A Stock Market Model with Systematic and Idiosyncratic Risks: A GMM Estimation Approach for Cross-Sectional Data

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Financial Market Structure

Financial assets:

- many risky assets called **stocks**
- one diversified portfolio of stocks called **market index**
- one riskless asset such as Treasury bill

Asset prices are quoted continuously, but we will ultimately focus on only two dates: \( t = 0 \) and \( t = T \)

**Simplification:**

between 0 and \( T \), risk-free interest rate \( r \) is constant
Dynamics of market index price:

\[
\frac{dM_t}{M_t} = \mu_m dt + \sigma_m dW_t
\]

where drift \( \mu_m \) is

\[
\mu_m = r + \delta \sigma_m
\]

- \( \sigma_m \): market volatility, \( \sigma_m > 0 \)
- \( \delta \): Sharpe ratio of market index
- \( \{W_t\} \): systematic risk, modeled as standard Brownian motion
Stock Price Dynamics

Dynamics of price of stock $i$ for $i = 1, 2, ...$:

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \beta_i \sigma_m dW_t + \sigma_i dZ_t^i$$

where drift $\mu_i$ is

$$\mu_i = r + \delta \beta_i \sigma_m + \gamma \sigma_i$$

- $\beta_i \sim UNI [\kappa_\beta, \kappa_\beta + \lambda_\beta]$: systematic risk loading of stock $i$
- $\sigma_i \sim UNI [0, \lambda_\sigma]$: idiosyncratic volatility of stock $i$
- $\gamma$: idiosyncratic risk premium
- $\{Z_t^i\}$: idiosyncratic risk, modeled as standard Brownian motion
Recall:

\[ \mu_i = r + \delta \beta_i \sigma_m + \gamma \sigma_i \]

If \( \gamma = 0 \), our price dynamics are in line with:

- ICAPM with constant invest. opportunity set: Merton (1973)
- APT with one market factor: Ross (1976)

**But** growing literature suggests that idiosyncratic risk is priced:

- Epstein & Schneider (2008): ambiguity premium

Green & Rydqvist (1997), Ang et al. (2006), Fu (2009): idiosyncratic premium is nonzero, but no consensus about sign
Objectives and Contribution

Estimating $\gamma$ helps inform debate over idiosyncratic risk premium:

- value of $\gamma$ affects construction of investment strategies

Estimating $\sigma_m$ from cross-sectional data is complementary to high-frequency time-series approach (e.g., Andersen et al., 2003):

- many pricing applications require volatility estimates

Remark:
Our estimation method differs from traditional regression technique of Fama & MacBeth (1973)
Using Itô’s lemma:

\[
\frac{S^i_T}{S^i_0} = \exp \left[ \left( \mu_i - \frac{1}{2} \beta_i^2 \sigma_m^2 - \frac{1}{2} \sigma_i^2 \right) T + \beta_i \sigma_m W_T + \sigma_i Z^i_T \right]
\]

\[
\frac{M_T}{M_0} = \exp \left[ \left( \mu_m - \frac{1}{2} \sigma_m^2 \right) T + \sigma_m W_T \right]
\]

where \( W_T, Z^i_T \) for \( i = 1, 2, ... \sim i.i.d. \ N(0, T) \)

Common shock \( W_T \) induces dependence across \( \frac{S^1_T}{S^1_0}, \frac{S^2_T}{S^2_0}, ... \)

But easy to see that \( \frac{S^1_T}{S^1_0}, \frac{S^2_T}{S^2_0}, ... \) are conditionally i.i.d. given \( \frac{M_T}{M_0} \)
Relationship to Econometrics Literature

Large literature on localized common shocks:

- general approach: Conley (1999)
- spatial, group, social effects: e.g., Kelejian & Prucha (1999), Lee (2007), Bramoullé et al. (2009)

Sparse literature on non-localized common shocks:


We build on Andrews (2003) to develop GMM estimation theory under non-localized common shock, which is induced by $W_T$
Let \( \theta = (\sigma_m, \gamma, \kappa_\beta, \lambda_\beta, \lambda_\sigma)' \). Note: \( \delta \) is not identifiable. Consider:

\[
g_i(\zeta; \theta) = \left(\frac{S_T^i}{S_0^i}\right)^{\zeta} - E_\theta \left[ \left(\frac{S_T^i}{S_0^i}\right)^{\zeta} | M_T/M_0 \right]
\]

Given constants \( \zeta_1, ..., \zeta_k \), let \( k \times 1 \) vector of moment restrictions be:

\[
g_i(\theta) = (g_i(\zeta_1; \theta), ..., g_i(\zeta_k; \theta))'
\]

**Theorem:**

For any finite \( \zeta \in \mathbb{R} \), \( E_\theta \left[ \left(\frac{S_T^i}{S_0^i}\right)^{\zeta} | M_T/M_0 \right] \) exists and can be expressed analytically. Moreover, it is continuously differentiable in \( \theta \) and all derivatives can be expressed analytically.
GMM Estimators

**One-step** estimation using \( k \times k \) nonstoch. positive definite \( \Sigma \):

\[
Q_{1,n} (\theta) = \left( n^{-1} \sum_i g_i(\theta) \right)' \Sigma^{-1} \left( n^{-1} \sum_i g_i(\theta) \right)
\]

\[
\hat{\theta}_{1,n} = \arg \min_{\theta \in \Theta} Q_{1,n} (\theta)
\]

**Two-step** estimation using \( \hat{\Sigma}_{1,n} = n^{-1} \sum_i g_i(\hat{\theta}_{1,n}) g_i(\hat{\theta}_{1,n})' \):

\[
Q_{2,n} (\theta) = \left( n^{-1} \sum_i g_i(\theta) \right)' \hat{\Sigma}_{1,n}^{-1} \left( n^{-1} \sum_i g_i(\theta) \right)
\]

\[
\hat{\theta}_{2,n} = \arg \min_{\theta \in \Theta} Q_{2,n} (\theta)
\]
Asymptotics

**Theorem:** Under very general regularity conditions:

\[ \hat{\theta}_{1,n} \to^p \theta_0 \]

\[ \hat{\theta}_{2,n} \to^p \theta_0 \]

**Theorem:** Under additional regularity conditions:

\[ \sqrt{n} \left( \hat{\theta}_{1,n} - \theta_0 \right) \to^d MN (0, V_{1,F_0}) \]

\[ \sqrt{n} \left( \hat{\theta}_{2,n} - \theta_0 \right) \to^d MN (0, V_{2,F_0}) \]

\( V_{1,F_0}, V_{2,F_0} \) are \( p \times p \) positive definite **stochastic** matrices
Consider testing $r$ parametric restrictions:

$$H_0 : a(\theta_0) = 0, \ H_A : a(\theta_0) \neq 0$$

Suppose:

- $r \times 1$ vector-function $a(\theta)$ is continuously differentiable
- $r \times p$ Jacobian $A(\theta_0) = \partial a(\theta_0) / \partial \theta'$ has full row rank

Under $H_0$, **Wald test statistic**

$$W \equiv na(\hat{\theta}_{2,n})' \left[ A(\hat{\theta}_{2,n}) V_{2,n} A(\hat{\theta}_{2,n})' \right]^{-1} a(\hat{\theta}_{2,n}) \xrightarrow{d} \chi^2(r)$$

**Remark:** result for $\hat{\theta}_{1,n}$ is analogous
Sources:

- stock data: Center for Research in Security Prices (CRSP)
- T-bill data: Federal Reserve Bank Reports (from WRDS)

CRSP provides extensive information on securities traded on NYSE, AMEX, and NASDAQ, but not all securities are used.

We only include regularly traded stocks of operating companies:

- closed-end funds, ETFs, financial REITs are dropped
- ADRs are included
- if company issues 2+ classes of shares, only class with largest number of outstanding shares is included
Preliminary Estimation Results

- Date 0: January 2\textsuperscript{nd}, 2008
- Date $T$: January 4\textsuperscript{th}, 2008
- Sample size $n = 5,413$
- Number of moments $k = 6$
- Index return $M_T/M_0 = 0.975$
- T-bill rate $r = 3.2\%$

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<th>Parameter</th>
<th>Estimate</th>
<th>Wald Statistic</th>
<th>P-value</th>
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<td>$1.03 \cdot 10^4$</td>
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<td>$\gamma$</td>
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<td>$1.45 \cdot 10^4$</td>
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Further Directions

Currently in progress:

- estimation of model parameters

Extensions of financial model:

- multi-factor stock price model
- stochastic volatility setting

Direction for future econometric research:

- MLE under common non-localized shocks
Thank you!
Questions?
Random variable $Y$ has **mixed normal distribution**

$$Y \sim MN \left(0, \eta^2 \right)$$

if characteristic function of $Y$ is

$$\phi_Y (t) \equiv E \left[ \exp \left( itY \right) \right] = E \left[ \exp \left( - \frac{1}{2} \eta^2 t^2 \right) \right]$$

where $\eta$ is random variable

$Y$ can be represented as

$$Y = \eta Z$$

where $Z \sim N \left(0, 1 \right)$ and $Z$ is **independent** of $\eta$
Inference: Formulas

\[
W \equiv n a (\hat{\theta}_{2,n})' \left[ A (\hat{\theta}_{2,n}) V_{2,n} A (\hat{\theta}_{2,n})' \right]^{-1} a (\hat{\theta}_{2,n})
\]

\[
V_{2,n} = \left[ G'_{2,n} \hat{\Sigma}_{2,n}^{-1} G_{2,n} \right]^{-1}
\]

\[
G_{2,n} = n^{-1} \sum_i \partial g_i (\hat{\theta}_{2,n}) / \partial \theta'
\]

\[
\hat{\Sigma}_{2,n} = n^{-1} \sum_i g_i (\hat{\theta}_{2,n}) \partial g_i (\hat{\theta}_{2,n})'
\]