Estimating Idiosyncratic Volatility and Its Effects on a Cross-Section of Returns

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Research Objective and Novelty

Goals:

- Develop a method to consistently estimate parameters of a financial model using a **single cross-section** of return data
- Apply the method to compute idiosyncratic volatility parameters, including **idiosyncratic volatility premium**

Novelty:

- Estimation method differs from the two-pass regression approach of Fama & MacBeth (1973)
- GMM estimation is implemented under strong **cross-sectional data dependence**

▸ econometric literature

Finance Literature

Why focus on stock-specific idiosyncratic volatility (**IV**)?

- Classical finance models (e.g., Sharpe, 1964; Lintner, 1965):
  - IV commands **no premium** in capital market equilibrium
- Growing literature indicates that **IV may be priced**:
  - Levy (1978), Merton (1987), Malkiel & Xu (2006)
  - Epstein & Schneider (2008)
  - Guo & Savickas (2010), Chabi-Yo (2011), Bhootra & Hur (2011)
- **No consensus** on IV premium in empirical literature:
  - Fu (2009), Huang et al. (2010): **positive** premium
  - Ang et al. (2006), Jiang et al. (2009): **negative** premium

Financial Model Structure

Financial assets:

- many risky assets called **stocks**
- one diversified portfolio of stocks called **market index**
- one riskless asset such as T-Bill

Asset prices are quoted continuously, but we will ultimately focus on only two dates: $t = 0$ and $t = T$

Simplifying assumption:

Between 0 and T, risk-free rate r is constant

Market Index Price Dynamics

Dynamics of market index:

$$\frac{dM_t}{M_t} = \mu_m dt + \sigma_m dW_t$$

where drift μ_m is:

$$\mu_m = r + \delta\sigma_m$$

- σ_m: market volatility, $\sigma_m > 0$
- δ: Sharpe ratio of market index, non-identifiable
- $\{W_t\}$: **systematic** risk source, modeled as Brownian motion

Stock Price Dynamics

Dynamics of stock i for $i = 1, 2, ...$:

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \beta_i \sigma_m dW_t + \sigma_i dZ_t^i$$

where drift μ_i is:

$$\mu_i = r + \delta \beta_i \sigma_m + \gamma \sigma_i$$

- $\beta_i \sim UNI\left[\kappa_\beta, \kappa_\beta + \lambda_\beta\right]$: beta of stock i
- $\sigma_i \sim UNI\left[0, \lambda_\sigma\right]$: idiosyncratic volatility of stock i
- γ: idiosyncratic volatility premium
- $\left\{Z_t^i\right\}$: **idiosyncratic** risk source, modeled as Brownian motion

Estimation Challenge

Using Itô's lemma:

$$\frac{S_T^i}{S_0^i} = \exp\left[\left(\mu_i - 0.5\beta_i^2\sigma_m^2 - 0.5\sigma_i^2\right)T + \beta_i\sigma_m W_T + \sigma_i Z_T^i\right]$$

$$\frac{M_T}{M_0} = \exp\left[\left(\mu_m - 0.5\sigma_m^2\right)T + \sigma_m W_T\right]$$

W_T, Z_T^i for $i = 1, 2, \ldots \sim i.i.d.\ N(0, T)$

Common shock W_T induces **dependence** among $\frac{S_T^1}{S_0^1}, \frac{S_T^2}{S_0^2}, \ldots \Rightarrow$

$\Rightarrow$ standard LLNs and CLTs are **not applicable**

But $\frac{S_T^1}{S_0^1}, \frac{S_T^2}{S_0^2}, \ldots$ are **conditionally i.i.d.** given $\frac{M_T}{M_0}$

GMM Implementation

Let $\boldsymbol{\theta} = \left(\sigma_m, \gamma, \kappa_\beta, \lambda_\beta, \lambda_\sigma\right)'$. We construct a function $g_i\left(\xi; \boldsymbol{\theta}\right)$:

$$g_i\left(\xi; \boldsymbol{\theta}\right) = \left(S_T^i / S_0^i\right)^{\xi} - E_{\boldsymbol{\theta}}\left[\left(S_T^i / S_0^i\right)^{\xi} | M_T / M_0\right]$$

Given constants $\xi_1, ..., \xi_k$, $k \times 1$ vector of **moment restrictions** is:

$$\underset{(k\times 1)}{\boldsymbol{g}_i}\left(\boldsymbol{\theta}\right) = \left(g_i\left(\xi_1; \boldsymbol{\theta}\right), ..., g_i\left(\xi_k; \boldsymbol{\theta}\right)\right)'$$

For any finite ξ, $E_{\boldsymbol{\theta}}\left[\left(S_T^i / S_0^i\right)^{\xi} | M_T / M_0\right]$ exists and can be expressed analytically

GMM Estimation

GMM objective function:

$$Q_n(\boldsymbol{\theta}) = \left(\frac{1}{n} \sum_i \boldsymbol{g}_i(\boldsymbol{\theta}) \right)' \boldsymbol{\Sigma}^{-1} \left(\frac{1}{n} \sum_i \boldsymbol{g}_i(\boldsymbol{\theta}) \right)$$

$\boldsymbol{\Sigma}$ is a $k \times k$ positive definite matrix

GMM estimator:

$$\widehat{\boldsymbol{\theta}}_n = \arg\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} Q_n(\boldsymbol{\theta})$$

As $n \to \infty$, $Q_n(\boldsymbol{\theta})$ converges to a **stochastic** function dependent on common shock

Properties of Estimator

Under general regularity conditions:

$$\widehat{\boldsymbol{\theta}}_n \longrightarrow^p \boldsymbol{\theta}_0$$

$\widehat{\boldsymbol{\theta}}_n$ is **consistent** as $n \longrightarrow \infty$

Note: two-pass regression requires $T \longrightarrow \infty$ (Shanken, 1992)

Under additional regularity conditions:

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \longrightarrow^d MN\left(\mathbf{0}, \mathbf{V}_{M_T/M_0}\right)$$

$\widehat{\boldsymbol{\theta}}_n$ is asymptotically **mixed normal**. $\mathbf{V}_{M_T/M_0}$ is **stochastic**

▸ jump to data ▸ inference ▸ mixed normality

Monte Carlo Design

Inputs:

- $\sigma_m = 0.20$, $\gamma = 0.50$
- $\kappa_\beta = -0.20$, $\lambda_\beta = 3.40$; $\lambda_\sigma = 0.50$
- $\delta = 0.50$, $r = 0.01$, $T = 1/12$

Identifiable parameters are $\boldsymbol{\theta} = (\sigma_m, \gamma, \kappa_\beta, \lambda_\beta, \lambda_\sigma)'$

Moment restrictions are of the form:

$$g_i(\xi; \boldsymbol{\theta}) = \left(S_T^i / S_0^i\right)^\xi - E_{\boldsymbol{\theta}}\left[\left(S_T^i / S_0^i\right)^\xi |M_T / M_0\right]$$

- vector $\boldsymbol{g}_i(\boldsymbol{\theta}) = (g_i(\xi_1; \boldsymbol{\theta}), ..., g_i(\xi_6; \boldsymbol{\theta}))'$
- vector $\boldsymbol{\xi} = (-1.5, -1, -0.5, 0.5, 1, 1.5)'$

Monte Carlo Results

	Sample size n (thousands)					
	25	50	250	1,000	10,000	True value
Panel A: Means						
σ_m	0.2526	0.2382	0.2205	0.2116	0.2011	0.2000
γ	0.5560	0.5339	0.5161	0.5076	0.5020	0.5000
κ_β	−0.1316	−0.1484	−0.1476	−0.1817	−0.1978	−0.2000
λ_β	3.6166	3.5798	3.4874	3.4722	3.4303	3.4000
λ_σ	0.4989	0.4996	0.4998	0.4999	0.5000	0.5000
Panel B: RMSEs						
σ_m	0.2327	0.2102	0.1382	0.1279	0.0658	
γ	0.2105	0.1582	0.0836	0.0488	0.0182	
κ_β	0.9925	0.8817	0.7330	0.4077	0.1410	
λ_β	1.4086	1.2965	0.8896	0.8310	0.4298	
λ_σ	0.0063	0.0046	0.0020	0.0010	0.0003	
Panel C: Test sizes, H_0 : parameter = true value, %						
σ_m	15.80	13.20	8.00	7.10	5.70	5.00
γ	7.30	5.50	5.40	5.60	5.30	5.00
κ_β	8.30	6.40	5.70	5.40	4.60	5.00
λ_β	10.60	9.60	5.60	5.50	4.70	5.00
λ_σ	3.80	3.10	4.60	3.80	4.50	5.00

additional MC results

Empirical Implementation: Data

- Stock data are from **CRSP** database:
  - include stocks of operating companies
  - include only one share class per company
  - exclude bankruptcy cases, closed-end funds, ETFs, REITs
  - use **weekly** returns, computed by compounding daily returns
- Market index is approximated by S&P 500 index
- Risk-free rate is derived from 4-week T-Bill
- We evaluate estimation method on data from two months:
  - **January 2008**: a month of relatively low market volatility
  - **October 2008**: a month of relatively high market volatility

Selected Empirical Results: Full Model Estimates

	January 22-29, 2008		October 23-30, 2008	
Parameter	Estimate	P-value	Estimate	P-value
σ_m	0.0537	0.00	0.0672	0.00
γ	-2.1117	0.34	-1.2936	0.56
κ_β	0.3417	0.74	-0.3058	0.74
λ_β	3.0475	0.00	2.8367	0.00
λ_σ	1.0580	0.00	1.7478	0.00

Notes:

- σ_m: market volatility
- γ: idiosyncratic volatility premium
- Beta of stock i, $\beta_i \sim i.i.d.UNI[\kappa_\beta, \kappa_\beta + \lambda_\beta]$
- Idiosyncratic volatility of stock i, $\sigma_i \sim i.i.d.UNI[0, \lambda_\sigma]$
- Moment order vector $\boldsymbol{\xi} = (-2, -1.5, -1, -0.5, 0.5, 1, 1.5, 2)'$

Selected Empirical Results: January 2008

Interval	Idiosyncratic volatility premium, γ		Average idiosyncratic volatility, $\lambda_\sigma/2$	
	Estimate	P-value	Estimate	P-value
January 02-09	-4.7251	0.00	0.5609	0.02
03-10	-5.0907	0.00	0.5370	0.00
04-11	-8.0336	0.00	0.4747	0.00
07-14	-0.8656	0.00	0.5359	0.00
08-15	-4.4627	0.00	0.5106	0.00
09-16	-9.1830	0.00	0.4816	0.00
⋮	⋮	⋮	⋮	⋮
Mean	-6.0666		0.5452	
Std. dev.	3.6396		0.0519	

▸ return decomposition

Selected Empirical Results: October 2008

Interval	Idiosyncratic volatility premium, γ		Average idiosyncratic volatility, $\lambda_\sigma/2$	
	Estimate	P-value	Estimate	P-value
October 01-08	-8.5104	0.00	0.8095	0.00
02-09	-8.4123	0.00	0.8858	0.00
03-10	-8.4921	0.00	0.8999	0.00
06-13	-7.8321	0.00	0.7532	0.00
07-14	-5.9003	0.00	0.7949	0.00
08-15	-0.8830	0.00	0.8595	0.01
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Mean	-5.5748		0.8372	
Std. dev.	3.9705		0.0725	

return decomposition

Summary

- We develop an econometric method to estimate a financial model featuring a common shock:
  - the method differs from the two-pass regression approach
  - consistent and asymptotically mixed normal estimates are obtained as the number of stocks (n) grows
  - estimation is implemented on a single return cross-section
- Findings using returns from January and October 2008:
  - IV premium is estimated to be negative
  - average cross-sectional IV estimates increase by 50% between January and October

Thank you!

Questions?

Relationship to Econometric Literature

Large literature on **localized** common shocks:

- general approach: Conley (1999)
- spatial, group, social effects: e.g., Kelejian & Prucha (1999), Lee (2007), Bramoullé et al. (2009)

Sparse literature on **non-localized** common shocks:

- Andrews (2003, 2005)

We build on Andrews (2003) to develop GMM estimation theory under a non-localized common shock in cross-sectional data

◂ back to goals

Mixed Normal Distribution

Random variable Y has **mixed normal distribution**:

$$Y \sim MN\left(0, \eta^2\right)$$

if characteristic function of Y is:

$$\phi_Y(t) \equiv E\left[\exp\left(itY\right)\right] = E\left[\exp\left(-\frac{1}{2}\eta^2 t^2\right)\right]$$

where η is random variable

Y can be represented as:

$$Y = \eta Z$$

where $Z \sim N(0,1)$ and Z is **independent** of η

back to properties

Inference and Specification Test

Consider testing r parametric restrictions:

$$H_0 : \underset{(r\times 1)}{\mathbf{a}\left(\boldsymbol{\theta}_0\right)} = \mathbf{0}$$

Let $\mathbf{A}(\cdot)$ be Jacobian of $\mathbf{a}(\cdot)$. Under H_0, **Wald test** statistic

$$W_n \equiv n\mathbf{a}\left(\widehat{\boldsymbol{\theta}}_n\right)' \left[\mathbf{A}\left(\widehat{\boldsymbol{\theta}}_n\right) \mathbf{V}_n \mathbf{A}\left(\widehat{\boldsymbol{\theta}}_n\right)'\right]^{-1} \mathbf{a}\left(\widehat{\boldsymbol{\theta}}_n\right) \rightarrow^d \chi^2(r)$$

OIR test can be implemented after two-step estimation. If the model is correctly specified:

$$J_n \equiv n \cdot Q_{2,n}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right) \rightarrow^d \chi^2(k-p)$$

◂ back to properties

Monte Carlo Results: 1-Step vs. 2-Step Estimation

	RMSE		|*Median–true value*|		|*Mean–true value*|		True
Parameter	1-step	2-step	1-step	2-step	1-step	2-step	value
σ_m	0.5631	0.7967	0.0916	0.1433	0.2770	0.2643	0.2000
γ	0.5337	0.7546	0.0014	0.0818	0.0223	0.1013	-2.0000
κ_β	1.3163	1.8470	0.0067	0.0514	0.0209	0.0611	0.5000
λ_β	2.4048	3.3329	0.3979	0.4720	0.0862	0.1034	3.0000
λ_σ	0.0277	0.0394	0.0039	0.0049	0.0052	0.0060	1.0000

Notes:

- Number of stocks $n = 5,500$
- Number of simulation rounds: $1,000$
- Risk-free rate $r = 0.01$
- Moment order vector $\boldsymbol{\xi} = (-2, -1.5, -1, -0.5, 0.5, 1, 1.5, 2)'$
- Moment restrictions of the form: $g_i(\xi; \boldsymbol{\theta}) = \left(S_T^i / S_0^i\right)^\xi - E_{\boldsymbol{\theta}}\left[\left(S_T^i / S_0^i\right)^\xi | M_T / M_0\right]$

back to MC results

Conditional Return Decomposition: January 2008

Interval	E	$\mathcal{S}$	$\mathcal{I}$	$\frac{E\text{-}e^{rT}\mathcal{S}}{E}$	$\frac{e^{rT}(\mathcal{S}\text{-}\mathcal{I})}{E}$
January 02-09	0.9486	0.9988	0.9492	-0.0535	0.0523
03-10	0.9647	1.0173	0.9477	-0.0552	0.0722
04-11	0.9762	1.0512	0.9280	-0.0776	0.1263
07-14	0.9870	0.9955	0.9909	-0.0092	0.0047
08-15	0.9833	1.0277	0.9561	-0.0459	0.0729
09-16	0.9857	1.0741	0.9172	-0.0903	0.1593
⋮	⋮	⋮	⋮	⋮	⋮
Mean	0.9890	1.0488	0.9430	-0.0615	0.1071
Std. dev.	0.0311	0.0337	0.0311	0.0346	0.0571

Notes:

- Conditional expected gross return $E = \exp(rT) \cdot \mathcal{S}(M_T/M_0) \cdot \mathcal{I}$
- Risk-free component: $\exp(rT)$
- Market risk component: $\mathcal{S}(M_T/M_0)$
- Idiosyncratic volatility component: $\mathcal{I}$

◂ back to January results

Conditional Return Decomposition: October 2008

Interval	E	$\mathcal{S}$	$\mathcal{I}$	$\frac{E - e^{rT}\mathcal{S}}{E}$	$\frac{e^{rT}(\mathcal{S}-\mathcal{I})}{E}$
October 01-08	0.8211	0.9384	0.8750	-0.1429	0.0773
02-09	0.8022	0.9267	0.8657	-0.1551	0.0760
03-10	0.8163	0.9463	0.8626	-0.1593	0.1026
06-13	0.9455	1.0605	0.8916	-0.1216	0.1787
07-14	0.9852	1.0797	0.9125	-0.0959	0.1698
08-15	0.9352	0.9494	0.9851	-0.0151	-0.0382
⋮	⋮	⋮	⋮	⋮	⋮
Mean	0.9438	1.0270	0.9184	-0.0929	0.1162
Std. dev.	0.0834	0.0564	0.0574	0.0654	0.0778

Notes:

- Conditional expected gross return $E = \exp(rT) \cdot \mathcal{S}(M_T/M_0) \cdot \mathcal{I}$
- Risk-free component: $\exp(rT)$
- Market risk component: $\mathcal{S}(M_T/M_0)$
- Idiosyncratic volatility component: $\mathcal{I}$

back to October results