A Stock Market Model with Systematic and Idiosyncratic Risks: A GMM Estimation Approach for Cross-Sectional Data

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Financial Market Structure

Financial assets:

- many risky assets called **stocks**
- one diversified portfolio of stocks called **market index**
- one riskless asset such as Treasury bill

Asset prices are quoted continuously, but we will ultimately focus on only two dates: $t = 0$ and $t = T$

Simplification:

between 0 and T, risk-free interest rate r is constant

Market Index Price Dynamics

Dynamics of market index price:

$$\frac{dM_t}{M_t} = \mu_m dt + \sigma_m dW_t$$

where drift μ_m is

$$\mu_m = r + \delta\sigma_m$$

- σ_m: market volatility, $\sigma_m > 0$
- δ: Sharpe ratio of market index
- $\{W_t\}$: systematic risk, modeled as standard Brownian motion

Stock Price Dynamics

Dynamics of price of stock i for $i = 1, 2, \ldots$:

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \beta_i \sigma_m dW_t + \sigma_i dZ_t^i$$

where drift μ_i is

$$\mu_i = r + \delta \beta_i \sigma_m + \gamma \sigma_i$$

- $\beta_i \sim UNI\left[\kappa_\beta, \kappa_\beta + \lambda_\beta\right]$: systematic risk loading of stock i
- $\sigma_i \sim UNI\left[0, \lambda_\sigma\right]$: idiosyncratic volatility of stock i
- γ: idiosyncratic risk premium
- $\left\{Z_t^i\right\}$: idiosyncratic risk, modeled as standard Brownian motion

Relationship to Finance Literature

Recall:

$$\mu_i = r + \delta\beta_i\sigma_m + \gamma\sigma_i$$

If $\gamma = 0$, our price dynamics are in line with:

- ICAPM with constant invest. opportunity set: Merton (1973)
- APT with one market factor: Ross (1976)

But growing literature suggests that idiosyncratic risk is priced:

- Merton (1987), Malkiel & Xu (2006): incomplete diversification
- Epstein & Schneider (2008): ambiguity premium

Green & Rydqvist (1997), Ang et al. (2006), Fu (2009): idiosyncratic premium is nonzero, but no consensus about sign

Objectives and Contribution

Estimating γ helps inform debate over idiosyncratic risk premium:

- value of γ affects construction of investment strategies

Estimating σ_m from cross-sectional data is complementary to high-frequency time-series approach (e.g., Andersen et al., 2003):

- many pricing applications require volatility estimates

Remark:

Our estimation method differs from traditional regression technique of Fama & MacBeth (1973)

Cross-Sectional Dependence of Observations

Using Itô's lemma:

$$\frac{S_T^i}{S_0^i} = \exp\left[\left(\mu_i - \frac{1}{2}\beta_i^2\sigma_m^2 - \frac{1}{2}\sigma_i^2\right)T + \beta_i\sigma_m W_T + \sigma_i Z_T^i\right]$$

$$\frac{M_T}{M_0} = \exp\left[\left(\mu_m - \frac{1}{2}\sigma_m^2\right)T + \sigma_m W_T\right]$$

where W_T, Z_T^i for $i = 1, 2, ... \sim i.i.d.\ N(0, T)$

Common shock W_T induces **dependence** across $\frac{S_T^1}{S_0^1}, \frac{S_T^2}{S_0^2}, ...$

But easy to see that $\frac{S_T^1}{S_0^1}, \frac{S_T^2}{S_0^2}, ...$ are **conditionally i.i.d.** given $\frac{M_T}{M_0}$

Relationship to Econometrics Literature

Large literature on **localized** common shocks:

- general approach: Conley (1999)
- spatial, group, social effects: e.g., Kelejian & Prucha (1999), Lee (2007), Bramoullé et al. (2009)

Sparse literature on **non-localized** common shocks:

- Andrews (2003, 2005)

We build on Andrews (2003) to develop GMM estimation theory under non-localized common shock, which is induced by W_T

GMM Implementation: Moment Restrictions

Let $\boldsymbol{\theta} = \left(\sigma_m, \gamma, \kappa_\beta, \lambda_\beta, \lambda_\sigma\right)'$. *Note:* δ is not identifiable. Consider:

$$g_i\left(\xi; \boldsymbol{\theta}\right) = \left(S_T^i / S_0^i\right)^{\xi} - E_{\boldsymbol{\theta}}\left[\left(S_T^i / S_0^i\right)^{\xi} | M_T / M_0\right]$$

Given constants $\xi_1, ..., \xi_k$, let $k \times 1$ vector of moment restrictions be:

$$\underset{(k\times 1)}{\boldsymbol{g}_i\left(\boldsymbol{\theta}\right)} = \left(g_i\left(\xi_1; \boldsymbol{\theta}\right), ..., g_i\left(\xi_k; \boldsymbol{\theta}\right)\right)'$$

Theorem:

For any finite $\xi \in \mathbb{R}$, $E_{\boldsymbol{\theta}}\left[\left(S_T^i / S_0^i\right)^{\xi} | M_T / M_0\right]$ exists and can be expressed analytically. Moreover, it is continuously differentiable in $\boldsymbol{\theta}$ and all derivatives can be expressed analytically

GMM Estimators

One-step estimation using $k \times k$ nonstoch. positive definite $\mathbf{\Sigma}$:

$$Q_{1,n}(\boldsymbol{\theta}) = \left(n^{-1}\sum_i \boldsymbol{g}_i(\boldsymbol{\theta})\right)' \mathbf{\Sigma}^{-1} \left(n^{-1}\sum_i \boldsymbol{g}_i(\boldsymbol{\theta})\right)$$

$$\widehat{\boldsymbol{\theta}}_{1,n} = \arg\min_{\boldsymbol{\theta}\in\mathbf{\Theta}} Q_{1,n}(\boldsymbol{\theta})$$

Two-step estimation using $\widehat{\mathbf{\Sigma}}_{1,n} = n^{-1}\sum_i \boldsymbol{g}_i\left(\widehat{\boldsymbol{\theta}}_{1,n}\right)\boldsymbol{g}_i\left(\widehat{\boldsymbol{\theta}}_{1,n}\right)'$:

$$Q_{2,n}(\boldsymbol{\theta}) = \left(n^{-1}\sum_i \boldsymbol{g}_i(\boldsymbol{\theta})\right)' \widehat{\mathbf{\Sigma}}_{1,n}^{-1} \left(n^{-1}\sum_i \boldsymbol{g}_i(\boldsymbol{\theta})\right)$$

$$\widehat{\boldsymbol{\theta}}_{2,n} = \arg\min_{\boldsymbol{\theta}\in\mathbf{\Theta}} Q_{2,n}(\boldsymbol{\theta})$$

Asymptotics

Theorem: Under very general regularity conditions:

$$\widehat{\boldsymbol{\theta}}_{1,n} \rightarrow^{p} \boldsymbol{\theta}_0$$

$$\widehat{\boldsymbol{\theta}}_{2,n} \rightarrow^{p} \boldsymbol{\theta}_0$$

Theorem: Under additional regularity conditions:

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_{1,n} - \boldsymbol{\theta}_0\right) \rightarrow^{d} MN\left(\mathbf{0}, \mathbf{V}_{1,\mathcal{F}_0}\right)$$

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_{2,n} - \boldsymbol{\theta}_0\right) \rightarrow^{d} MN\left(\mathbf{0}, \mathbf{V}_{2,\mathcal{F}_0}\right)$$

$\mathbf{V}_{1,\mathcal{F}_0}$, $\mathbf{V}_{2,\mathcal{F}_0}$ are $p \times p$ positive definite **stochastic** matrices

▸ mixed normality

Inference

Consider testing r parametric restrictions:

$$H_0 : \mathbf{a}(\boldsymbol{\theta}_0) = \mathbf{0}, \; H_A : \mathbf{a}(\boldsymbol{\theta}_0) \neq \mathbf{0}$$

Suppose:

- $r \times 1$ vector-function $\mathbf{a}(\boldsymbol{\theta})$ is continuously differentiable
- $r \times p$ Jacobian $\mathbf{A}(\boldsymbol{\theta}_0) = \partial \mathbf{a}(\boldsymbol{\theta}_0) / \partial \boldsymbol{\theta}'$ has full row rank

Under H_0, **Wald test statistic**

$$W \equiv n\mathbf{a}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)' \left[\mathbf{A}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right) \mathbf{V}_{2,n} \mathbf{A}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)'\right]^{-1} \mathbf{a}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right) \rightarrow^d \chi^2(r)$$

Remark: result for $\widehat{\boldsymbol{\theta}}_{1,n}$ is analogous

▸ formulas

Data

Sources:

- stock data: Center for Research in Security Prices (**CRSP**)
- T-bill data: Federal Reserve Bank Reports (from WRDS)

CRSP provides extensive information on securities traded on NYSE, AMEX, and NASDAQ, but not all securities are used

We only include regularly traded stocks of **operating companies**:

- closed-end funds, ETFs, financial REITs are dropped
- ADRs are included
- if company issues 2+ classes of shares, only class with largest number of outstanding shares is included

Preliminary Estimation Results

- Date 0: January 2nd, 2008
- Date T: January 4th, 2008
- Sample size $n = 5,413$
- Number of moments $k = 6$
- Index return $M_T / M_0 = 0.975$
- T-bill rate $r = 3.2\%$

Parameter	Estimate	Wald Statistic	P-value
σ_m	0.07^*	$1.03 \cdot 10^4$	0.00
γ	6.48^*	$2.24 \cdot 10^2$	0.00
κ_β	-0.19	0.07	0.80
λ_β	3.34	1.90	0.17
λ_σ	11.15^*	$1.45 \cdot 10^4$	0.00

Further Directions

Currently in progress:

- estimation of model parameters

Extensions of financial model:

- multi-factor stock price model
- stochastic volatility setting

Direction for future econometric research:

- MLE under common non-localized shocks

Thank you!

Questions?

Mixed Normal Distribution

Random variable Y has **mixed normal distribution**

$$Y \sim MN\left(0, \eta^2\right)$$

if characteristic function of Y is

$$\phi_Y(t) \equiv E\left[\exp\left(itY\right)\right] = E\left[\exp\left(-\frac{1}{2}\eta^2 t^2\right)\right]$$

where η is random variable

Y can be represented as

$$Y = \eta Z$$

where $Z \sim N(0,1)$ and Z is **independent** of η

◂ return to asymptotics

Inference: Formulas

$$W \equiv n\mathbf{a}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)'\left[\mathbf{A}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)\mathbf{V}_{2,n}\mathbf{A}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)'\right]^{-1}\mathbf{a}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)$$

$$\mathbf{V}_{2,n} = \left[\mathbf{G}_{2,n}'\widehat{\boldsymbol{\Sigma}}_{2,n}^{-1}\mathbf{G}_{2,n}\right]^{-1}$$

$$\mathbf{G}_{2,n} = n^{-1}\sum_i \partial g_i\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)/\partial\boldsymbol{\theta}'$$

$$\widehat{\boldsymbol{\Sigma}}_{2,n} = n^{-1}\sum_i g_i\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)g_i\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)'$$

◂ return to inference