} < w_* < \overline{w}$.

### 4.1 No State Verification: $\gamma = \infty$

Suppose $\gamma = \infty$. That is, suppose state verification is infeasible.

**Lemma 1**

(i) A feasible and incentive compatible contract that promises the risk averse agent expected utility $w$ exists if and only if $w \in [w_*, \overline{w}]$. (ii) An optimal contract that promises expected utility $w$ to the risk averse agent exists if and only if $w \in [w_*, \overline{w}]$. (iii) The contract where $M_t(h^t) = c$, for all $t$ and $h^t$ and some $c \in (-\theta_0, \theta_1]$, is not optimal.

So if a feasible and incentive compatible contract exists, then an optimal contract also exists, and, as long as $\beta > 0$, it is not a simple rental contract (in which the risk averse agent makes a constant payment to the risk neutral agent). This result will be used in the proof of Proposition 13.

Let $\{M_{NV}(\theta_i, w), W_{NV}(\theta_i, w), \theta_i \in \Theta, w \in [w_*, \overline{w}]\}$ denote the optimal contract under $\gamma = \infty$.

### 4.2 Costly State Verification: $\gamma \in [0, \infty)$

**Lemma 2**

(i) A feasible and incentive compatible contract that promises the risk averse agent expected utility $w$ exists if and only if $w \in [\underline{w}, \overline{w}]$. (ii) An optimal contract that promises expected utility $w$ to the risk averse agent exists if and only if $w \in [\underline{w}, \overline{w}]$. (iii) It is never optimal to have $S_t = \emptyset$ and $M_t = c$, for all $t$ and some $c \in (-\theta_0, \theta_1]$.

Lemmas 1 and 2 show that monitoring expands the set of attainable expected utilities. Specifically, suppose $w \in [\underline{w}, w^*)$. Then $w$ is attainable under $\gamma < \infty$ but not under $\gamma = \infty$. This implies that the availability of monitoring makes it possible for the risk neutral agent to use more effectively the risk averse agent's future utility as an incentive device, leading potentially to efficiency gains.

---

Note that if $\beta = 0$, i.e., in the case of static contracting, then for all $w \in [w_*, \overline{w}]$, the rental contract is the only feasible and incentive compatible contract. Now obviously if $c = -\theta_0$, then the contract where $M_t = -\theta_0$ is optimal. In fact, it is the only feasible and incentive compatible contract that delivers expected utility $\overline{w}$ to the risk averse agent. Also note that if $\theta_0 > 0$, then a corollary of Lemma 1 is that as long as $\beta > 0$, the autarky contract where $M_t(h^t) = 0$, for all $h^t$, is never efficient, no matter how small $\beta$ is.
Let \( U^*: [\underline{w}, \overline{w}] \rightarrow R \) be the risk neutral agent’s value function. Then \( U^* \) and the optimal contract \( \{ S^*(w), M^*(\theta, w), W^*(\theta, w), \theta \in \Theta, w \in [\underline{w}, \overline{w}] \} \) can be characterized by the following functional equation:

\[
U^*(w) = \max_{\{ S(w), M(\theta, w), W(\theta, w) \}} \left[ \sum_{i=1}^{N} \pi_i \{ M(\theta_i, w) + \beta U^*[W(\theta_i, w)] \} - \mu(S(w)) \gamma \right] \quad (18)
\]

subject to \( S(w) \subseteq \Theta, (5), (6) \), and

\[
M(\theta_i, w) \in [-\theta_0, \theta_i], \quad W(\theta_i, w) \in [\underline{w}, \overline{w}], \quad \forall \theta_i \in \Theta. \quad (19)
\]

In the special case of complete information where \( \gamma = 0, 1 \) have \( S^*(w) = \Theta, M^*(\theta_i, w) = M_{FB}(\theta_i, w), \) and \( W^*(\theta_i, w) = W_{FB}(\theta_i, w) \) where \( \{ M_{FB}(\theta, w), \theta \in \Theta \} \) maximizes \( \sum \pi_i M(\theta_i, w) \) subject to \( M(\theta_i, w) \in [-\theta_0, \theta_i], \) and \( \sum \pi_i e[\theta_i - M(\theta_i, w)] = (1 - \beta) w \), and \( W_{FB}(\theta_i, w) = w \). Let \( U_{FB}(w), w \in [\underline{w}, \overline{w}], \) denote the risk neutral agent’s value function in this case. Obviously, for all \( \gamma > 0, U_{NV}(w) \leq U^*(w) \leq U_{FB}(w), \forall w \in [\underline{w}, \overline{w}], \) and \( U^*(w) \leq U_{FB}(w), \forall w \in [\underline{w}, w_*]. \)

**Proposition 12** Assume \( \underline{w} > -\infty \), then there exists \( w_1 \in (\underline{w}, \overline{w}) \) such that \( \{ \theta_1, \ldots, \theta_{N-1} \} \subseteq S^*(w) \) if \( w \leq w_1 \).

Suppose \( \underline{w} > -\infty \). Suppose some state \( \theta_i < \theta_N \) is not monitored. Then when \( w \) gets sufficiently close to \( \underline{w} \), the promise-keeping constraint is necessarily violated, for the agent can report \( \theta_i > \theta_N \) in state \( \theta_N \) to achieve an expected utility which is higher than \( w \).

Thus monitoring in all income states except the highest is optimal when \( w \) is sufficiently low. Clearly, \( [\underline{w}, w_1] \) is a set of absorbing states of the risk averse agent’s expected utilities. 13

Note that the proof of Proposition 12 depends critically on the assumption \( \underline{w} > -\infty \), that is, the risk averse agent’s utility is bounded from below. Suppose the risk averse agent’s utility is not bounded from below. Then our numerical example in Section 4.3 shows that the opposite may be optimal: monitoring will not occur for \( w \) sufficiently low.

Next, I look at the other end of the set of the risk averse agent’s expected utilities. I show that there is a well specified sufficient condition under which monitoring is not optimal if \( w \) is sufficiently high. Let \( w \in [w_*, \overline{w}] \). (If \( \overline{w} = \infty \), then let \( w \in [w_*, \overline{w}] \).) Define \( \overline{M}(w) \in (-\infty, \theta_1) \) by

\[
(1 - \beta) w = \sum \pi_i e[\theta_i - \overline{M}(w)]
\]

Given \( w \in [w_*, \overline{w}] \), \( \overline{M}(w) \) exists. Next, for all \( S \subseteq \Theta \) and all \( w \in [\underline{w}, \overline{w}] \), let

\[
U^*_S(w) \equiv \max_{\{ M(\theta_i, w), W(\theta_i, w) \}} \sum_{i=1}^{N} \pi_i \{ M(\theta_i, w) + \beta U^*[W(\theta_i, w)] \}
\]

\footnote{This result depends on the assumption \( \alpha = \beta \). If \( \alpha \neq \beta \), then Proposition 12 still holds, but \( \overline{w} \) will be the only absorbing state in the set of \( [\underline{w}, w_1] \).}
subject to (19), and (5), (6). That is, \( U^*_R(w) \) is the value of the contract to the risk neutral agent if he adopts the following strategy: the monitoring region is \( S \subseteq \Theta \) in the current period, and optimally determined in all future periods. Obviously then

\[
U^*_R(w) > \overline{M}(w)/(1 - \beta),
\]

for \( U^*_R(w) \geq U^*_N(w) > \overline{M}(w)/(1 - \beta), \) where the last inequality follows from (iii) of Lemma 1.

Now \( S^*(w) = \emptyset \) if and only if

\[
U^*_S(w) - \mu(S) \gamma \leq U^*_R(w), \forall S \subseteq \Theta, S \neq \emptyset.
\]

But

\[
U^*_S(w) - \mu(S) \gamma \leq U^*_FB(w) - \min\{\pi_i\} \gamma, \forall S \subseteq \Theta, S \neq \emptyset,
\]

for \( U^*_S(w) \leq U^*_FB(w) \) and \( \mu(S) \gamma \geq \min\{\pi_i\} \gamma \). Therefore \( S^*(w) = \emptyset \) holds if the following holds:

\[
U^*_FB(w) - \min\{\pi_i\} \gamma \leq \overline{M}(w)/(1 - \beta).
\]

Now \((1 - \beta)U^*_FB(w) = \sum \pi_i M^*_FB(\theta_i, w). \) So \( S^*(w) = \emptyset \) if

\[
R(w) \equiv \sum \pi_i M^*_FB(\theta_i, w) - \overline{M}(w) \leq (1 - \beta) \min\{\pi_i\} \gamma, \tag{20}
\]

where \( R(w) = \sum \pi_i [\theta_i - \overline{M}(w)] - \sum \pi_i [\theta_i - M^*_FB(\theta_i, w)] \) is the risk premium that will make the risk averse agent indifferent between the simple fixed payment contract and the first best contract. This leads to the following result.

**Proposition 13** Suppose \( \theta_0 = \infty \). Suppose the risk averse agent’s utility function \( v(\cdot) \) is such that \( R(w) \to \overline{M} \), as \( w \to \overline{w} \), where \( \overline{M} \) is a constant and \( \overline{M} < (1 - \beta) \min\{\pi_i\} \gamma \). Then there exists \( w_2 \in (w, \overline{w}) \) such that \( S^*(w) = \emptyset \) for all \( w \geq w_2 \).

A special case occurs when \( R(w) \to 0 \), as \( w \to \overline{w} \). Then for any given monitoring cost \( \gamma \), the risk averse agent is not monitored as long as his expected utility is above some cutoff level.

Now suppose the assumptions of Propositions 12 and 13 are satisfied. Then the dynamics of the optimal contract is such that the risk averse agent will be fully monitored if he becomes sufficiently poor, he will not be monitored if he becomes sufficiently rich.

A corollary of the above analysis is:

**Corollary 3** Let \( w \in [w_s, \overline{w}] \). There exists \( \gamma \) such that if \( \gamma > \gamma(w) \), then \( S^*(w) = \emptyset \).

20
I now introduce a result that is stronger than the above corollary and generalizes the result of Proposition 10.

For expositional purposes, in the following of this subsection we will write an optimal contract as $\sigma^*(\gamma) = \{S^*(w)(\gamma), M^*(\theta, w)(\gamma), W^*(\theta, w)(\gamma), \theta \in \Theta, w \in [\underline{w}, \overline{w}]\}$, to emphasize that the cost of verification is $\gamma$.

**Proposition 14** There exists $\gamma > 0$ such that if $\gamma > \gamma$, then (a) $S^*(w)(\gamma) = \emptyset$, for all $w \in [\underline{w}, \overline{w}]$, and (b) $U^*(w)(\gamma) = U_{NV}(w)$, $M^*(\theta, w)(\gamma) = M_{NV}(\theta, w)$, and $W^*(\theta, w)(\gamma) = W_{NV}(\theta, w)$, for all $w \in [\underline{w}, \overline{w}]$ and all $\theta \in \Theta$.

Thus if $\gamma \geq \gamma$ and the risk averse agent’s initial expected utility is in $[\underline{w}, \overline{w}]$, then no verification will take place along all possible paths of the expected utility of the risk averse agent that are dictated by the optimal contract. Finally, as in the case of CARA (Proposition 9), I can show that as the cost of verification goes to zero, the efficient compensation scheme converges to the first best.

**Proposition 15** Let $w \in [\underline{w}, \overline{w}]$ and $\theta \in \Theta$. Then $U^*(w)(\gamma) \to U_{FB}(w)$, $M^*(\theta, w)(\gamma) \to M_{FB}(\theta, w)$, and $W^*(\theta, w)(\gamma) \to W_{FB}(\theta, w)$, as $\gamma \to 0$.

4.3 A Numerical Example: log utility

Propositions 12 and 13 do not deal with the case of $\underline{w} = -\infty$. In this subsection I consider numerically the case where the risk averse agent’s period utility function takes the form $v(e) = \log(e)$. This utility function exhibits decreasing absolute risk aversion. Specifically, the degree of absolute risk aversion goes to infinity as $e$ goes to zero, and it goes to zero as $e$ goes to infinity. Now according to Proposition 13, for any given $\gamma > 0$, $S^*(\gamma) = \emptyset$ if $w$ is sufficiently high. And, according to the insight provided by Proposition 5 and Figures 1 and 3, at least for some $\gamma > 0$, $S^*(w) = \emptyset$ for $w$ sufficiently small. Note that in the case of $\log$, $\underline{w} = -\infty$, so the assumption of Proposition 12 is not satisfied.

I compute the optimal contract for the following parameterization: $N = 2$, $\theta_1 = 0.75$, $\theta_2 = 1.25$; $\pi_1 = \pi_2 = 0.5$; $\alpha = \beta = 0.95$. The optimal contract is shown in Figures 4 to 6. In all figures here the horizontal axis represents $w$, the risk averse agent’s expected utility. Figure (4) shows the optimal monitoring policy, where $1^\theta$ indicates $S^*(w) = \{\theta_1\}$ and $0^\theta$ indicates $S^*(w) = \emptyset$. Figure (5) shows the optimal law of motion for the risk averse agent’s expected utility. Here the vertical axis represents the risk averse agent’s expected utility next period. Figure (6) shows the optimal payment schemes.

In Figure 4, there are two cut-off levels of $w$, $w^*_1$ and $w^*_2$. The risk averse agent is not monitored either when he is too rich ($w$ above $w^*_2$) or when he is too poor ($w$ below $w^*_1$). This is consistent with the insight of Proposition 5: intertemporal incentives are relatively more efficient than monitoring when the risk averse agent’s degree of risk aversion is sufficiently
The limiting distribution of the risk averse agent’s expected utility is degenerate: either it converges to a constant in the interval \((w_1^*, w_2^*)\), where monitoring takes place whenever \(\theta_1\) is reported and the optimal payment scheme is first best, or it diverges to minus infinity, depending on the risk averse agent’s initial expected utility, \(w_0\). Specifically, if \(w_0\) is higher than \(w_2^*\), then after some transient dynamics the risk averse agent’s expected utility will converge to a constant in the interval \((w_1^*, w_2^*)\) and stay there. If \(w_0\) is between \(w_1^*\) and \(w_2^*\), then the risk averse agent’s expected utility never moves and stays at its initial level. Finally, if \(w_0\) is lower than \(w_1^*\), then with a positive probability the risk averse agent’s expected utility converges to a constant in the interval \((w_1^*, w_2^*)\), and otherwise it drifts stochastically toward minus infinity. Finally, suppose in the model monitoring is not feasible. Then the risk averse agent’s expected utility goes to minus infinity with probability one, just like in many of the existing models of dynamic contracting without CSV. CSV makes an important and interesting difference here.

5 Concluding Remarks

In this paper, I have studied a problem of dynamic risk sharing with costly state verification. The set of characterizations that I can provide for the optimal contract, especially the optimal monitoring policy, is far from being complete. In particular, some of the characterizations I obtained for the CARA case with two income levels (Propositions 4, 5) seem intuitive and interesting, but they are hard to be fully generalized. One reason is that monitoring causes non-monotonicity in the optimal contract that makes analysis difficult. Finally, some of the conclusions I have obtained suggest that it is important to imbed the current model in a dynamic general equilibrium framework if one is interested in, for instance, evaluating the long run implications of dynamic CSV.
6 Appendix

Proof of Proposition 1 Let $U^*(w) = \frac{1}{1-\alpha} \log(-w) + \frac{1}{\alpha} [\phi(S^*) - \mu(S^*)\gamma]$. Let $U(w) = \frac{1}{1-\alpha} m(w), \text{where} \ m(w) \text{ solves} \ \frac{1}{1-\alpha} \sum_{i=1}^{N} \pi_i [-\exp(-\theta_i + m(w))] = w. \text{That is,} \ U(w) \text{ is the expected lifetime utility of the risk neutral agent if the contract is such that he receives a fixed payment} \ m(w) \text{ every period which promises the risk averse agent the lifetime expected utility of} \ w: \ \frac{1}{1-\alpha} \log(-w) - \frac{1}{\alpha} \log(\frac{1}{1-\alpha} \pi_i [\exp(-\theta_i)] + \frac{1}{\alpha} \log(1 - \beta).

Next, let $U(w) \equiv U^{bf}(w), \ \forall w \in (-\infty, 0)$, where $U^{bf}(w)$ is the risk neutral agent's value function associated with the first best allocation: $U^{bf}(w) = \frac{1}{1-\alpha} \left[ \log(-w) + \log(1 - \alpha) + \bar{\theta} + \frac{\alpha}{1-\alpha} \log(\frac{a}{\beta}) \right]$. Obviously, for all $w$, $U(w) \leq U^*(w) \leq U(w)$. It is straightforward to show that the operator $\Gamma$ is monotone in the sense that $U_1 \leq U_2$ implies $\Gamma(U_1) \leq \Gamma(U_2)$. We therefore have: $\Gamma^n(U^*) \leq U^*(w) \leq \Gamma^n(U)(w), \ \forall n \geq 1$. Thus to show $U^*(w) = U^*(w)$ we need only show that $\lim_{n \to \infty} \Gamma^n(U^*) \equiv \lim_{n \to \infty} \Gamma^n(U)(w) = U^*(w)$. But to show the above, it is sufficient to show that for any function $U(w) = \frac{1}{1-\alpha} \log(-w) + B$, where $B$ is any constant, the following holds: $\Gamma^n(U)(w) \to U^*(w)$, as $n \to \infty$. Make the following transformation: $\exp[M(\theta_1, w)] \equiv m(-w), W(\theta_1, w) \equiv w$. Then it is easy to obtain $\Gamma^1(U)(w) = \log(-w)/(1 - \alpha) + [\phi(S^*) - \mu(S^*)\gamma] + \alpha B$. And for all $n \geq 2$, $\Gamma^n(U)(w) = \log(-w)/(1 - \alpha) + (1 + \alpha + \ldots + \alpha^{n-1}) [\phi(S^*) - \mu(S^*)\gamma] + \alpha^n B$, which in turn converges to $\log(-w)/(1 - \alpha) + [\phi(S^*) - \mu(S^*)\gamma]$, as $n \to \infty$. Thus we have shown that $U^*(w) = U^*(w)$.

The rest of the proposition follows trivially.

Proof of Proposition 2 The Bellman equation is now written:

$$U^*(w) = \max \{U_0(w), U_1(w), U_2(w), U_1,2(w)\},$$

where

$$(P0) \ U_0(w) = \max_{\{M(\theta, w), W(\theta, w)\}_{i=1,2}} \sum_{i=1}^{2} \pi_i[M(\theta_i, w) + \alpha U^*(W(\theta_i, w))],$$

subject to

$$-e^{-\dot{\theta}_1 + M(\theta_1, w)} + \beta W(\theta_1, w) \geq -e^{-\dot{\theta}_1 + M(\theta_2, w)} + \beta W(\theta_2, w), \quad (A.1)$$

$$-e^{-\dot{\theta}_2 + M(\theta_2, w)} + \beta W(\theta_2, w) \geq -e^{-\dot{\theta}_2 + M(\theta_1, w)} + \beta W(\theta_1, w), \quad (A.2)$$

$$\sum_{i=1}^{2} \pi_i[-e^{-\dot{\theta}_i + M(\theta_i, w)} + \beta W(\theta_i, w)] = w. \quad (A.3)$$

$$(P1) \ U_1(w) = \max \sum_{i=1}^{2} \pi_i[M(\theta_i, w) + \alpha U^*(W(\theta_i, w))] - \pi_1 \gamma, \text{subject to} (A.1), (A.3).$$

$$(P2) \ U_2(w) = \max \sum_{i=1}^{2} \pi_i[M(\theta_i, w) + \alpha U^*(W(\theta_i, w))] - \pi_2 \gamma, \text{subject to} (A.2), (A.3).$$
\[ (P12) \quad U_{12}(w) = \max \sum \pi_i [M(\theta_i, w) + \alpha U^*(W(\theta_i, w))] - \gamma, \text{ subject to (A.3)}. \]

Note that by Proposition 1, \( U^*(w) = \frac{1}{1-\alpha} \log(-w) + B, \) where \( B \) is some constant.

We need only show that both \( U_2(w) \) and \( U_{12} \) are dominated either by \( U_0(w) \) or by \( U_1(w) \), and that in the problem \((P0)\), the first incentive constraint is not binding while the second is. The remainder of the proof shows this, and is divided into 2 steps, (A) and (B).

Step (A). We show that the incentive constraint is not binding in problem \((P1)\). Suppose we ignore the incentive constraint and solve the problem. Then we have:

\[-\theta_1 + M(\theta_1, w) = -\theta_2 + M(\theta_2, w), \quad W(\theta_1, w) = W(\theta_2, w). \quad (A.4)\]

Clearly the incentive constraint is satisfied given the above.

Step (B). Consider problem \((P0)\). First, the two incentive constraints imply:

\[ M(\theta_2, w) \geq M(\theta_1, w), \quad W(\theta_2, w) \geq W(\theta_1, w). \]

Second, we show that at least one of the two incentive constraints is binding. Suppose not, then the solution to problem \((P0)\) is such that \((A.4)\) holds, leading to a violation of constraint \((A.2)\). Third, we show that only one of the two incentive constraints binds. Rewrite the two incentive constraints as:

\[-e^{-\theta_2}[e^{M(\theta_2, w)} - e^{M(\theta_1, w)}] \leq \beta [W(\theta_2, w) - W(\theta_1, w)] \leq -e^{-\theta_1}[e^{M(\theta_2, w)} - e^{M(\theta_1, w)}] \]

where the first inequality corresponds to constraint \((A.2)\) and the second corresponds to constraint \((A.1)\). Now given \( M(\theta_2, w) \geq M(\theta_1, w) \) and \( W(\theta_2, w) \geq W(\theta_1, w), \) it is obvious that if \((A.2)\) holds with equality then \((A.1)\) is satisfied, and if \((A.1)\) holds with equality then \((A.2)\) is satisfied. Fourth, we show that \((A.2)\) holds with equality. Suppose not. Then constraint \((A.1)\) binds but \((A.2)\) does not, contradicting our result in Step (A).

By Step (A), \( U_{12}(w) < U_1(w), \forall w. \) By Step (B), \( U_2(w) < U_0(w), \forall w. \) Thus the proposition is proven.

**Proof of Proposition 3** We first show that \( M^*(\theta_2, w) > M^*(\theta_1, w) \), and \( W^*(\theta_2, w) > W^*(\theta_1, w) \). In fact, we need only show that \( m_2^* > m_1^* \), \( w_2^* < w_1^* \). By manipulating the incentive constraints we have \( m_2^* \geq m_1^* \), \( w_2^* \leq w_1^* \). Therefore we need only show \( m_2^* \neq m_1^* \) and \( w_2^* \neq w_1^* \). Suppose \( m_2 = m_1 \), then by lowering the value of \( m_1 \) and raising the value of \( m_2 \) the objective can be strictly improved. Similarly, \( w_2 = w_1 \) is not optimal.

Next we show that \( C^*(\theta_2, w) > C^*(\theta_1, w) \). We need only show that \( e^{-\theta_2} m_2^* < e^{-\theta_1} m_1^* \), for \( C^*(\theta_i, \cdot) = \theta_i - \log(m_i^*) - \log(-w). \) Now \( \phi(\theta) = \sum \pi_i [\log m_i + \frac{\alpha}{1-\alpha} \log w_i] \text{ subject to } \exp(-\theta_2)m_1 + \beta w_1 = \exp(-\theta_2)m_2 + \beta w_2 \text{ and } \sum \pi_i[\exp(-\theta_1)m_i + \beta w_i] = 1. \) We then have the following first
order conditions: \( \frac{\pi_1}{e^{\mu_m^1}} = -\lambda e^{-(\delta_2 - \delta_1)} - \pi_1 \mu, \) \( \frac{\pi_2}{e^{\mu_m^2}} = \lambda - \pi_2 \mu, \) and
\( \frac{\lambda e^{-(\delta_2 - \delta_1)} + \lambda}{\pi_1^2} \) for \( \phi = 1, 2 \) where \( \lambda > 0. \) We thus have \( \frac{1}{e^{\mu_m^1}} - \frac{1}{e^{\mu_m^2}} = \frac{\lambda e^{-(\delta_2 - \delta_1)} + \lambda}{\pi_1^2} > 0, \) and thus \( e^{-(\delta_2 - \delta_1)} < e^{-\delta_1} m_1^* \). Notice the above first order conditions also implies \( \mu > 0. \)

Next, we show that equations (10) and (11) hold. (11) holds trivially, so we need only show (10), or \( \exp[-\theta_1 + M^*()] \geq (1 - \alpha) \hat{\delta} W^*(), e, w, \) or \( e^{-(\delta_1} m_1^* < (1 - \alpha) \hat{\delta} w_1^*, \) or \( \frac{1}{\lambda - \mu_2 - \pi_1 \mu} < \frac{1}{\lambda - \pi_1 \mu}, \) or \( e^{-(\delta_2 - \delta_1)} < 1, \) which clearly holds.

Third, to show that (12) holds, one simply needs to substitute (10) and (11) into (9), the promise keeping constraint. This completes the proof.

**Proof of Proposition 4** To prepare for the proof, let \( \hat{\theta}_1 \equiv \exp(-\theta_1) m_i, \) \( \hat{w}_1 \equiv \beta w_1, \) \( \hat{\alpha} \equiv \alpha + \frac{\alpha}{1 - \alpha}, \) \( \hat{e} \equiv \exp(-\theta_2 + \theta_1), \) Then \( \phi(\hat{\theta}) = \phi(\hat{\theta}) + \hat{\theta} - \hat{\alpha} \log(\beta), \) where \( \phi(\hat{\theta}) \equiv \max_{\hat{m}_i, \hat{w}_1, i=1,2} \sum \pi_i \log \hat{m}_i + \frac{\hat{\alpha}}{\alpha} \log \hat{w}_1 \) subject to \( \hat{m}_i > 0, \hat{w}_1 > 0, \) \( \hat{m}_2 + \hat{w}_2 = \exp(-\theta_2 + \theta_1) m_2 + \hat{w}_1, \) and \( \sum \pi_i [\hat{m}_i + \hat{w}_1] = 1. \) Note that at the optimum of the above problem, it holds that \( \hat{w}_2 = \hat{\alpha} m_2 \) and

\[
\hat{m}_1 = \frac{1 - (1 + \hat{\alpha}) \hat{m}_2^*}{(1 - \hat{e}) \pi_1}, \quad \hat{w}_1 = \frac{[\hat{e} + (1 - \hat{e}) \pi_1] (1 + \hat{\alpha}) \hat{m}_2^* - \hat{e}}{(1 - \hat{e}) \pi_1}.
\]

Similarly, \( \phi(\{\theta_1\}) = \phi(\{\theta_1\}) + \hat{\theta} - \hat{\alpha} \log(\beta), \) where \( \phi(\{\theta_1\}) \equiv \max_{\tilde{m}_i, \tilde{w}_1, i=1,2} \sum \pi_i \log \tilde{m}_i + \hat{\alpha} \log \tilde{w}_1 \) subject to \( \tilde{m}_i > 0, \tilde{w}_1 > 0, \) and \( \sum \pi_i [\tilde{m}_i + \tilde{w}_1] = 1. \) We then have: \( \gamma^* = \left[ \phi(\{\theta_1\}) - \phi(\hat{\theta}) \right]/\pi_1, \) where it is easy to verify that \( \phi(\{\theta_1\}) = \hat{\alpha} \log(\hat{\alpha}) - (1 + \hat{\alpha}) \log(1 + \hat{\alpha}). \)

It is now clear that \( \gamma^* \) is constant in \( \beta. \) We show that \( \gamma^* \) decreases as \( \alpha \) increases. Note that given \( \hat{\alpha} \) is increasing in \( \alpha, \) we need only show that \( \gamma^* \) is decreasing in \( \hat{\alpha}. \) To show that \( \gamma^* \) decreases with \( \hat{\alpha}, \) we need only show that \( \tilde{Z} \equiv \phi(\{\theta_1\}) - \phi(\hat{\theta}) \right]/\pi_1, \) where it is easy to derive that \( \frac{\partial \tilde{Z}[\{\theta_1\}]}{\partial \hat{\alpha}} = \phi(\{\theta_1\}) - \log(1 + \hat{\alpha}). \) By the envelope theorem, we have:

\[
\frac{\partial \phi(\hat{\theta})}{\partial \hat{\alpha}} = \pi_1 \log \hat{w}_1^* + (1 - \pi_1) \log \hat{w}_2^*
\]

\[
= \pi_1 \log \left[ \frac{1 - (1 + \hat{\alpha}) \hat{m}_2^*}{(1 - \hat{e}) \pi_1} \right] + (1 - \pi_1) \log(\hat{\alpha} \hat{m}_2^*),
\]

where \( \hat{m}_2^* \) solves the following maximization problem:

\[
\max_{\hat{m}_2 > 0} \pi_1 \log \left[ \frac{1 - (1 + \hat{\alpha}) \hat{m}_2^*}{(1 - \hat{e}) \pi_1} \right] + \frac{\hat{\alpha} \pi_1 \log \left[ \frac{1 + \hat{e}}{(1 - \hat{e}) \pi_1} \right] (1 + \hat{\alpha}) \hat{m}_2^* - \hat{e}}{(1 - \hat{e}) \pi_1} + [1 - \pi_1] \left( \hat{\alpha} \log \hat{\alpha} + (1 + \hat{\alpha}) \log m_2^* \right)
\]

with the following first order condition:

\[
\frac{\pi_1}{(1 + \hat{\alpha}) \hat{m}_2^* - 1} + \frac{\hat{\alpha} \pi_1 \left[ \hat{e} + (1 - \hat{e}) \pi_1 \right] (1 + \hat{\alpha}) \hat{m}_2^* - \hat{e}}{m_2^*} + \frac{1 - \pi_1}{m_2^*} = 0.
\]

25
Let \( x = (1 + \hat{\alpha}) \hat{m}^*, d \equiv \frac{\hat{\epsilon}}{x + (1 - x) \hat{m}^*}, \) and \( a \equiv \frac{1 - \pi_1}{\pi_1}. \) We have: \( d < x < 1, \) \( 0 < d < 1, \) and \( a \geq 0. \) With these definitions, \( Z < 0 \) holds if and only if

\[
\pi_1 \log \hat{\alpha} - \pi_1 \log(1 + \hat{\alpha}) - \pi_1 \log \left( \frac{\hat{\epsilon}(1 - \pi_1) + \pi_1}{(1 - \hat{\epsilon}) \pi_1} \right) - \pi_1 \log(x - d) - (1 - \pi_1) \log x < 0,
\]
or

\[
\log \left( \frac{\hat{\alpha}}{1 + \hat{\alpha}} \right) - \log \left( \frac{\hat{\epsilon}(1 - \pi_1) + \pi_1}{(1 - \hat{\epsilon}) \pi_1} \right) - \log(x - d) - a \log x < 0,
\]
or

\[
\frac{\hat{\alpha}}{1 + \hat{\alpha}} \cdot \frac{\hat{\epsilon}(1 - \pi_1) + \pi_1}{(1 - \hat{\epsilon}) \pi_1} < (x - d)x^a.
\]

But it is straightforward to verify that \( \frac{\hat{\epsilon}(1 - \pi_1) + \pi_1}{(1 - \hat{\epsilon}) \pi_1} = \frac{1}{1 - x}, \) so \( Z < 0 \) holds if and only if \( x^a(x - d) > (1 - d) \frac{\hat{\alpha}}{1 + \hat{\alpha}}, \) where \( x \) satisfies the following Euler equation:

\[
\frac{\hat{\alpha}}{x - d} + a(1 + \hat{\alpha}) \frac{1}{x} = \frac{1}{1 - x}.
\]

Now let \( h(a) \equiv x^a(x - d) - (1 - d) \frac{\hat{\alpha}}{1 + \hat{\alpha}}. \) First, suppose \( a = 0, \) i.e., suppose \( \pi_1 = 1. \) Then the above Euler equation yields \( x = \frac{\hat{\alpha} + d}{\hat{\alpha}} \) and \( h(a) = 0. \) Second, suppose \( a > 0. \) Given that \( d < x < 1, \) from the above Euler equation we have that \( x \) must increase as \( a \) increases. Thus to prove the proposition we need only show that \( h(a) \) is increasing in \( a. \) In fact,

\[
h'(a) = (a + 1)x^a - dax^{a-1} = ax^a \left[ \frac{a + 1}{a} - \frac{d}{x} \right] > 0,
\]

where note that \( x > d. \) Thus the first statement in part (ii) of the proposition is proven. The proof of the second statement is covered in the proof of Proposition 10, which states a more general result. In the same manner part (iii) of the proposition can be shown to hold as well. Thus the proposition is proven.

**Proof of Proposition 5** In the one period model, in the case of monitoring, the risk neutral agent’s value function is \( U_m(w) \equiv 3 - c_0(w) - \pi_1 \gamma \) where \( c_0(w) \) solves \( w \equiv -\exp[-\rho c_0(w)]. \) In the case of no monitoring, the risk neutral agent’s value function is \( U_{nm}(w) = M(w) \) where \( M(w) \) solves \( w = -\sum \pi_i \exp[-\rho(\theta_i - M(w))]. \) And hence

\[
\gamma^* = \frac{U_m(w) - U_{nm}(w)}{\pi_1} = \frac{\tilde{\theta} - c_0(w) - M(w)}{\pi_1}.
\]

But the above equations imply that \( -\exp(-\rho \tilde{\theta}) = -\sum \pi_i \exp(-\rho(\theta_i - \tilde{\theta}) - M(w) - c_0(w) + \tilde{\theta}). \) And so \( \tilde{\theta} - c_0(w) - M(w), \) the risk premium, increases as \( \rho \) increases, and hence \( \gamma^* \) increases as \( \rho \) increases. This proves the first half of the proposition.
We now prove the proposition’s results for the dynamic model. To simplify notation, let $\alpha = \beta$ throughout the rest of the proof. When $v(c) = -\exp(-\rho c)$, it is straightforward to show that $\gamma^* = \left[ \hat{\phi}(\{\theta_1\}) - \hat{\phi}(\{\theta_1\}) \right] (\pi_1 \rho)$, where $\hat{\phi}(\theta) = \max_{(\bar{m}_i, \bar{w}_i), i = 1, 2} \sum_{i=1, 2} \pi_i \left[ \log \bar{m}_i + \frac{\beta}{1 - \beta} \log \bar{w}_i \right]$ subject to $\bar{m}_i > 0$, $\bar{w}_i > 0$, $\bar{m}_2 + \bar{w}_2 = \exp[-\rho (\theta_2 - \theta_1)] \bar{m}_1 + \bar{w}_1$ and $\sum_{i=1, 2} \pi_i [\bar{m}_i + \bar{w}_i] = 1$, and that $\hat{\phi}(\{\theta_1\})$ is similarly defined, except that the maximization problem is only subject to $\sum_{i=1, 2} \pi_i [\bar{m}_i + \bar{w}_i] = 1$. Here, $\hat{\phi}(\{\theta_1\}) = \log(1 - \beta) + \frac{\beta}{1 - \beta} \log(\beta)$, which is independent of $\rho$.

For each $\rho > 0$, let $\mu_\rho$ be the lagrange multipliers associated with constraints $\bar{m}_2 + \bar{w}_2 = \exp[-\rho (\theta_2 - \theta_1)] \bar{m}_1 + \bar{w}_1$ and $\sum_{i=1, 2} \pi_i [\bar{m}_i + \bar{w}_i] = 1$, respectively. We have the following first-order conditions:

$$\frac{\pi_1}{\bar{m}_1} + \mu_\rho \exp[-\rho (\theta_2 - \theta_1)] + \lambda_\rho \pi_1 = 0, \quad (A.5)$$

$$\frac{\pi_2}{\bar{m}_2} - \mu_\rho + \lambda_\rho \pi_2 = 0. \quad (A.6)$$

By the envelope theorem, $\frac{\partial \gamma^*}{\partial \rho} = -\mu_\rho (\theta_2 - \theta_1) \exp[-\rho (\theta_2 - \theta_1)] < 0$. Now

$$\frac{\partial \gamma^*}{\partial \rho} = \frac{1}{\pi_1} \left[ \frac{\hat{\phi}(\theta) - \hat{\phi}(\{\theta_1\})}{\rho^2} - \frac{\partial \hat{\phi}(\theta)}{\partial \rho} \right].$$

First consider the case where $\rho$ is sufficiently small. For $\rho \to 0$, we have $\hat{\phi}(\theta) \to \hat{\phi}(\{\theta_1\})$ because $\exp[-\rho (\theta_2 - \theta_1)] \to 1$ as $\rho \to 0$. Therefore

$$\lim_{\rho \to 0} \left[ \frac{\hat{\phi}(\theta) - \hat{\phi}(\{\theta_1\})}{\rho^2} - \frac{\partial \hat{\phi}(\theta)}{\partial \rho} \right] = -\frac{1}{2} \lim_{\rho \to 0} \frac{\partial \hat{\phi}(\theta)}{\partial \rho} > 0.$$

Next we consider the situation where $\rho$ is sufficiently large. Let $\rho \to \infty$, then $\hat{\phi}(\theta) \to c$, where $c < 0$ is some constant. Therefore, $\frac{\partial \hat{\phi}(\theta)}{\partial \rho} \to 0$, as $\rho \to \infty$. On the other hand, we have $-\frac{\partial \hat{\phi}(\theta)}{\partial \rho} = (\theta_2 - \theta_1) \mu_\rho \exp[-\rho (\theta_2 - \theta_1)]$, Thus the proof is complete if we can show the following holds: $\mu_\rho \exp[-\rho (\theta_2 - \theta_1)] \to \infty$, as $\rho \to \infty$. Suppose not, then it follows that $\mu_\rho \to \infty$ and by equation (A.5), it also follows that $\lambda_\rho \to -\infty$. But it is straightforward to show that $m_2^* \not\to 0$, as $\rho \to \infty$, $i = 1, 2$. Thus we have derived a contradiction to equation (A.6).

Lastly, we show that (iii) holds. Let $\tilde{\phi}(\theta) = \max_{(\bar{m}_i, \bar{w}_i), i = 1, 2} \sum_{i=1, 2} \pi_i \left[ \log \bar{m}_i + \frac{\beta}{1 - \beta} \log \bar{w}_i \right]$ subject to $\bar{m}_i > 0$, $\bar{w}_i > 0$, (A.11), (A.12), and $\bar{m}_1 = \bar{m}_2 = \frac{1 - \beta}{\beta} \bar{w}_2$. Or equivalently,

$$\tilde{\phi}(\theta) = \max_{\bar{w}_1, \bar{w}_2} \sum_{i=1, 2} \pi_i \left[ \frac{1 - \beta}{\beta} \log \bar{w}_i + \left( 1 + \pi_2 \frac{1 - \beta}{\beta} \right) \log \bar{w}_2 + \log \left( \frac{1 - \beta}{\beta} \right) \right]$$

subject to $\bar{w}_i > 0$, and $\bar{w}_1 = \frac{1}{\beta} \left[ 1 - (1 - \beta) e^{-\rho (\theta_2 - \theta_1)} \right] \bar{w}_2$, and $\pi_1 \bar{w}_1 + \left( \frac{1 - \beta}{\beta} + \pi_2 \right) \bar{w}_2 = 1$. Thus it is obvious that $\hat{\phi}(\theta)$ is bounded when $\rho \to \infty$.
And so \( \gamma^* = \frac{\hat{\delta}(\theta_1) - \hat{\delta}(\theta)}{\pi_1 \rho} \leq \frac{\hat{\delta}(\theta)}{\pi_1 \rho} \to 0 \), as \( \rho \to \infty \). This completes the proof of the proposition.

**Proof of Corollary 1** Let \( \theta_i, \theta_j \in S^* \) and \( \theta_i < \theta_j \). Then, in the optimization that defines \( \phi(S) \), the incentive constraints for \( \theta_i \) and \( \theta_j \) can be written:

\[
\begin{align*}
    w_i &\leq b_i \equiv \min_{h \in S^*} \left[ \exp(-\theta_i m_h^* + \beta w_h^*) \right], \\
    w_j &\leq b_j \equiv \min_{h \in S^*} \left[ \exp(-\theta_j m_h^* + \beta w_h^*) \right],
\end{align*}
\]

Obviously, \( b_i > b_j \). Therefore either \( w_i^* = w_j^* \) (the constraint \( w_j \leq b_j \) is not binding) or \( w_i^* > w_j^* \) (the constraint \( w_j \leq b_j \) is binding) must hold. Thus I have shown \( W^*(\theta_i, w) < W^*(\theta_j, w) \). That \( C^*(\theta_i, w) < C^*(\theta_j, w) \) follows immediately from Proposition 6. The last inequality holds because for \( \theta_i, \theta_j \in S^* \), \( C^*(\theta_i, w) \leq C^*(\theta_j, w) \) implies that \( e^{-\theta_i - \theta_j} m_i \leq e^{-\theta_j - \theta_j} m_j \), and hence \( m_i < m_j \). Thus \( M^*(\theta_i, w) < M^*(\theta_j, w) \).

**Proof of Proposition 7** We first show (ii). Suppose (ii) is false and that \( \theta_i \in S^* \). Given that \( W^*(\theta_i, w) < W^*(\theta_j, w), \forall j \neq i \), we have: \( w_i > w_j \), \( \forall j \neq i \). Now decrease the value of \( w_i \) by \( \delta > 0 \) and increase the values of \( w_j \) by \( \Delta > 0 \), for all \( j \neq i \), where \( \delta \) and \( \Delta \) are chosen to be sufficiently small and to keep the promise-keeping constraint (9) satisfied. This will increase the value of the risk neutral agent. A contradiction.

Next, we prove (i). The idea is essentially the same although there are a few more details. Let \( x_j \equiv e^{-\theta_i} m_j \), \( \forall j \). Then the maximization problem defining the right hand side of (7) is

\[
\max_{x_j, w_j} \sum_{j=1}^{N} \pi_j \left[ \log x_j + \frac{\alpha}{\pi_i} \log w_j \right] \quad \text{subject to} \quad x_j > 0, w_j > 0, \forall j, x_i + \beta w_i \leq \exp(-\theta_j + \theta_k) x_j + \beta w_k, \forall j, \forall \theta_k \notin S, \text{ and } \sum_{j=1}^{N} \pi_j x_j + \beta w_j = 1.
\]

Suppose \( \theta_i \in S^* \) and \( C^*(\theta_i, w) < C^*(\theta_j, w), \forall j \neq i \). Then \( x_i > x_j \), \( \forall j \neq i \). Let \( x'_i = x_i - e^{-\theta_i} \delta \) and \( x'_j = x_j + e^{-\theta_j} \Delta \), \( \forall j \neq i \), where \( \delta, \Delta > 0 \), and are chosen to satisfy \( x'_i > 0, \forall j \), and \( x'_i + \beta w_i \leq \exp(-\theta_j + \theta_k) x'_j + \beta w_k, \forall j, \forall \theta_k \notin S \), and \( \sum_{j=1}^{N} \pi_j [x'_j + \beta w_j] = 1 \). It is then straightforward to verify that \( \delta \) and \( \Delta \) need only satisfy \( e^{-\theta_i} \delta = \sum_{j \neq i} \pi_j x'_j e^{-\theta_j} \Delta \). Given the above, the result of (i) is proven if we can show that

\[
\pi_i \log \left[ x_i - \sum_{j \neq i} \frac{\pi_j}{\pi_i} e^{-\theta_j} \Delta \right] + \sum_{j \neq i} \pi_j \log \left[ x_j + e^{-\theta_j} \Delta \right] > \pi_i \log x_i + \sum_{j \neq i} \pi_j \log x_j.
\]

Or equivalently,

\[
f(\Delta) = \pi_i \log \left[ 1 - \sum_{j \neq i} \frac{\pi_j}{\pi_i} x'_j e^{-\theta_j} \Delta \right] + \sum_{j \neq i} \pi_j \log \left[ 1 + \frac{e^{-\theta_j} \Delta}{x_j} \right] > 0.
\]
Since $f(0) = 0$, we need only show that for $\Delta$ small and positive, $f'(\Delta) > 0$. But

$$f'(\Delta) = \frac{\pi_i - \sum_{j \neq i} \pi_j e^{-\theta_j}}{\pi_i x_i - \sum_{j \neq i} \pi_j e^{-\theta_j} \Delta} + \frac{\sum_{j \neq i} \pi_j e^{-\theta_j}}{x_i + e^{-\theta_j} \Delta}$$

is continuous in $\Delta$ and

$$f'(0) = -\frac{1}{x_i} \sum_{j \neq i} \pi_j e^{-\theta_j} + \sum_{j \neq i} \frac{\pi_j e^{-\theta_j}}{x_j} > -\frac{1}{x_i} \sum_{j \neq i} \pi_j e^{-\theta_j} + \sum_{j \neq i} \frac{\pi_j e^{-\theta_j}}{x_i} = 0.$$ 

So (i) is proven, completing the proof of the proposition.

**Proof of Proposition 8** Let $H(S, \gamma) \equiv \phi(S) - \mu(S)\gamma$, for all $S \subseteq \Theta$ and $\gamma \in [0, \infty]$. Given that $H(S, \gamma)$ is linear and downward sloping in $\gamma$, and given $\mu(S_1) > \mu(S_2)$, there exists $\gamma^*$ such that $H(S_1, \gamma) > H(S_2, \gamma)$, $\forall \gamma < \gamma^*$, and $H(S_1, \gamma) < H(S_2, \gamma)$, $\forall \gamma > \gamma^*$. We can now claim that $\gamma_2 \leq \gamma^* \leq \gamma_1$. Suppose not, then either (i) $\gamma^* \leq \gamma_1, \gamma_2$, or (ii) $\gamma_1, \gamma_2 \leq \gamma^*$, or (iii) $\gamma_1 \leq \gamma^* \leq \gamma_2$. In case (i), $S^*(\gamma_1) = S_1$ cannot hold; in case (ii), $S^*(\gamma_2) = S_2$ cannot hold; in case (iii), both $S^*(\gamma_1) = S_1$ and $S^*(\gamma_2) = S_2$ are violated. The proposition is proven.

**Proof of Proposition 9** We first show (i). Suppose $S^*(w) = \{\theta_1, \ldots, \theta_N\}$. Then it is straightforward to show that for all $i$, $m_i^* = (1 - \beta) \exp(\theta_i)$, and $w_i^* = 1$. We then have, for all $i$, $\exp(-\theta_i) m_i^* + \beta w_i^* \leq \exp(-\theta_i) m_N^* + \beta w_N^*$. This indicates that monitoring $\theta_N$ is not necessary. So (i) is shown.

To show (ii), we first show that

$$\phi(\Theta - \{\theta_N\}) > \phi(S), \quad \forall S \neq \Theta - \{\theta_N\}, \Theta. \quad (A.7)$$

From the proof of (i) we have $\phi(\Theta - \{\theta_N\}) = \phi(\Theta)$, conditional on $S = \Theta - \{\theta_N\}$, it must hold that $m_i^* = (1 - \beta) \exp(\theta_i)$, $w_i^* = 1$, $i = 1, \ldots, N$, which in turn implies that the optimal contract is first best. Next, conditional on $S = \Theta - \{\theta_N\}$ or $S = \Theta$, it is easily seen that if $i > j$, then $\exp(-\theta_i) m_i^* + \beta w_i^* > \exp(-\theta_j) m_j^* + \beta w_j^*$. On the other hand, conditional on $S \neq \Theta - \{\theta_N\}$ and on $S \neq \Theta$, the following inequality $\exp(-\theta_i) m_i^* + \beta w_i^* \leq \exp(-\theta_j) m_j^* + \beta w_j^*$ must hold for some $i$ and $j$ with $i > j$. Thus we have shown that (A.7) holds.

Now let $\gamma$ be the solution to the following equation:

$$\phi(\Theta - \{\theta_N\}) - \mu(\Theta - \{\theta_N\}) \gamma = \max_{S \neq \Theta - \{\theta_N\}, \Theta} \left[ \phi(S) - \mu(S)\gamma \right]. \quad (A.8)$$

Note that such a $\gamma$ exists and $\gamma > 0$. In fact, at $\gamma = 0$, the left hand side of the above equation is strictly greater than the right hand side, whereas as $\gamma \to \infty$, the reverse holds. Furthermore, $\gamma$ is unique. To see this, notice that the left hand side of equation (16) is linear in $\gamma$ while the right hand side is piecewise linear in $\gamma$. Finally, since $\phi(\Theta - \{\theta_N\}) > \phi(S^*)$, where $S^*$ is
the optimal monitoring policy for any given $\gamma$, we have that at any given $\gamma$, the slope of the left hand side of ($A$,$B$), with respect to $\gamma$, is strictly greater than the slope of the right hand side. This proves the proposition.

**Proof of Proposition 10** Let $S(\gamma)$ denote a verification strategy when the monitoring cost is $\gamma$. Let $\bar{\gamma}$ be the solution to the following equation: 

$$\max_{S(\gamma) \neq \theta_N} [\phi(S(\gamma)) - \mu(S(\gamma))] = \phi(0).$$

Such a solution exists for the left hand side of the equation is strictly greater than the right hand side at $\gamma = 0$ whereas the reverse holds as $\gamma \to \infty$. $\bar{\gamma}$ is also unique for the left hand side of the equation is piecewise linear and strictly downward sloping in $\gamma$. In other words,

$$\bar{\gamma} = \frac{\phi(S^*(\bar{\gamma})) - \phi(0)}{\mu(S^*(\bar{\gamma}))} \leq \frac{\phi(\Theta) - \phi_*}{\min\{\pi_i, i \neq N\}},$$

where since $S^*(\bar{\gamma}) \neq \theta_N$, $\min\{\pi_i, i \neq N\} \leq \mu(S^*(\bar{\gamma}))$, and

$$\phi(S^*(\bar{\gamma})) \leq \phi(\Theta) = \max_{\{m_i, w_i\}} \sum \pi_i \left( \log m_i + \frac{\alpha}{1 - \alpha} \log w_i \right)$$

subject to $m_i > 0, w_i > 0$, and $\sum \pi_i [\exp(-\theta_i)m_i + \beta w_i] = 1$. And $\phi(0) \geq \phi_* = \max_{\{m_i, w_i\}} \left( \log m + \frac{\alpha}{1 - \alpha} \log w \right)$ subject to $m > 0, w > 0$, and $\sum \pi_i \exp(-\theta_i)m + \beta w = 1$. It is straightforward to show that $\phi(\Theta) = \bar{\theta} + \log(1 - \alpha) + \frac{\alpha}{1 - \alpha} \log(\beta)$ and $\phi_* = \log(1 - \alpha) - \log[\sum \pi_i \exp(-\rho \theta_i)] + \frac{\alpha}{1 - \alpha} \log(\beta)$. It then follows immediately that

$$\bar{\gamma} \leq \frac{\sum \pi_i \theta_i + \log[\sum \pi_i \exp(-\theta_i)]}{\min\{\pi_i, i \neq N\}},$$

which in turn implies immediately $\bar{\gamma} \to 0$, as $|\theta_N - \theta_1| \to 0$.

We now show (ii) holds. We first prove the following Lemma:

**Lemma** \(\phi(0) \to \phi(\Theta), \text{ as } \alpha \to 1.\)

We have $\phi(\Theta) = \bar{\theta} + \log(1 - \alpha) + \frac{\alpha}{1 - \alpha} \log(\alpha/\beta)$, and it also holds with the best contract (i.e., suppose constraint ($12$) needs not be satisfied in the problem of ($11$)), that for all $i$, $e^{-\theta_i} m_i \bar{f}_i = 1 - \alpha$, $w_i \bar{f}_i = \alpha \beta$. Now consider a scheme \(\{m_i(\alpha), w_i(\alpha)\}\) where $m_i(\alpha) = m_i^{\bar{f}_i}$, $\forall i$, but $w_i(\alpha)$ are chosen to be such that \(\{m_i(\alpha), W_i(\alpha)\}\) satisfies the incentive constraints $e^{-\theta_i} m_i(\alpha) + \beta w_i(\alpha) \leq e^{-\theta_i} m_j(\alpha) + \beta w_j(\alpha)$, $\forall i, j$ and $\sum \pi_i [e^{-\theta_i} m_i(\alpha) + \beta w_i(\alpha)] = 1$. The above equation can be written $\sum \pi_i w_i(\alpha) = \frac{\alpha}{\beta}$. Now

$$\phi(0) \geq \sum \pi_i \log(m_i^{\bar{f}_i}) + \frac{\alpha}{1 - \alpha} \sum \pi_i \log[w_i(\alpha)]$$

$$= \bar{\theta} + \log(1 - \alpha) + \frac{\alpha}{1 - \alpha} \sum \pi_i \log[w_i(\alpha)]$$

$$= \phi(\Theta) - \frac{\alpha}{1 - \alpha} \log\left(\frac{\alpha}{\beta}\right) + \frac{\alpha}{1 - \alpha} \sum \pi_i \log[w_i(\alpha)],$$
which in turn implies $0 \leq \phi(\Theta) - \phi(0) \leq \frac{\alpha}{1 - \alpha} \left[ \log \left( \frac{\alpha}{\beta} \right) - \sum \pi_i \log[w_i(a)] \right]$. Therefore to establish the lemma, we need only show

$$\frac{1}{1 - \alpha} \left[ \log \left( \frac{\alpha}{\beta} \right) - \sum \pi_i \log[w_i(a)] \right] \to 0, \text{ as } \alpha \to 1.$$

By the incentive constraints, for all $i > 1$, $e^{-\delta_i} m_i^f + \beta w_i(a) \leq e^{-\delta_i} m_i^{fb} + \beta w_{i-1}(a)$, and $e^{-\delta_i-1} m_i^{fb} + \beta w_{i-1}(a) \leq e^{-\delta_{i-1}} m_i^f + \beta w_i(a)$, which imply, again for all $i > 1$, $1 - \beta \left[ 1 - e^{-\delta_{i-1}} \right] \leq w_i(a) - w_{i-1}(a) \leq \frac{1}{\beta} \left[ e^{-\delta_i + \delta_{i-1}} - 1 \right]$. Thus we have $w_i(a) - w_{i-1}(a) = (1 - \alpha) H_i(a)$, where

$$\frac{1}{\beta} \left[ 1 - e^{-\delta_{i-1}} \right] \leq H(a) \leq \frac{1}{\beta} \left[ e^{-\delta_i + \delta_{i-1}} - 1 \right].$$

With the above, we have $w_i(a) = w_i(a) + (1 - \alpha) K_i(a)$, $i = 1, 2, \ldots, n$, where $K_i(a) = \sum H_i(a)$ and $H_1(a) = 0$. Given that $\sum \pi_i w_i(a) = \alpha / \beta$, we then have: $w_1(a) = \alpha / \beta - (1 - \alpha) K(a)$, where $K(a) = \sum \pi_i K_i(a)$. By Taylor’s expansion,

$$\log[w_1(a)] = \log \left( \frac{1 - \alpha) K(a)}{\beta} \right) - \frac{(1 - \alpha) K(a)}{\beta - \xi(1 - \alpha) K(a)},$$

where $\xi \in [0, 1]$, and for $i > 1$,

$$\log[w_i(a)] = \log[w_1(a)] + \frac{(1 - \alpha) K_i(a)}{w_1(a) + \xi_i(1 - \alpha) K_i(a)}$$

$$= \log \left( \frac{\alpha}{\beta} \right) - \frac{(1 - \alpha) K(a)}{\beta} - \xi_i(1 - \alpha) K_i(a) + \frac{(1 - \alpha) K_i(a)}{w_1(a) + \xi_i(1 - \alpha) K_i(a)},$$

where $\xi_i \in [0, 1]$. Thus

$$\frac{1}{1 - \alpha} \left[ \log \left( \frac{\alpha}{\beta} \right) - \sum \pi_i \log[w_i(a)] \right]$$

$$= \frac{1}{1 - \alpha} \left[ \log \left( \frac{\alpha}{\beta} \right) - \log \left( \frac{\alpha}{\beta} \right) + \frac{(1 - \alpha) K(a)}{\beta} - \sum \frac{\pi_i(1 - \alpha) K_i(a)}{w_1(a) + \xi_i(1 - \alpha) K_i(a)} \right]$$

$$= \frac{K(a)}{\beta} - \xi_i(1 - \alpha) K_i(a) - \sum \frac{\pi_i K_i(a)}{w_1(a) + \xi_i(1 - \alpha) K_i(a)}$$

$$\to 0, \text{ as } \alpha \to 1,$$

for $K(a), K_i(a) \to 0$, and $w_1(a) \to 1 / \beta$ as $\alpha \to 1$. This proves the lemma.

We now prove (ii) of the proposition. Fix $\beta$. Given the lemma, and given that $\phi(0) \leq \phi(S) \leq \phi(\Theta), \forall S, \forall \alpha$, we have $\phi(S) \to \phi(\Theta)$, as $\alpha \to 1, \forall S$. Now for all $\alpha$, let $\gamma(\alpha)$ denote the highest monitoring cost at which the optimal monitoring strategy is not an empty set, and let $S^*(\alpha)$ denote just that monitoring strategy. It then holds that at $\gamma = \gamma(\alpha)$, the risk neutral agent is indifferent between the two verification strategies: $S = \emptyset$ and $S =$
\( S^* \). This implies: 
\[ \mu(S^*(\alpha)) \gamma(\alpha) = \phi(S^*(\alpha)) - \phi(\theta) \to 0, \text{ as } \alpha \to 1. \]
But \( \mu(S^*(\alpha)) \geq \min\{\pi_i\} > 0 \), therefore \( \gamma(\alpha) \to 0 \) as \( \alpha \to 1 \). The proposition is proven.

**Proof of Proposition 11** Let \( \theta_p, \theta_q \in \Theta \), and \( \theta_p < \theta_q \). Let \( S \in \Theta \) be any monitoring strategy such that \( \theta_q \) is monitored but \( \theta_p \) is not. That is: \( S = \{\theta_q\} \cup A \), \( \Theta - S = \{\theta_p\} \cup B \), where \( A, B \subseteq \Theta \). To prove the proposition we need only show that the following monitoring strategy \( S' \), where \( S' = \{\theta_p\} \cup A \), \( \Theta - S' = \{\theta_q\} \cup B \), weakly dominates \( S \). Suppose \( S \) is the monitoring strategy. Incentive constraints that involve \( \theta_p \) and \( \theta_q \) are: 
\[ \bar{m}_i + \bar{w}_i \leq e^{-\theta_i + \theta_p} \bar{m}_p + \bar{w}_p, \ \forall i \neq p, q, \text{ and } \bar{m}_q + \bar{w}_q \leq e^{-\theta_q + \theta_p} \bar{m}_p + \bar{w}_p, \]
where \( \bar{m}_i = \exp(-\theta_i) m_i \) and \( \bar{w}_i = \beta w_i \) as in the proof of Proposition 5.

Suppose instead \( S' \) is the monitoring strategy, then incentive constraints that involve \( \theta_p \) and \( \theta_q \) are: 
\[ \bar{m}_i + \bar{w}_i \leq e^{-\theta_i + \theta_q} \bar{m}_q + \bar{w}_q, \ \forall i \neq p, q, \text{ and } \bar{m}_p + \bar{w}_p \leq e^{-\theta_p + \theta_q} \bar{m}_q + \bar{w}_q. \]
But \( \theta_p < \theta_q \), so \(-\theta_i + \theta_q > -\theta_i + \theta_p\), and 
\[ -\theta_0 + \theta_q > -\theta_0 + \theta_p; \]
hence the second set of constraints contains a larger domain than the first. In other words, \( S' \) weakly dominates \( S \), taking into consideration that \( \theta_p - \pi_q \). The proposition is proven.

**Proof of Lemma 1** First notice that for all \( w \in [\underline{w}, \overline{w}] \), there exists some \( c(w) \in [-\theta_0, \theta_0] \) such that \( w = \frac{1}{U'} \sum \pi_i v(\theta_i - c(w)) \) due to continuity. The contract in which the borrower pays \( c(w) \) to the lender each period is feasible and incentive compatible and delivers expected utility \( w \) to the borrower. Suppose \( w < \underline{w} \), then there is no feasible and incentive compatible contract that delivers \( w \) to the borrower, for the borrower can report the lowest endowment every period to guarantee a higher ex ante expected utility weakly above \( \underline{w} \). Suppose \( w > \overline{w} \), then the resource constraint will be violated.

Next, for each \( w \in [\underline{w}, \overline{w}] \), we show that the set of expected utilities of the lender that can be achieved by a feasible and incentive compatible contract that promises expected utility \( w \) to the agent is compact. Let \( \Phi[w] \) denote this set. We need only show that \( \Phi[w] \) is closed since it is clearly bounded. This is left for the reader (see Wang 2000b). See Wang (2000b) for the proof of (iii).

**Proof of Lemma 2** Notice that for all \( w \in [\underline{w}, \overline{w}] \), the contract where \( S_i(h_{i-1}) = \Theta \), \( M_i(h_i) = M_{FB}(\theta_i, w) \), \( \forall h_i \in H_i \), is feasible and incentive compatible. And of course any \( w \) outside the interval \([\underline{w}, \overline{w}]\) is simply not feasible. That shows (a). (b) of the proposition can be shown by following the arguments in the proof of (b) in Proposition 4.1. To show (c), notice that for the so specified contract to be feasible, we have \( w \in [\underline{w}, \overline{w}] \). Then by (c) of Proposition 4.1 we have: 
\[ c/(1 - \beta) < U_{NV}(w) \leq U^*(w). \]
This completes the proof of the proposition.

**Proof of Proposition 12** Suppose not. Then there exists a sequence \( \{w_n\}_{n=1}^{\infty} \), where \( w_n \to \overline{w} \) as \( n \to \infty \), such that \( \{\theta_1, ..., \theta_{N-1}\} \not\subseteq S^*(\overline{w}) \).
We then have: for all \( n \), there is some \( \theta_i, n \neq \theta_N \) such that \( \theta_i, n \not\in S(w_n) \). But this cannot be incentive compatible when \( n \) becomes sufficiently large as \( w_n \to w \), because the agent can always misreport \( \theta_N \) as \( \theta_i, n \) to obtain an expected utility \( u'_n \) which is strictly greater than \( w_n \).

**Proof of Proposition 14** We first show that there exists \( \gamma \) such that (a) holds. This takes two steps.

**Step 1.** Fix \( w \in [w_\ast, \overline{w}] \). We have:

\[
\exists \gamma(w) > 0 \text{ such that } S^\ast(w)(\gamma) = \emptyset, \quad \forall \gamma \geq \gamma(w). \tag{A.9}
\]

This is directly from Corollary 2. The following is a slightly different proof of the same result which is useful for step 2. First, notice that since \( w \in [w_\ast, \overline{w}] \), we have \( U^\ast(w)(\gamma) \geq \theta_1/(1 - \beta), \) for all \( \gamma \). Second, \( U^\ast(w')(\gamma) \leq \sum \pi_i \theta_i/(1 - \beta), \) for all \( w' \in [w_\ast, \overline{w}] \) and all \( \gamma \). Third, by the Bellman equation, we have for all \( \gamma, U^\ast(w)(\gamma) = E_0\{M^\ast(\theta, w)(\gamma) + \beta U^\ast(W^\ast(\theta, w)(\gamma))\} - \mu[S^\ast(w)(\gamma)]\gamma \). Now suppose (A.9) is not true. Then there is a sequence \( \{\gamma_n\} \) such that \( \gamma_n \to \infty \) as \( n \to \infty \) and \( S^\ast(w)(\gamma_n) \neq \emptyset \) for all \( n \). We then have \( \mu[S^\ast(w)(\gamma_n)]\gamma_n \to \infty \) as \( n \to \infty \). Given that \( E_0\{M^\ast(\theta, w)(\gamma) + \beta U^\ast(W^\ast(\theta, w)(\gamma))\} \), is bounded from above, and that \( U^\ast(w)(\gamma_n) \) is bounded from below, we thus have a contradiction.

**Step 2.** We now show that the \( \gamma \) exists. For each \( w \in [w_\ast, \overline{w}] \), let \( \gamma(w) \equiv \inf\{\gamma(w) : S^\ast(w)(\gamma) = \emptyset, \quad \forall \gamma \geq \gamma(w)\} - \Delta \), where \( \Delta > 0 \) is some small constant. Suppose the proposition does not hold. Then there exists a sequence \( \{w_n\} \) where \( w_n \in [w_\ast, \overline{w}] \), such that \( \gamma(w_n) \to \infty \) as \( n \to \infty \). Now

\[
\mu[S^\ast(w_n)(\gamma(w_n)) + \gamma(w_n)] = E_0\{M^\ast(\theta, w_n)(\gamma(w_n)) + \beta U^\ast(W^\ast(\theta, w_n)(\gamma(w_n))\}) - U^\ast(w_n)(\gamma(w_n))\}
\]

where \( \mu[S^\ast(w_n)(\gamma(w_n))] \geq \min\{\pi_i\} \). So, as \( \gamma \) goes to \( \infty \), the left hand side of the above equation is unbounded, whereas the right hand side is bounded, by step 1. A contradiction.

Next, We show there exists \( \gamma \) such that (b) of the proposition holds. This also takes two steps. Step 1. We show that for all \( z > 0 \), there exists \( \overline{z}(z) > 0 \) such that if \( \gamma \geq \overline{z}(z) \) then \( U^\ast(w)(\gamma) < -z \), for all \( w \in [\overline{w}, w_\ast] \). This can be established as follows. We have for any expected utility \( w \) to be achievable by a contract that involves no verification at all, it is necessary that \( w \in [w_\ast, \overline{w}] \). It then follows that for all \( w \in [w_\ast, \overline{w}] \), there exists \( N(w) \geq 1 \) and some history \( h^{N(w)-1} \) such that \( S_{N(w)}(h^{N(w)-1}) \neq \emptyset \). Here \( N(w) \) is independent of \( \gamma \). Therefore as \( \gamma \to \infty \), \( \max\{U^\ast(w)(\gamma), w \in [\overline{w}, w_\ast]\} \to -\infty \).

Step 2. Step 1 implies that there exists \( \overline{\gamma}_1 > 0 \) such that if \( \gamma \geq \overline{\gamma}_1 \) then \( W^\ast(\theta, w)(\gamma) \not\in [w_\ast, \overline{w}] \), for all \( w \in [w_\ast, \overline{w}] \). Further, by (a) of the this Proposition, there exists \( \overline{\gamma}_2 > 0 \) such that if \( \gamma \geq \overline{\gamma}_2 \) then \( S^\ast(w)(\gamma) = \emptyset \), for all \( w \in [w_\ast, \overline{w}] \). Let \( \gamma \equiv \max\{\overline{\gamma}_1, \overline{\gamma}_2\} \). Then for all \( \gamma \geq \overline{\gamma} \), the Bellman equation with costly state verification and with \( w \) restricted on the domain \([w_\ast, \overline{w}] \) is precisely the same as the Bellman equation with no state verification. The proposition is proven.
Proof of Proposition 15. We first show $U^*(w)(\gamma) \rightarrow U_{FB}(w)$, as $\gamma \rightarrow 0$. The following contract is feasible and incentive compatible: $S(w)(\gamma) = \Theta$ and $M(\theta, w)(\gamma) = M_{FB}(\theta, w)$, for all $\theta$ and all $w$. Therefore $U_{FB}(w) - \gamma \leq U^*(w) \leq U_{FB}(w), \forall w$. It follows immediately that $U^*(w)(\gamma) \rightarrow U_{FB}(w)$, as $\gamma \rightarrow 0$.

Next, we show that $M^*(\theta, w)(\gamma) \rightarrow M_{FB}(\theta, w)$, as $\gamma \rightarrow 0$. The proof for $W^*(\theta, w)(\gamma) \rightarrow W_{FB}(\theta, w)$ is similar and is left for the reader. Fix $w$. Let $\{\gamma_n\}_{n=1}^{\infty}$ be any sequence of verification costs satisfying $\gamma_n \rightarrow 0, as n \rightarrow \infty$. We need to show that $M^*(\theta, w)(\gamma_n) \rightarrow M_{FB}(\theta, w)$, as $n \rightarrow \infty$, for all $\theta$. Suppose not, then since for all $\theta \in \Theta$, the sequence $\{M^*(\theta, w)(\gamma_n)\}_{n=1}^{\infty}$ is bounded, there is a $\theta$ and a subsequence $\{\gamma_{n_q}\}$ of $\{\gamma_n\}$ such that $M^*(\theta, w)(\gamma_{n_q}) \rightarrow \ell \neq M_{FB}(\theta, w)$, for $n_q \rightarrow \infty$. Without loss of generality assume the subsequence $\{\gamma_{n_q}\}$ is the sequence $\{\gamma_n\}$ itself. Then there exists $N$ large enough such that for all $n \geq N$, $|M^*(\theta, w)(\gamma_n) - M_{FB}(\theta, w)| \geq \frac{1}{n} |w - M_{FB}(\theta, w)| \equiv \delta > 0$. Now given $U^*(w)(\gamma_n) \leq U_{FB}(w), \forall w \in [\underline{w}, \overline{w}]$ and $\forall \gamma > 0$, we have, for $n \geq N$, $U^*(w)(\gamma_n) \leq \max_{\theta \in \Theta} \{\gamma_n \in [0, \theta], W(\theta, w)(\gamma_n) \in [\underline{w}, \overline{w}]\}$, for all $\theta \in \Theta$ and $\forall \theta - M(\theta, w)(\gamma_n) \geq \epsilon$. Since $S(w)(\gamma_n) = \Theta$ is always feasible, we then have: $U^*(w)(\gamma_n) \leq X(\delta), X(\delta) \equiv \max_{\theta \in \Theta} \{\gamma_n \in [0, \theta], W(\theta, w)(\gamma_n) \in [\underline{w}, \overline{w}]\}$, for all $\theta \in \Theta$ and $\forall \theta - M(\theta, w)(\gamma_n) \geq \epsilon$. But $X(\delta) < U_{FB}(w)$, $\forall n \geq N$. Then we have: $U^*(w)(\gamma_n) \leq U_{FB}(w) - \epsilon, \forall n \geq N$. But this is a contradiction to $U^*(w)(\gamma_n) \rightarrow U_{FB}(w), n \rightarrow \infty$. The proposition is proven.

7 References


Figure 1: $\gamma^*$ as a function of $\rho$. 
Figure 2: $\tau$ as a function of $\alpha$. 
Figure 3: $\tau$ as a function of $\rho$. 
Figure 4: The case of $v(e) = \log(e)$: The optimal monitoring policy.
Figure 5: The case of $v(c) = \log(c)$: The optimal law of motion for $w$. 
Figure 6: The case of $v(c) = \log(c)$: The optimal payment scheme.