What Can We Learn from a Cross-Section of Returns?

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Independence in Cross-Sectional Data

Cross-sectional econometricians typically assume observations are **independent** and (often, also) identically distributed (i.i.d.)

Independence allows for straightforward derivation of asymptotic properties of cross-sectional extremum estimators:

- **asymptotic properties**: consistency and asymptotic normality
- **extremum estimators**: MLE and GMM (including OLS and IV)

In many cross-sectional settings **independence breaks down**
Observations are dependent if population units are affected by common shocks.

Examples:
- Oil price shocks affect production costs of many firms.
- Interest rate shocks affect consumption decisions of many households.

Empirical evidence in finance literature:
- Returns on stocks are driven by common factors.
Localized vs. Non-Localized Common Shocks

**Localized** shock:
- dependence between observations fades with **distance**
- distance may be geographical, socioeconomic, time-wise, etc.

**Non-localized** shock:
- dependence between observations does not fade

Consider observations $X_1, X_2, ..., X_{100}, ...$
- localized shock: $X_1, X_{100}$ are “less dependent” than $X_1, X_2$
- non-localized shock: no such relationship exists
Localized common shocks:

- general approach: Conley (1999)
- spatial effects: e.g., Kelejian & Prucha (1999)
- group effects: e.g., Lee (2007)
- social effects: e.g., Bramoullé et al. (2009)

Non-localized common shocks:

- Andrews (2005)
We propose GMM estimators for non-linear cross-sectional model with non-localized common shock.

We specify regularity conditions under which GMM estimators are:

- consistent
- asymptotically mixed normal

We show that inference can still be conducted using conventional Wald tests.

We provide financial application to demonstrate methodology.
Outline

- Econometric Framework
- Application
- Further Directions
Preliminaries: Martingale Difference Sequence

Sequence of random variables \( \{Y_i\} \) on probability space \((\Omega, \mathcal{F}, P)\) is **martingale difference sequence** (m.d.s.) with respect to filtration \( \{\mathcal{F}_i\} \) if:

(i) \( Y_i \) is measurable with respect to \( \mathcal{F}_i \) for all \( i \)

(ii) \( E[|Y_i|] < \infty \) for all \( i \)

(iii) \( E[Y_j|F_i] = 0 \) a.s. for all \( j > i \)
Preliminaries: Mixed-Normal Distribution

Random variable $Y$ has **mixed normal distribution**

$$Y \sim MN \left( 0, \eta^2 \right)$$

if characteristic function of $Y$ is

$$\phi_Y (t) \equiv E \left[ \exp (itY) \right] = E \left[ \exp \left( -\frac{1}{2} \eta^2 t^2 \right) \right]$$

where $\eta$ is random variable

$Y$ can be represented as

$$Y = \eta Z$$

where $Z \sim N (0, 1)$ and $Z$ is **independent** of $\eta$
Setup: Data Structure

Data generating process provides observations $X_0, X_1, X_2, ...$

Data structure:

- $X_0$ is driven by systematic (common) risk
- $X_i$, $i = 1, 2, ...$, is driven by systematic and idiosyncratic risk

Examples:

- aggregate per capita income vs. individual incomes
- stock market return vs. individual stock returns

We interpret systematic risk as non-localized common shock

$\Rightarrow \{X_i\}$ is neither ergodic stationary nor mixingale
Setup: Conditionally I.I.D. Observations

Let $X_0, X_1, X_2, ...$ be defined on probability space $(\Omega, \mathcal{F}, P)$

**Assumption:**

$X_1, X_2, ...$ are **conditionally i.i.d.** given $\sigma$-field $\mathcal{F}_0 \equiv \sigma (X_0)$

$\sigma (X_0)$: $\sigma$-field **generated by** $X_0$ (i.e., by systematic risk)

This assumption is **very** mild (Andrews, 2005):

- arbitrary dependence across population units
- different effects of systematic risk on population units
- heterogeneity across population units
Setup: Parameters and Moment Restrictions

**Goal**: estimate and do inference on true $p \times 1$ parameter vector $\theta_0$

Parameter set is $\Theta$:

- $\theta_0 \in \Theta$
- $\Theta$ is compact and convex subset of $\mathbb{R}^p$

Economic model provides $k$ moment restrictions

$$g_i(\theta) \equiv g(X_i; \theta, X_0) \text{ for } i = 1, 2, ...$$

For example, 1st component of $g_i(\theta)$ may be:

$$g_i^{(1)}(\theta) = X_i - E_{\theta}[X_i|X_0]$$
**GMM Estimators**

**One-step** estimation using \( k \times k \) nonstoch. positive definite \( \Sigma \):

\[
Q_{1,n} (\theta) = \left( \frac{1}{n} \sum_{i=1}^{n} g_i (\theta) \right)' \Sigma^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} g_i (\theta) \right)
\]

\[
\hat{\theta}_{1,n} = \arg \min_{\theta \in \Theta} Q_{1,n} (\theta)
\]

**Two-step** estimation using \( \hat{\Sigma}_{1,n} = \frac{1}{n} \sum_{i=1}^{n} g_i (\hat{\theta}_{1,n}) g_i (\hat{\theta}_{1,n})' \):

\[
Q_{2,n} (\theta) = \left( \frac{1}{n} \sum_{i=1}^{n} g_i (\theta) \right)' \hat{\Sigma}_{1,n}^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} g_i (\theta) \right)
\]

\[
\hat{\theta}_{2,n} = \arg \min_{\theta \in \Theta} Q_{2,n} (\theta)
\]
Regularity Conditions for Consistency

- $g_i(\theta)$ is a.s. continuous and differentiable on $\Theta$

- $E\left[\sup_{\theta \in \Theta} \|g_i(\theta)\|^2\right] < \infty$

- $E\left[\sup_{\theta \in \Theta} \left\|\frac{\partial g_i(\theta)}{\partial \theta'}\right\|^2\right] < \infty$

- $E \left[ g_i(\theta_0) | \mathcal{F}_0 \right] = 0$ a.s. and $E \left[ g_i(\theta) | \mathcal{F}_0 \right] \neq 0$ a.s. if $\theta \neq \theta_0$

- $k \times k$ stochastic matrix $\Sigma_{\mathcal{F}_0} = E \left[ g_i(\theta_0) g_i(\theta_0)' | \mathcal{F}_0 \right]$ is a.s. positive definite

Remarks:

- $\| \cdot \|$ is Euclidean norm

- $\mathcal{F}_0 \equiv \sigma (X_0)$
Theorem:
Under regularity conditions:

\[ \hat{\theta}_{1,n} \rightarrow^p \theta_0 \]

\[ \hat{\theta}_{2,n} \rightarrow^p \theta_0 \]

as \( n \rightarrow \infty \)

Proof applies law of large numbers for conditionally i.i.d. random variables.
Consistency: Proof Sketch

We adapt argument due to Andrews (2003) but clarify several details

Sketch:

- infer existence and measurability of estimator from standard theorem
- show pointwise convergence of objective
- show stochastic equicontinuity of objective
- establish uniform convergence of objective
- establish unique minimum of objective in the limit at $\theta_0$ a.s.
- use above results to prove convergence of estimator to $\theta_0$
Regularity Conditions for Asymptotic Mixed Normality

Additional regularity conditions:

- there is open ball $\mathcal{N} \subset \Theta$ centered at $\theta_0$ such that $g_i(\theta)$ is a.s. twice differentiable on $\mathcal{N}$ and $E \left[ \sup_{\theta \in \mathcal{N}} \left\| \frac{\partial^2 g_i(\theta)}{\partial \theta \partial \theta'} \right\| \right] < \infty$

- $k \times p$ stochastic matrix $G_{\mathcal{F}_0} = E \left[ \frac{\partial g_i(\theta_0)}{\partial \theta} | \mathcal{F}_0 \right]$ has full column rank a.s.

Remark:
We also need to show that $\{g_i(\theta_0)\}$ is m.d.s. with respect to some filtration $\{\mathcal{F}_i\}$

We can take $\mathcal{F}_i = \sigma(X_0, X_1, ..., X_i)$. Observe that if $j > i$:

$$E \left[ g_j(\theta_0) | \mathcal{F}_i \right] = E \left[ g(X_j; \theta_0, X_0) | \sigma(X_0, X_1, ..., X_i) \right] =$$

$$= E \left[ g(X_j; \theta_0, X_0) | \sigma(X_0) \right] \equiv E \left[ g_j(\theta_0) | \mathcal{F}_0 \right] = 0$$
Asymptotic Mixed Normality: Result

**Theorem:**
Under regularity conditions:

\[
\sqrt{n} \left( \hat{\theta}_{1,n} - \theta_0 \right) \xrightarrow{d} \text{MN} \left( 0, V_{1,F_0} \right)
\]

\[
\sqrt{n} \left( \hat{\theta}_{2,n} - \theta_0 \right) \xrightarrow{d} \text{MN} \left( 0, V_{2,F_0} \right)
\]

as \( n \to \infty \)

\( V_{1,F_0}, V_{2,F_0} \) are \( p \times p \) a.s. positive definite stochastic matrices:

\[
V_{1,F_0} = \left[ G'_{F_0} \Sigma^{-1} G_{F_0} \right]^{-1} G'_{F_0} \Sigma^{-1} \Sigma_{F_0} \Sigma^{-1} G_{F_0} \left[ G'_{F_0} \Sigma^{-1} G_{F_0} \right]^{-1}
\]

\[
V_{2,F_0} = \left[ G'_{F_0} \Sigma_{F_0} G_{F_0} \right]^{-1}
\]
Asymptotic Mixed Normality: Proof Sketch

Proof utilizes conventional techniques:

- mean-value expand \( \frac{1}{n} \sum_{i=1}^{n} g_i \left( \hat{\theta}_{1,n} \right) \) around \( \theta_0 \) in f.o.c.

- show that \( G_{1,n} \equiv \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_i \left( \hat{\theta}_{1,n} \right)}{\partial \theta} \rightarrow p G_{F_0} \)

- invoke c.l.t. for m.d.s. to show that

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_i \left( \theta_0 \right) \xrightarrow{d} \left[ \Sigma_{F_0} \right]^{\frac{1}{2}} Z
\]

- invoke Slutsky’s theorem to establish final result with \( V_{1,F_0} \)

- repeat steps for \( \hat{\theta}_{2,n} \) and simplify to obtain \( V_{2,F_0} \)
Asymptotic Inference

Consider testing $r$ parameter restrictions:

$$H_0 : \mathbf{a}(\theta_0) = \mathbf{0}, \quad H_A : \mathbf{a}(\theta_0) \neq \mathbf{0}$$

Suppose:

- $r \times 1$ vector-function $\mathbf{a}(\theta)$ is continuously differentiable on $\Theta$
- $r \times p$ Jacobian $\mathbf{A}(\theta_0) = \partial \mathbf{a}(\theta_0) / \partial \theta'$ has full row rank

then, it can be shown that under $H_0$, **Wald test statistic**

$$W \equiv na\left(\hat{\theta}_{2,n}\right)' \left[\mathbf{A}\left(\hat{\theta}_{2,n}\right) \mathbf{V}_{2,n} \mathbf{A}\left(\hat{\theta}_{2,n}\right)'ight]^{-1} \mathbf{a}\left(\hat{\theta}_{2,n}\right) \rightarrow^d \chi^2(r)$$

Remark: result for $\hat{\theta}_{1,n}$ is analogous
Financial Market Structure

Financial assets:

- many risky assets called **stocks**
- one diversified portfolio of stocks called **market index**
- one riskless asset (e.g., Treasury bill)

Asset prices are quoted continuously, but we will ultimately focus on only two dates: \( t = 0 \) and \( t = T \)

**Simplification:**
between 0 and \( T \), risk-free interest rate \( r \) is constant
Dynamics of market index price:

\[
\frac{dM_t}{M_t} = \mu_m dt + \sigma_m dW_t
\]

where drift \( \mu_m \) is

\[
\mu_m = r + \delta \sigma_m
\]

- \( \sigma_m \): market volatility, \( \sigma_m > 0 \)
- \( \delta \): Sharpe ratio of market index
- \( \{W_t\} \): systematic risk, modeled as standard Brownian motion
Stock Price Dynamics

Dynamics of price of stock $i$ for $i = 1, 2, \ldots$:

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \beta_i \sigma_m dW_t + \sigma_i dZ_t^i$$

where drift $\mu_i$ is

$$\mu_i = r + \delta \beta_i \sigma_m + \gamma \sigma_i$$

- $\beta_i$: systematic risk loading ("beta") of stock $i$
- $\sigma_i$: idiosyncratic volatility of stock $i$
- $\gamma$: idiosyncratic risk premium
- $\{Z_t^i\}$: idiosyncratic risk, modeled as standard Brownian motion
Additional Assumptions

{\mathcal{W}_t}, \{Z^1_t\}, \{Z^2_t\}, \ldots \text{ are mutually independent processes}

We specify $\beta_i$ and $\sigma_i$ for $i = 1, 2, \ldots$ as random variables:

$$\beta_i \sim \text{i.i.d. \ UNI} [\kappa_\beta, \kappa_\beta + \lambda_\beta], \ \lambda_\beta > 0$$

$$\sigma_i \sim \text{i.i.d. \ UNI} [0, \lambda_\sigma], \ \lambda_\sigma > 0$$

Remark:

Using cross-sectional data, it is impossible to estimate $\beta_i$ and $\sigma_i$
Relationship to Finance Literature

Recall:

$$\mu_i = r + \delta \beta_i \sigma_m + \gamma \sigma_i$$

If $\gamma = 0$, our price dynamics are in line with:

- ICAPM with constant invest. opportunity set: Merton (1973)
- APT with one market factor: Ross (1976)

**But** growing literature suggests that idiosyncratic risk is priced:

- Epstein & Schneider (2008): ambiguity premium

Green & Rydqvist (1997), Ang et al. (2006), Fu (2009): idiosyncratic premium is nonzero, but no consensus about sign
Contribution to Empirical Finance Literature

Estimating $\gamma$ helps inform debate over idiosyncratic risk premium:

- value of $\gamma$ affects construction of investment strategies

Estimating $\sigma_m$ from cross-sectional data is complementary to high-frequency time-series approach (e.g., Andersen et al., 2003):

- many pricing applications require volatility estimates

Remark:

Our estimation method differs from traditional regression technique of Fama & MacBeth (1973)
GMM Implementation: Observations

Using Itô’s lemma:

\[
\frac{S^i_T}{S^i_0} = \exp \left[ \left( \mu_i - \frac{1}{2} \beta^2_i \sigma^2_m - \frac{1}{2} \sigma^2_i \right) T + \beta_i \sigma_m W_T + \sigma_i Z^i_T \right]
\]

\[
\frac{M_T}{M_0} = \exp \left[ \left( \mu_m - \frac{1}{2} \sigma^2_m \right) T + \sigma_m W_T \right]
\]

where \( W_T, Z^i_T \) for \( i = 1, 2, ... \) \(~\text{i.i.d.}~ N(0, T)\)

Interpretation: \( X_0 = \frac{M_T}{M_0}, X_1 = \frac{S^1_T}{S^1_0}, X_2 = \frac{S^2_T}{S^2_0}, ... \)

Remark:

Easy to see that \( \frac{S^1_T}{S^1_0}, \frac{S^2_T}{S^2_0}, ... \) are conditionally i.i.d. given \( \frac{M_T}{M_0} \)
GMM Implementation: Moment Restrictions

**Theorem:**

Let $\mathcal{F}_0 = \sigma (M_T / M_0)$ and $\theta = (\sigma_m, \gamma, \kappa_\beta, \lambda_\beta, \lambda_\sigma)'$. For any finite $\xi \in \mathbb{R}$, $E_\theta \left[ \left( \frac{S_T^i}{S_0^i} \right)^{\xi} \middle| \mathcal{F}_0 \right]$ exists and can be expressed analytically. Moreover, it is continuously differentiable in $\theta$ and all derivatives can be expressed analytically.

Given constants $\xi_1, ..., \xi_k$, let $k \times 1$ vector of moment restrictions be

$$g_i (\theta) = (g_i (\xi_1; \theta), ..., g_i (\xi_k; \theta))'$$

where $l^{th}$ component of $g_i (\theta)$ is

$$g_i^{(l)} (\theta) \equiv g_i (\xi_l; \theta) = \left( \frac{S_T^i}{S_0^i} \right)^{\xi_l} - E_\theta \left[ \left( \frac{S_T^i}{S_0^i} \right)^{\xi_l} \middle| \mathcal{F}_0 \right]$$

**Remark:** $\delta$ is not identifiable.
Data Sources

Sources:

- stock data: Center for Research in Security Prices (CRSP)
- T-bill data: Federal Reserve Bank Reports (from WRDS)
- index data: Yahoo! Finance and Bloomberg

All raw data are daily. We use data from two months:

- January 2008: low market volatility month
- October 2008: high market volatility month

CRSP provides extensive information on assets traded on NYSE, AMEX, and NASDAQ, but not all assets are stocks of companies.
Data: Details on Stocks

We only include securities that are regularly traded stocks of operating companies:

- exclude closed-end funds, ETFs, mortgage/financial REITs
- include ADRs (stocks of foreign companies traded on U.S. exchanges)
- if company issues two or more classes of shares, include class with largest number of outstanding shares

Average daily number of included distinct securities:

- January 2008: 5,452
- October 2008: 5,245
Further Directions

Currently in progress:

- estimation of model parameters

Direction for future econometric research:

- MLE under common shocks

Extensions of financial application:

- multi-factor stock price model
- stochastic volatility setting
Thank you!

Questions?
Appendix Outline I

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\( \sigma \)-Field

Collection \( \mathcal{F} \) of subsets of set \( \Omega \) is said to be \( \sigma \)-field on \( \Omega \) if \( \mathcal{F} \) has following properties:

(i) \( \Omega \in \mathcal{F} \)

(ii) if \( A \in \mathcal{F} \), then its complement \( A^c \in \mathcal{F} \)

(iii) if \( A_n \in \mathcal{F} \) for \( n = 1, 2, \ldots \), then \( \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \)
Let $(\Omega, \mathcal{F}, P)$ be probability space. **Filtration** on $(\Omega, \mathcal{F}, P)$ is family $\{\mathcal{F}_i\}$ of $\sigma$-fields such that:

(i) $\mathcal{F}_i \subseteq \mathcal{F}$ for every $i$

(ii) $\mathcal{F}_i \subseteq \mathcal{F}_j$ if $i < j$
Appendix

Interchangeable R.V.’s and de Finetti’s Theorem

Random variables $X_1, X_2, \ldots, X_n$ are called **interchangeable** (exchangeable) if their joint cumulative distribution function (c.d.f.) is symmetric function, i.e., if their c.d.f. is invariant under permutations.

Collection of random variables $\{X_i\}_{i=1}^{\infty}$ is interchangeable if every finite subset of them is interchangeable.

de Finetti’s Theorem:

Collection of random variables $\{X_i\}_{i=1}^{\infty}$ on probability space $(\Omega, \mathcal{F}, P)$ are interchangeable if and only if they are **conditionally independent and identically distributed** given some $\sigma$-field $\mathcal{G}$.

How to Generate $\sigma$-Fields

If $\mathcal{G}$ is any collection of subsets of $\Omega$, there always exists smallest $\sigma$-field $\mathcal{F}$ on $\Omega$ such that $\mathcal{G} \subseteq \mathcal{F}$.

If $X : \Omega \rightarrow \mathbb{R}^n$ is any function, then $\sigma$-field generated by $X$, denoted as $\sigma(X)$, is smallest $\sigma$-field on $\Omega$ containing all sets

$$X^{-1}(U), \text{ where } U \subseteq \mathbb{R}^n \text{ is open}$$

Remark:

Random variable $X$ is measurable with respect to $\sigma$-field $\sigma(X)$, as well as any $\sigma$-field containing $\sigma(X)$

See Rudin (1987, p. 12), Øksendal (1995, pp. 6-7)

forward to $\sigma$-field
return to conditional i.i.d.'ness
Law of Large Numbers for Conditionally I.I.D. R.V.’s

Let random variables $X_1, X_2, ...$ be defined on probability space $(\Omega, \mathcal{F}, P)$. Suppose there exists $\sigma$-field $\mathcal{F}_0 \subset \mathcal{F}$ such that, conditional on $\mathcal{F}_0$, $X_1, X_2, ...$ are i.i.d. Let $h(\cdot)$ be vector-valued function that satisfies $E\|h(X_i)\| < \infty$, where $\|\cdot\|$ is Euclidean norm. Then:

$$\frac{1}{n} \sum_{i=1}^{n} h(X_i) \xrightarrow{p} E(h(X_i) | \mathcal{F}_0) \quad \text{as} \quad n \to \infty$$

Remark:

$E(h(X_i) | \mathcal{F}_0)$ is random variable

Appendix

Central Limit Theorem for M.D.S.

Let \( \{S_{ni}, F_{ni}, 1 \leq i \leq k_n, n \geq 1\} \) be zero-mean, square-integrable martingale array with differences \( X_{ni} \), and let \( \eta^2 \) be a.s. finite r.v. Suppose that:

(i) \( \max_i |X_{ni}| \to^p 0 \)
(ii) \( \sum_i X_{ni}^2 \to^p \eta^2 \)
(iii) \( E(\max_i X_{ni}^2) \) is bounded in \( n \)

and \( \sigma \)-fields are nested: \( F_{n,i} \subseteq F_{n+1,i} \). Then:

\[
S_{nk_n} = \sum_i X_{ni} \to^d Z \text{ (stably),}
\]

where r.v. \( Z \) has characteristic function \( E[\exp (-\frac{1}{2} \eta^2 t^2)] \)

Remark: \( Z \) has mixed normal distribution

Ergodic Stationarity

Sequence of random variables \( \{X_i\} \) is (strictly) **stationary** if, for any finite integer \( r \) and any set of subscripts \( i_1, i_2, ..., i_r \), joint distribution of \((X_i, X_{i_1}, X_{i_2}, ..., X_{i_r})\) depends on \( i_1 - i, i_2 - i, ..., i_r - i \) and does **not** depend on \( i \)

Stationary sequence \( \{X_i\} \) is **ergodic stationary** if, for any two bounded functions \( f : \mathbb{R}^{k+1} \rightarrow \mathbb{R} \) and \( g : \mathbb{R}^{l+1} \rightarrow \mathbb{R} \):

\[
\lim_{n \to \infty} \left| Ef \left( X_i, ..., X_{i+k} \right) \cdot g \left( X_{i+n}, ..., X_{i+l+n} \right) \right| = \\
= \left| Ef \left( X_i, ..., X_{i+k} \right) \right| \cdot \left| Eg \left( X_i, ..., X_{i+l} \right) \right|
\]
Mixingale

Let \( \{X_i\} \) be sequence of random variables and let \( \{\mathcal{F}_i\} \) be filtration on probability space \((\Omega, \mathcal{F}, P)\).

Let \( \| \cdot \|_p \) denote \( L^p(P) \) norm: \( \|X_i\|_p = \left( E|X_i|^p \right)^{\frac{1}{p}} \)

Sequence \( \{X_i, \mathcal{F}_i\} \) is \textbf{\( L^1 \)-mixingale} if there exist nonnegative constants \( \{c_i\} \) and \( \{\psi_m\} \) such that \( \psi_m \to 0 \) as \( m \to \infty \) and for all \( i \) and \( m \geq 0 \):

(i) \( \|E(X_i|\mathcal{F}_{i-m})\|_1 \leq c_i\psi_m \)

(ii) \( \|X_i - E(X_i|\mathcal{F}_{i+m})\|_1 \leq c_i\psi_{m+1} \)

Remark: condition (ii) usually holds trivially, because \( X_i \) is almost always measurable with respect to \( \mathcal{F}_i \).

See McLeish (1975), Andrews (1988)
Stochastic Equicontinuity (I)

Let $B (\theta, \delta)$ denote closed ball of radius $\delta > 0$ centered at $\theta$. Sequence of functions $\{G_n (\theta)\}$ is **stochastically equicontinuous** on $\Theta$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$
\lim \sup_{n \to \infty} P \left( \sup_{\theta \in \Theta} \sup_{\theta' \in B (\theta, \delta)} |G_n (\theta') - G_n (\theta)| > \epsilon \right) < \epsilon
$$

Assumption SE-1 of Andrews (1992, p. 246):

(a) $G_n (\theta) = \hat{Q}_n (\theta) - Q_n (\theta)$, where $Q_n (\cdot)$ is nonrandom function that is continuous in $\theta$ uniformly over $\Theta$

(b) $|\hat{Q}_n (\theta') - \hat{Q}_n (\theta)| \leq B_n h (d (\theta', \theta))$ for any $\theta', \theta \in \Theta$ a.s. for some random variable $B_n$ and some nonrandom function $h$ such that $h (y) \downarrow 0$ as $y \downarrow 0$, where $d$ is metric on $\Theta$

(c) $B_n = O_p (1)$
Stochastic Equicontinuity (II)

Lemma 1 of Andrews (1992, p. 246). If \( \{ G_n(\theta) \} \) satisfies Assumption SE-1, then \( \{ G_n(\theta) \} \) is stochastically equicontinuous on \( \Theta \).

Theorem 1 of Andrews (1992, p. 244). Suppose that:

(i) \( \Theta \) is totally bounded metric space
(ii) \( G_n(\theta) \rightarrow^p 0 \) for all \( \theta \in \Theta \) (pointwise)
(iii) \( \{ G_n(\theta) \} \) is stochastically equicontinuous on \( \Theta \)

then \( G_n(\theta) \) converges uniformly in probability to 0:

\[
\sup_{\theta \in \Theta} |G_n(\theta)| \rightarrow^p 0
\]

Remark: total boundedness is weaker than compactness
Asymptotic Inference: Formulas

\[ W \equiv n a (\hat{\theta}_{2,n})' \left[ A (\hat{\theta}_{2,n}) V_{2,n} A (\hat{\theta}_{2,n})' \right]^{-1} a (\hat{\theta}_{2,n}) \]

\[ V_{2,n} = \left[ G'_{2,n} \hat{\Sigma}_{2,n}^{-1} G_{2,n} \right]^{-1} \]

\[ G_{2,n} = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_i}{\partial \theta'} (\hat{\theta}_{2,n}) \]

\[ \hat{\Sigma}_{2,n} = \frac{1}{n} \sum_{i=1}^{n} g_i (\hat{\theta}_{2,n}) g_i (\hat{\theta}_{2,n})' \]
Conditional Moment Formula (I)

\[
E_\theta \left[ \left( \frac{S_T^i}{S_0^i} \right)^{\xi} \mid \mathcal{F}_0 \right] = \exp [r\xi T] \cdot A [x_A, y_A] \cdot B [x_B, y_B]
\]

where

\[
x_A = \xi \left( \ln \left( \frac{M_T}{M_0} \right) + \left[ \frac{1}{2} \sigma_m^2 - r \right] T \right)
\]

\[
y_A = -\frac{1}{2} \xi \sigma_m^2 T
\]

\[
x_B = \xi \gamma T
\]

\[
y_B = \frac{1}{2} \xi (\xi - 1) T
\]
Conditional Moment Formula (II)

If $\xi < 0$, $A \left[ x_A, y_A \right] = \frac{\sqrt{\pi}}{2\lambda_\beta \sqrt{y_A}} \exp \left[ -\frac{x_A^2}{4y_A} \right] \times \left( \text{erfi} \left[ \frac{x_A}{2\sqrt{y_A}} + (\kappa_\beta + \lambda_\beta) \sqrt{y_A} \right] - \text{erfi} \left[ \frac{x_A}{2\sqrt{y_A}} + \kappa_\beta \sqrt{y_A} \right] \right)$

If $\xi > 0$, $A \left[ x_A, y_A \right] = \frac{\sqrt{\pi}}{2\lambda_\beta \sqrt{-y_A}} \exp \left[ -\frac{x_A^2}{4y_A} \right] \times \left( \text{erf} \left[ \frac{x_A}{2\sqrt{-y_A}} - \kappa_\beta \sqrt{-y_A} \right] - \text{erf} \left[ \frac{x_A}{2\sqrt{-y_A}} - (\kappa_\beta + \lambda_\beta) \sqrt{-y_A} \right] \right)$

If $\xi = 0$, $A \left[ x_A, y_A \right] = 1$

$\text{erf} \left[ \cdot \right]$ is error function: $\text{erf} \left[ z \right] = \frac{2}{\sqrt{\pi}} \int_0^z \exp \left( -t^2 \right) dt$

$\text{erfi} \left[ \cdot \right]$ is imaginary error function: $\text{erfi} \left[ z \right] = \frac{2}{\sqrt{\pi}} \int_0^z \exp \left( t^2 \right) dt$
Conditional Moment Formula (III)

If $\zeta < 0$ or $\zeta > 1$, \( B \left[ x_B, y_B \right] = \frac{\sqrt{\pi}}{2\lambda\sigma\sqrt{y_B}} \exp \left[ -\frac{x_B^2}{4y_B} \right] \times \)

\[ \times \left( \text{erfi} \left[ \frac{x_B}{2\sqrt{y_B}} + \lambda\sigma\sqrt{y_B} \right] - \text{erfi} \left[ \frac{x_B}{2\sqrt{y_B}} \right] \right) \]

If $0 < \zeta < 1$, \( B \left[ x_B, y_B \right] = \frac{\sqrt{\pi}}{2\lambda\sigma\sqrt{-y_B}} \exp \left[ -\frac{x_B^2}{4y_B} \right] \times \)

\[ \times \left( \text{erf} \left[ \frac{x_B}{2\sqrt{-y_B}} \right] - \text{erf} \left[ \frac{x_B}{2\sqrt{-y_B}} - \lambda\sigma\sqrt{-y_B} \right] \right) \]

If $\zeta = 1$ and $x_B \neq 0$, \( B \left[ x_B, y_B \right] = \frac{\exp[\lambda\sigma x_B] - 1}{\lambda\sigma x_B} \)

If $\zeta = 1$ and $x_B = 0$ or if $\zeta = 0$, \( B \left[ x_B, y_B \right] = 1 \)