What Can We Learn from a Cross-Section of Returns?

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Independence in Cross-Sectional Data

Cross-sectional econometricians typically assume observations are **independent** and (often, also) identically distributed (i.i.d.)

Independence allows for straightforward derivation of asymptotic properties of cross-sectional extremum estimators:

- *asymptotic properties:* consistency and asymptotic normality
- *extremum estimators:* MLE and GMM (including OLS and IV)

In many cross-sectional settings **independence breaks down**

Common Shocks in Cross-Sectional Data

Observations are **dependent** if population units are affected by **common shocks**

Examples:

- oil price shocks affect production costs of many firms
- interest rate shocks affect consumption decisions of many households

Empirical evidence in finance literature:

- returns on stocks are driven by common factors

Localized vs. Non-Localized Common Shocks

Localized shock:

- dependence between observations fades with **distance**
- distance may be geographical, socioeconomic, time-wise, etc.

Non-localized shock:

- dependence between observations does not fade

Consider observations $X_1, X_2, ..., X_{100}, ...$:

- localized shock: X_1, X_{100} are "less dependent" than X_1, X_2
- non-localized shock: no such relationship exists

Econometrics Literature

Localized common shocks:

- general approach: Conley (1999)
- spatial effects: e.g., Kelejian & Prucha (1999)
- group effects: e.g., Lee (2007)
- social effects: e.g., Bramoullé et al. (2009)

Non-localized common shocks:

- Andrews (2003)
- Andrews (2005)

Workplan

We propose GMM estimators for non-linear cross-sectional model with non-localized common shock

We specify regularity conditions under which GMM estimators are:

- consistent
- asymptotically mixed normal

We show that inference can still be conducted using conventional Wald tests

We provide financial application to demonstrate methodology

Outline

- Econometric Framework
- Application
- Further Directions

Preliminaries: Martingale Difference Sequence

Sequence of random variables $\{Y_i\}$ on probability space $(\Omega, \mathcal{F}, P)$ is **martingale difference sequence** (m.d.s.) with respect to filtration $\{\mathcal{F}_i\}$ if:

(i) Y_i is measurable with respect to $\mathcal{F}_i$ for all i

(ii) $E\left[|Y_i|\right] < \infty$ for all i

(iii) $E[Y_j|F_i] = 0$ a.s. for all $j > i$

forward to filtration

Preliminaries: Mixed-Normal Distribution

Random variable Y has **mixed normal distribution**

$$Y \sim MN\left(0, \eta^2\right)$$

if characteristic function of Y is

$$\phi_Y(t) \equiv E\left[\exp\left(itY\right)\right] = E\left[\exp\left(-\frac{1}{2}\eta^2 t^2\right)\right]$$

where η is random variable

Y can be represented as

$$Y = \eta Z$$

where $Z \sim N(0,1)$ and Z is **independent** of η

Setup: Data Structure

Data generating process provides observations $X_0, X_1, X_2, \ldots$

Data structure:

- X_0 is driven by systematic (common) risk
- X_i, $i = 1, 2, \ldots$, is driven by systematic **and** idiosyncratic risk

Examples:

- aggregate per capita income *vs.* individual incomes
- stock market return *vs.* individual stock returns

We interpret systematic risk as non-localized common shock
$\Rightarrow \{X_i\}$ is neither ergodic stationary nor mixingale

forward to erg. stationarity forward to mixingale

Setup: Conditionally I.I.D. Observations

Let $X_0, X_1, X_2, \ldots$ be defined on probability space $(\Omega, \mathcal{F}, P)$

Assumption:

$X_1, X_2, \ldots$ are **conditionally i.i.d.** given σ-field $\mathcal{F}_0 \equiv \sigma(X_0)$

$\sigma(X_0)$: σ-field **generated by** X_0 (i.e., by systematic risk)

This assumption is **very** mild (Andrews, 2005):

when sample units are randomly drawn, it is compatible with:

- arbitrary dependence across population units
- different effects of systematic risk on population units
- heterogeneity across population units

forward to generation of σ-fields   forward to de Finetti's theorem

Setup: Parameters and Moment Restrictions

Goal: estimate and do inference on true $p \times 1$ parameter vector $\boldsymbol{\theta}_0$

Parameter set is $\boldsymbol{\Theta}$:

- $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$
- $\boldsymbol{\Theta}$ is compact and convex subset of $\mathbb{R}^p$

Economic model provides k **moment restrictions**

$$\boldsymbol{g}_i(\boldsymbol{\theta}) \equiv \underset{(k\times 1)}{\boldsymbol{g}}(X_i; \boldsymbol{\theta}, X_0) \text{ for } i = 1, 2, \ldots$$

For example, 1st component of $\boldsymbol{g}_i(\boldsymbol{\theta})$ may be:

$$g_i^{(1)}(\boldsymbol{\theta}) = X_i - E_{\boldsymbol{\theta}}[X_i | X_0]$$

GMM Estimators

One-step estimation using $k \times k$ nonstoch. positive definite $\boldsymbol{\Sigma}$:

$$Q_{1,n}(\boldsymbol{\theta}) = \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{g}_i(\boldsymbol{\theta})\right)' \boldsymbol{\Sigma}^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{g}_i(\boldsymbol{\theta})\right)$$

$$\widehat{\boldsymbol{\theta}}_{1,n} = \arg\min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} Q_{1,n}(\boldsymbol{\theta})$$

Two-step estimation using $\widehat{\boldsymbol{\Sigma}}_{1,n} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{g}_i\left(\widehat{\boldsymbol{\theta}}_{1,n}\right) \boldsymbol{g}_i\left(\widehat{\boldsymbol{\theta}}_{1,n}\right)'$:

$$Q_{2,n}(\boldsymbol{\theta}) = \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{g}_i(\boldsymbol{\theta})\right)' \widehat{\boldsymbol{\Sigma}}_{1,n}^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{g}_i(\boldsymbol{\theta})\right)$$

$$\widehat{\boldsymbol{\theta}}_{2,n} = \arg\min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} Q_{2,n}(\boldsymbol{\theta})$$

Regularity Conditions for Consistency

- $\boldsymbol{g}_i(\boldsymbol{\theta})$ is a.s. continuous and differentiable on $\boldsymbol{\Theta}$
- $E\left[\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \|\boldsymbol{g}_i(\boldsymbol{\theta})\|^2\right] < \infty$
- $E\left[\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left\|\partial \boldsymbol{g}_i(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}'\right\|^2\right] < \infty$
- $E[\boldsymbol{g}_i(\boldsymbol{\theta}_0) | \mathcal{F}_0] = \mathbf{0}$ a.s. and $E[\boldsymbol{g}_i(\boldsymbol{\theta}) | \mathcal{F}_0] \neq \mathbf{0}$ a.s. if $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$
- $k \times k$ stochastic matrix $\boldsymbol{\Sigma}_{\mathcal{F}_0} = E\left[\boldsymbol{g}_i(\boldsymbol{\theta}_0)\boldsymbol{g}_i(\boldsymbol{\theta}_0)' | \mathcal{F}_0\right]$ is a.s. positive definite

Remarks:

- $\|\cdot\|$ is Euclidean norm
- $\mathcal{F}_0 \equiv \sigma(X_0)$

Consistency: Result

Theorem:

Under regularity conditions:

$$\widehat{\boldsymbol{\theta}}_{1,n} \longrightarrow^p \boldsymbol{\theta}_0$$

$$\widehat{\boldsymbol{\theta}}_{2,n} \longrightarrow^p \boldsymbol{\theta}_0$$

as $n \longrightarrow \infty$

Proof applies law of large numbers for conditionally i.i.d. random variables

forward to l.l.n.

skip to asy. mixed normality

Consistency: Proof Sketch

We adapt argument due to Andrews (2003) but clarify several details

Sketch:

- infer existence and measurability of estimator from standard theorem
- show pointwise convergence of objective
- show stochastic equicontinuity of objective
- establish uniform convergence of objective
- establish unique minimum of objective in the limit at $\boldsymbol{\theta}_0$ a.s.
- use above results to prove convergence of estimator to $\boldsymbol{\theta}_0$

▸ forward to stoch. equicontinuity ▸ forward to l.l.n.

Regularity Conditions for Asymptotic Mixed Normality

Additional regularity conditions:

- there is open ball $\mathcal{N} \subset \mathbf{\Theta}$ centered at $\boldsymbol{\theta}_0$ such that $\mathbf{g}_i(\boldsymbol{\theta})$ is a.s. twice differentiable on $\mathcal{N}$ and $E\left[\sup_{\boldsymbol{\theta}\in\mathcal{N}}\left\|\frac{\partial^2 \mathbf{g}_i(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right\|\right] < \infty$
- $k \times p$ stochastic matrix $\mathbf{G}_{\mathcal{F}_0} = E\left[\partial \mathbf{g}_i(\boldsymbol{\theta}_0)/\partial\boldsymbol{\theta}' | \mathcal{F}_0\right]$ has full column rank a.s.

Remark:

We also need to show that $\{\mathbf{g}_i(\boldsymbol{\theta}_0)\}$ is **m.d.s.** with respect to some filtration $\{\mathcal{F}_i\}$

We can take $\mathcal{F}_i = \sigma(X_0, X_1, ..., X_i)$. Observe that if $j > i$:

$$E\left[\mathbf{g}_j(\boldsymbol{\theta}_0)|\mathcal{F}_i\right] \equiv E\left[\mathbf{g}\left(X_j;\boldsymbol{\theta}_0, X_0\right)|\sigma(X_0, X_1, ..., X_i)\right] =$$

$$= E\left[\mathbf{g}\left(X_j;\boldsymbol{\theta}_0, X_0\right)|\sigma(X_0)\right] \equiv E\left[\mathbf{g}_j(\boldsymbol{\theta}_0)|\mathcal{F}_0\right] = \mathbf{0}$$

Asymptotic Mixed Normality: Result

Theorem:

Under regularity conditions:

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_{1,n} - \boldsymbol{\theta}_0\right) \to^d MN\left(\mathbf{0}, \mathbf{V}_{1,\mathcal{F}_0}\right)$$

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_{2,n} - \boldsymbol{\theta}_0\right) \to^d MN\left(\mathbf{0}, \mathbf{V}_{2,\mathcal{F}_0}\right)$$

as $n \to \infty$

$\mathbf{V}_{1,\mathcal{F}_0}$, $\mathbf{V}_{2,\mathcal{F}_0}$ are $p \times p$ a.s. positive definite **stochastic** matrices:

$$\mathbf{V}_{1,\mathcal{F}_0} = \left[\mathbf{G}'_{\mathcal{F}_0}\boldsymbol{\Sigma}^{-1}\mathbf{G}_{\mathcal{F}_0}\right]^{-1}\mathbf{G}'_{\mathcal{F}_0}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_{\mathcal{F}_0}\boldsymbol{\Sigma}^{-1}\mathbf{G}_{\mathcal{F}_0}\left[\mathbf{G}'_{\mathcal{F}_0}\boldsymbol{\Sigma}^{-1}\mathbf{G}_{\mathcal{F}_0}\right]^{-1}$$

$$\mathbf{V}_{2,\mathcal{F}_0} = \left[\mathbf{G}'_{\mathcal{F}_0}\boldsymbol{\Sigma}_{\mathcal{F}_0}^{-1}\mathbf{G}_{\mathcal{F}_0}\right]^{-1}$$

skip to asy. inference

Asymptotic Mixed Normality: Proof Sketch

Proof utilizes conventional techniques:

- mean-value expand $\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{g}_i\left(\widehat{\boldsymbol{\theta}}_{1,n}\right)$ around $\boldsymbol{\theta}_0$ in f.o.c.
- show that $\mathbf{G}_{1,n} \equiv \frac{1}{n}\sum_{i=1}^{n} \frac{\partial \boldsymbol{g}_i\left(\widehat{\boldsymbol{\theta}}_{1,n}\right)}{\partial \boldsymbol{\theta}'} \to^{p} \mathbf{G}_{\mathcal{F}_0}$
- invoke c.l.t. for m.d.s. to show that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \boldsymbol{g}_i\left(\boldsymbol{\theta}_0\right) \to^{d} \left[\boldsymbol{\Sigma}_{\mathcal{F}_0}\right]^{\frac{1}{2}} \mathbf{Z}$$

- invoke Slutsky's theorem to establish final result with $\mathbf{V}_{1,\mathcal{F}_0}$
- repeat steps for $\widehat{\boldsymbol{\theta}}_{2,n}$ and simplify to obtain $\mathbf{V}_{2,\mathcal{F}_0}$

forward to c.l.t.

Asymptotic Inference

Consider testing r parameter restrictions:

$$H_0 : \mathbf{a}(\boldsymbol{\theta}_0) = \mathbf{0},\ H_A : \mathbf{a}(\boldsymbol{\theta}_0) \neq \mathbf{0}$$

Suppose:

- $r \times 1$ vector-function $\mathbf{a}(\boldsymbol{\theta})$ is continuously differentiable on $\boldsymbol{\Theta}$
- $r \times p$ Jacobian $\mathbf{A}(\boldsymbol{\theta}_0) = \partial \mathbf{a}(\boldsymbol{\theta}_0) / \partial \boldsymbol{\theta}'$ has full row rank

then, it can be shown that under H_0, **Wald test statistic**

$$W \equiv n\mathbf{a}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)' \left[\mathbf{A}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)\mathbf{V}_{2,n}\mathbf{A}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)'\right]^{-1} \mathbf{a}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right) \to^d \chi^2(r)$$

Remark: result for $\widehat{\boldsymbol{\theta}}_{1,n}$ is analogous

▸ forward to formulas

Financial Market Structure

Financial assets:

- many risky assets called **stocks**
- one diversified portfolio of stocks called **market index**
- one riskless asset (e.g., Treasury bill)

Asset prices are quoted continuously, but we will ultimately focus on only two dates: $t = 0$ and $t = T$

Simplification:

between 0 and T, risk-free interest rate r is constant

Market Index Price Dynamics

Dynamics of market index price:

$$\frac{dM_t}{M_t} = \mu_m dt + \sigma_m dW_t$$

where drift μ_m is

$$\mu_m = r + \delta\sigma_m$$

- σ_m: market volatility, $\sigma_m > 0$
- δ: Sharpe ratio of market index
- $\{W_t\}$: systematic risk, modeled as standard Brownian motion

Stock Price Dynamics

Dynamics of price of stock i for $i = 1, 2, \ldots$:

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \beta_i \sigma_m dW_t + \sigma_i dZ_t^i$$

where drift μ_i is

$$\mu_i = r + \delta \beta_i \sigma_m + \gamma \sigma_i$$

- β_i: systematic risk loading ("*beta*") of stock i
- σ_i: idiosyncratic volatility of stock i
- γ: idiosyncratic risk premium
- $\{Z_t^i\}$: idiosyncratic risk, modeled as standard Brownian motion

Additional Assumptions

$\{W_t\}$, $\{Z_t^1\}$, $\{Z_t^2\}$, ... are mutually independent processes

We specify β_i and σ_i for $i = 1, 2, ...$ as random variables:

$$\beta_i \sim i.i.d.\ UNI\left[\kappa_\beta, \kappa_\beta + \lambda_\beta\right],\ \lambda_\beta > 0$$

$$\sigma_i \sim i.i.d.\ UNI\left[0, \lambda_\sigma\right],\ \lambda_\sigma > 0$$

Remark:

Using cross-sectional data, it is impossible to estimate β_i and σ_i

Relationship to Finance Literature

Recall:

$$\mu_i = r + \delta\beta_i\sigma_m + \gamma\sigma_i$$

If $\gamma = 0$, our price dynamics are in line with:

- ICAPM with constant invest. opportunity set: Merton (1973)
- APT with one market factor: Ross (1976)

But growing literature suggests that idiosyncratic risk is priced:

- Merton (1987), Malkiel & Xu (2006): incomplete diversification
- Epstein & Schneider (2008): ambiguity premium

Green & Rydqvist (1997), Ang et al. (2006), Fu (2009): idiosyncratic premium is nonzero, but no consensus about sign

Contribution to Empirical Finance Literature

Estimating γ helps inform debate over idiosyncratic risk premium:

- value of γ affects construction of investment strategies

Estimating σ_m from cross-sectional data is complementary to high-frequency time-series approach (e.g., Andersen et al., 2003):

- many pricing applications require volatility estimates

Remark:

Our estimation method differs from traditional regression technique of Fama & MacBeth (1973)

GMM Implementation: Observations

Using Itô's lemma:

$$\frac{S_T^i}{S_0^i} = \exp\left[\left(\mu_i - \frac{1}{2}\beta_i^2\sigma_m^2 - \frac{1}{2}\sigma_i^2\right)T + \beta_i\sigma_m W_T + \sigma_i Z_T^i\right]$$

$$\frac{M_T}{M_0} = \exp\left[\left(\mu_m - \frac{1}{2}\sigma_m^2\right)T + \sigma_m W_T\right]$$

where W_T, Z_T^i for $i = 1, 2, ... \sim i.i.d.\ N(0, T)$

Interpretation: $X_0 = \frac{M_T}{M_0}, X_1 = \frac{S_T^1}{S_0^1}, X_2 = \frac{S_T^2}{S_0^2}, ...$

Remark:

Easy to see that $\frac{S_T^1}{S_0^1}, \frac{S_T^2}{S_0^2}, ...$ are **conditionally i.i.d.** given $\frac{M_T}{M_0}$

GMM Implementation: Moment Restrictions

Theorem:

Let $\mathcal{F}_0 = \sigma\left(M_T/M_0\right)$ and $\boldsymbol{\theta} = \left(\sigma_m, \gamma, \kappa_\beta, \lambda_\beta, \lambda_\sigma\right)'$. For any finite $\xi \in \mathbb{R}$, $E_{\boldsymbol{\theta}}\left[\left(S_T^i/S_0^i\right)^{\xi} |\mathcal{F}_0\right]$ exists and can be expressed analytically. Moreover, it is continuously differentiable in $\boldsymbol{\theta}$ and all derivatives can be expressed analytically

Given constants $\xi_1, ..., \xi_k$, let $k \times 1$ vector of moment restrictions be

$$\boldsymbol{g}_i\left(\boldsymbol{\theta}\right) = \left(g_i\left(\xi_1; \boldsymbol{\theta}\right), ..., g_i\left(\xi_k; \boldsymbol{\theta}\right)\right)'$$

where l^{th} component of $\boldsymbol{g}_i\left(\boldsymbol{\theta}\right)$ is

$$\boldsymbol{g}_i^{(l)}\left(\boldsymbol{\theta}\right) \equiv g_i\left(\xi_l; \boldsymbol{\theta}\right) = \left(S_T^i/S_0^i\right)^{\xi_l} - E_{\boldsymbol{\theta}}\left[\left(S_T^i/S_0^i\right)^{\xi_l} |\mathcal{F}_0\right]$$

Remark: δ is not identifiable

▸ forward to formulas

Data Sources

Sources:

- stock data: Center for Research in Security Prices (**CRSP**)
- T-bill data: Federal Reserve Bank Reports (from WRDS)
- index data: Yahoo! Finance and Bloomberg

All raw data are daily. We use data from two months:

- January 2008: low market volatility month
- October 2008: high market volatility month

CRSP provides extensive information on assets traded on NYSE, AMEX, and NASDAQ, but not all assets are stocks of companies

Data: Details on Stocks

We only include securities that are regularly traded stocks of operating companies:

- exclude closed-end funds, ETFs, mortgage/financial REITs
- include ADRs (stocks of foreign companies traded on U.S. exchanges)
- if company issues two or more classes of shares, include class with largest number of outstanding shares

Average daily number of included distinct securities:

- January 2008: 5,452
- October 2008: 5,245

Further Directions

Currently in progress:

- estimation of model parameters

Direction for future econometric research:

- MLE under common shocks

Extensions of financial application:

- multi-factor stock price model
- stochastic volatility setting

Thank you!

Questions?

Appendix Outline I

σ-Field

Collection $\mathcal{F}$ of subsets of set Ω is said to be **σ-field** on Ω if $\mathcal{F}$ has following properties:

(i) $\Omega \in \mathcal{F}$

(ii) if $A \in \mathcal{F}$, then its complement $A^c \in \mathcal{F}$

(iii) if $A_n \in \mathcal{F}$ for $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

return to filtration

return to de Finetti's theorem

return to generation of σ-fields

Filtration

Let $(\Omega, \mathcal{F}, P)$ be probability space. **Filtration** on $(\Omega, \mathcal{F}, P)$ is family $\{\mathcal{F}_i\}$ of σ-fields such that:

(i) $\mathcal{F}_i \subseteq \mathcal{F}$ for every i

(ii) $\mathcal{F}_i \subseteq \mathcal{F}_j$ if $i < j$

forward to σ-field

return to m.d.s.

return to mixingale

Interchangeable R.V.'s and de Finetti's Theorem

Random variables $X_1, X_2, \ldots, X_n$ are called **interchangeable** (exchangeable) if their joint cumulative distribution function (c.d.f.) is symmetric function, i.e., if their c.d.f. is invariant under permutations

Collection of random variables $\{X_i\}_{i=1}^{\infty}$ is interchangeable if every finite subset of them is interchangeable

de Finetti's Theorem:

Collection of random variables $\{X_i\}_{i=1}^{\infty}$ on probability space $(\Omega, \mathcal{F}, P)$ are interchangeable if and only if they are **conditionally independent and identically distributed** given some σ-field $\mathcal{G}$

See Chow & Teicher (1997, pp. 232-234)

▸ forward to σ-field

◂ return to conditional i.i.d.'ness

How to Generate σ-Fields

If $\mathcal{G}$ is any collection of subsets of Ω, there always exists smallest σ-field $\mathcal{F}$ on Ω such that $\mathcal{G} \subset \mathcal{F}$

If $X : \Omega \to \mathbb{R}^n$ is any function, then σ-field **generated by** X, denoted as $\sigma(X)$, is smallest σ-field on Ω containing all sets

$$X^{-1}(U), \text{ where } U \subset \mathbb{R}^n \text{ is open}$$

Remark:

Random variable X is measurable with respect to σ-field $\sigma(X)$, as well as any σ-field containing $\sigma(X)$

See Rudin (1987, p. 12), Øksendal (1995, pp. 6-7)

▸ forward to σ-field

◂ return to conditional i.i.d.'ness

Law of Large Numbers for Conditionally I.I.D. R.V.'s

Let random variables $X_1, X_2, \ldots$ be defined on probability space $(\Omega, \mathcal{F}, P)$. Suppose there exists σ-field $\mathcal{F}_0 \subset \mathcal{F}$ such that, **conditional on** $\mathcal{F}_0$, $X_1, X_2, \ldots$ are i.i.d. Let $h(\cdot)$ be vector-valued function that satisfies $E\|h(X_i)\| < \infty$, where $\|\cdot\|$ is Euclidean norm. Then:

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i) \rightarrow^{p} E(h(X_i)\,|\mathcal{F}_0) \text{ as } n \rightarrow \infty$$

Remark:

$E(h(X_i)\,|\mathcal{F}_0)$ is random variable

See Andrews (2005, p. 1557), Hall & Heyde (1980, p. 202)

return to consistency result

return to consistency proof

Central Limit Theorem for M.D.S.

Let $\{S_{ni}, \mathcal{F}_{ni}, 1 \le i \le k_n, n \ge 1\}$ be zero-mean, square-integrable martingale array with differences X_{ni}, and let η^2 be a.s. finite r.v. Suppose that:

(i) $\max_i |X_{ni}| \to^p 0$

(ii) $\sum_i X_{ni}^2 \to^p \eta^2$

(iii) $E\left(\max_i X_{ni}^2\right)$ is bounded in n

and σ-fields are nested: $\mathcal{F}_{n,i} \subseteq \mathcal{F}_{n+1,i}$. Then:

$$S_{nk_n} = \sum_i X_{ni} \to^d Z \text{ (stably)},$$

where r.v. Z has characteristic function $E\left[\exp\left(-\frac{1}{2}\eta^2 t^2\right)\right]$

Remark: Z has **mixed normal** distribution

See Hall & Heyde (1980, pp. 58-59)

◂ return to proof of asy. mixed normality

Ergodic Stationarity

Sequence of random variables $\{X_i\}$ is (strictly) **stationary** if, for any finite integer r and any set of subscripts $i_1, i_2, ..., i_r$, joint distribution of $(X_i, X_{i_1}, X_{i_2}, ..., X_{i_r})$ depends on $i_1 - i, i_2 - i, ..., i_r - i$ and does **not** depend on i

Stationary sequence $\{X_i\}$ is **ergodic stationary** if, for any two bounded functions $f : \mathbb{R}^{k+1} \to \mathbb{R}$ and $g : \mathbb{R}^{l+1} \to \mathbb{R}$:

$$\lim_{n\to\infty} |Ef(X_i, ..., X_{i+k}) \cdot g(X_{i+n}, ..., X_{i+l+n})| = \\ = |Ef(X_i, ..., X_{i+k})| \cdot |Eg(X_i, ..., X_{i+l})|$$

return to data structure

Mixingale

Let $\{X_i\}$ be sequence of random variables and let $\{\mathcal{F}_i\}$ be filtration on probability space $(\Omega, \mathcal{F}, P)$

Let $\|\cdot\|_p$ denote $L^p(P)$ norm: $\|X_i\|_p = \left(E|X_i|^p\right)^{\frac{1}{p}}$

Sequence $\{X_i, \mathcal{F}_i\}$ is **L^1-mixingale** if there exist nonnegative constants $\{c_i\}$ and $\{\psi_m\}$ such that $\psi_m \to 0$ as $m \to \infty$ and for all i and $m \geq 0$:

(i) $\|E(X_i|\mathcal{F}_{i-m})\|_1 \leq c_i \psi_m$

(ii) $\|X_i - E(X_i|\mathcal{F}_{i+m})\|_1 \leq c_i \psi_{m+1}$

Remark: condition (ii) usually holds trivially, because X_i is almost always measurable with respect to $\mathcal{F}_i$

See McLeish (1975), Andrews (1988)

▸ forward to filtration

◂ return to data structure

Stochastic Equicontinuity (I)

Let $B(\theta, \delta)$ denote closed ball of radius $\delta > 0$ centered at θ. Sequence of functions $\{G_n(\theta)\}$ is **stochastically equicontinuous** on Θ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n \to \infty} P\left(\sup_{\theta \in \Theta} \sup_{\theta' \in B(\theta, \delta)} \left| G_n(\theta') - G_n(\theta) \right| > \epsilon \right) < \epsilon$$

Assumption SE-1 of Andrews (1992, p. 246):

(a) $G_n(\theta) = \hat{Q}_n(\theta) - Q_n(\theta)$, where $Q_n(\cdot)$ is nonrandom function that is continuous in θ uniformly over Θ

(b) $\left| \hat{Q}_n(\theta') - \hat{Q}_n(\theta) \right| \leq B_n h(d(\theta', \theta))$ for any $\theta', \theta \in \Theta$ a.s. for some random variable B_n and some nonrandom function h such that $h(y) \downarrow 0$ as $y \downarrow 0$, where d is metric on Θ

(c) $B_n = O_p(1)$

continue

Stochastic Equicontinuity (II)

Lemma 1 of Andrews (1992, p. 246). If $\{G_n(\theta)\}$ satisfies Assumption SE-1, then $\{G_n(\theta)\}$ is stochastically equicontinuous on Θ

Theorem 1 of Andrews (1992, p. 244). Suppose that:

(i) Θ is totally bounded metric space

(ii) $G_n(\theta) \rightarrow^p 0$ for all $\theta \in \Theta$ (pointwise)

(iii) $\{G_n(\theta)\}$ is stochastically equicontinuous on Θ

then $G_n(\theta)$ converges **uniformly** in probability to 0:

$$\sup_{\theta \in \Theta} |G_n(\theta)| \rightarrow^p 0$$

Remark: total boundedness is weaker than compactness

return to consistency proof

Asymptotic Inference: Formulas

$$W \equiv n\mathbf{a}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)'\left[\mathbf{A}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)\mathbf{V}_{2,n}\mathbf{A}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)'\right]^{-1}\mathbf{a}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)$$

$$\mathbf{V}_{2,n} = \left[\mathbf{G}_{2,n}'\widehat{\boldsymbol{\Sigma}}_{2,n}^{-1}\mathbf{G}_{2,n}\right]^{-1}$$

$$\mathbf{G}_{2,n} = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial \boldsymbol{g}_i\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)}{\partial \boldsymbol{\theta}'}$$

$$\widehat{\boldsymbol{\Sigma}}_{2,n} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{g}_i\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)\boldsymbol{g}_i\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)'$$

◂ return to asy. inference

Conditional Moment Formula (I)

$$E_{\boldsymbol{\theta}}\left[\left(S_T^i/S_0^i\right)^{\xi}|\mathcal{F}_0\right] = \exp\left[r\xi T\right]\cdot A\left[x_A, y_A\right]\cdot B\left[x_B, y_B\right]$$

where

$$x_A = \xi\left(\ln\left(M_T/M_0\right) + \left[\frac{1}{2}\sigma_m^2 - r\right]T\right)$$

$$y_A = -\frac{1}{2}\xi\sigma_m^2 T$$

$$x_B = \xi\gamma T$$

$$y_B = \frac{1}{2}\xi\left(\xi - 1\right)T$$

▸ continue

Conditional Moment Formula (II)

If $\xi < 0$, $A\left[x_A, y_A\right] = \frac{\sqrt{\pi}}{2\lambda_\beta\sqrt{y_A}} \exp\left[-\frac{x_A^2}{4y_A}\right] \times$

$$\times \left(\mathrm{erfi}\left[\frac{x_A}{2\sqrt{y_A}} + \left(\kappa_\beta + \lambda_\beta\right)\sqrt{y_A}\right] - \mathrm{erfi}\left[\frac{x_A}{2\sqrt{y_A}} + \kappa_\beta\sqrt{y_A}\right]\right)$$

If $\xi > 0$, $A\left[x_A, y_A\right] = \frac{\sqrt{\pi}}{2\lambda_\beta\sqrt{-y_A}} \exp\left[-\frac{x_A^2}{4y_A}\right] \times$

$$\times \left(\mathrm{erf}\left[\frac{x_A}{2\sqrt{-y_A}} - \kappa_\beta\sqrt{-y_A}\right] - \mathrm{erf}\left[\frac{x_A}{2\sqrt{-y_A}} - \left(\kappa_\beta + \lambda_\beta\right)\sqrt{-y_A}\right]\right)$$

If $\xi = 0$, $A\left[x_A, y_A\right] = 1$

$\mathrm{erf}\left[\cdot\right]$ is error function: $\mathrm{erf}\left[z\right] = \frac{2}{\sqrt{\pi}} \int_0^z \exp\left(-t^2\right) dt$

$\mathrm{erfi}\left[\cdot\right]$ is imaginary error function: $\mathrm{erfi}\left[z\right] = \frac{2}{\sqrt{\pi}} \int_0^z \exp\left(t^2\right) dt$

continue

Conditional Moment Formula (III)

If $\xi < 0$ or $\xi > 1$, $B\left[x_B, y_B\right] = \frac{\sqrt{\pi}}{2\lambda_\sigma \sqrt{y_B}} \exp\left[-\frac{x_B^2}{4y_B}\right] \times$

$$\times \left(\operatorname{erfi}\left[\frac{x_B}{2\sqrt{y_B}} + \lambda_\sigma \sqrt{y_B}\right] - \operatorname{erfi}\left[\frac{x_B}{2\sqrt{y_B}}\right]\right)$$

If $0 < \xi < 1$, $B\left[x_B, y_B\right] = \frac{\sqrt{\pi}}{2\lambda_\sigma \sqrt{-y_B}} \exp\left[-\frac{x_B^2}{4y_B}\right] \times$

$$\times \left(\operatorname{erf}\left[\frac{x_B}{2\sqrt{-y_B}}\right] - \operatorname{erf}\left[\frac{x_B}{2\sqrt{-y_B}} - \lambda_\sigma \sqrt{-y_B}\right]\right)$$

If $\xi = 1$ and $x_B \neq 0$, $B\left[x_B, y_B\right] = \frac{\exp[\lambda_\sigma x_B] - 1}{\lambda_\sigma x_B}$

If $\xi = 1$ and $x_B = 0$ or if $\xi = 0$, $B\left[x_B, y_B\right] = 1$

return to GMM implementation